Problem 9.9 asks us to use boundary layer theory to find the leading order solution to the initial value problem $\varepsilon y''(x) + ay'(x) + by(x) = 0$ with $y(0) = y'(0) = 1$ and $a > 0$. Then we are to compare to the exact solution. The problem is ambiguous as to whether a and b are functions or constants. For clarity's sake, we assume a and b are constants, but all the following work can be generalized for nonconstant a and b as well.

Since $a > 0$, the boundary layer occurs at $x = 0$, where the initial conditions are specified. We set $x = \varepsilon\xi$,[1] and then in the inner region,

$$\frac{1}{\varepsilon}y_{\text{in}}''(\xi) + \frac{a}{\varepsilon}y_{\text{in}}'(\xi) + by_{\text{in}}(\xi) = 0. \tag{1}$$

Thus, to leading order, $y_{\text{in}}''(\xi) \sim -ay_{\text{in}}'(\xi)$, which has solution $y_{\text{in}}(\xi) = C_0 + C_1 e^{-a\xi}$. Solving for C_0 and C_1, we see that $y(0) = 1 \implies C_0 + C_1 = 1$, and

$$y'(x)\bigg|_{x=0} = 1 \implies y'(\xi)\bigg|_{\xi=0} = \varepsilon \implies -aC_1 = \varepsilon \implies C_1 = -\varepsilon/a = O(\varepsilon).$$

This means that $C_1 e^{-a\xi} = O(\varepsilon)$ shouldn't appear at this order in the expansion, and $C_1 = 0$. We should throw this information out because we have already thrown out information at $O(\varepsilon)$ in solving the equation, and we have no guarantee that the $O(\varepsilon)$ value for C_1 is actually correct to $O(\varepsilon)$.

So there is no boundary layer at leading order! The inner solution $y_{\text{in}}(x) = C_0 = 1$ does not change rapidly, and it will cancel when we match, just leaving the outer solution. This is okay and happens occasionally when you get lucky.

In the outer region, we have $y_{\text{out}} \sim -\frac{b}{a}y_{\text{out}}$, so $y_{out}(x) = Ce^{-bx/a} + O(\varepsilon)$. Matching to the inner solution, $C = 1$, so $y_{\text{uniform}}(x) = e^{-bx/a} + O(\varepsilon)$. We note that $y_{\text{uniform}}'(0) = -\frac{b}{a} \neq 1$ in general.

Although the problem does not ask for it, we can also go to the next order in our asymptotic expansion. And even though no boundary layer appeared at leading order, one will appear at $O(\varepsilon)$.

Going back to the inner region, let $y_{\text{in}}(\xi) = Y_0(\xi) + \varepsilon Y_1(\xi) + O(\varepsilon^2)$. We already computed that $Y_0(\xi) = 1$. Now looking at (1) at $O(1)$, $Y_1''(\xi) + aY_1'(\xi) + bY_0(\xi) = 0 \implies Y_1''(\xi) + aY_1'(\xi) = -b$. Solving, $Y_1(\xi) = C_2 + C_3 e^{-a\xi} - \frac{b}{a}\xi$. We have initial conditions $Y_1(0) = 0$, but since $y'(0) = 1$ was not satisfied, we note that $Y_1'(0) = 1$. Thus, $C_2 + C_3 = 0$ and $-aC_3 - \frac{b}{a} = 1$, so

$$C_3 = -\frac{1}{a} - \frac{b}{a^2} \qquad \text{and} \qquad C_2 = -C_3 = \frac{1}{a} + \frac{b}{a^2}.$$

Therefore,

$$y_{\text{in}}(\xi) = 1 + \varepsilon\left(\left(\frac{1}{a} + \frac{b}{a^2}\right)\left(1 - e^{-a\xi}\right) - \frac{b}{a}\xi\right) + O(\varepsilon^2),$$

so

$$y_{\text{in}}(x) = 1 + \varepsilon\left(\left(\frac{1}{a} + \frac{b}{a^2}\right)\left(1 - e^{-ax/\varepsilon}\right) - \frac{b}{a}\frac{x}{\varepsilon}\right) + O(\varepsilon^2).$$

Turning to the outer solution, let $y_{\text{out}}(x) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$. We already found that $y_0(x) = e^{-bx/a}$. At order ε, the outer equation reads:

$$y_0''(x) + ay_1'(x) + by_1(x) = 0 \implies y_1'(x) + \frac{b}{a}y_1(x) = -\frac{1}{a}e^{-bx/a}.$$

[1] We know that we can use ε instead of ε^p for some unknown constant p because a is positive near zero, and $p \neq 1$ only occurs when $a(x) \to 0$ at the boundary layer.

Therefore,

$$y_1(x) = C_4 e^{-bx/a} - \frac{x}{a} e^{-bx/a},$$

and

$$y_{\text{out}}(x) = e^{-bx/a} + \varepsilon \left(\left(C_4 - \frac{x}{a} \right) e^{-bx/a} \right) + O(\varepsilon^2).$$

Now we match. As $x \to 0+$, the outer solution goes to

$$y_{\text{out}}(x) \sim 1 + C_4 \varepsilon + O(\varepsilon^2),$$

while as $\xi \to \infty$, the inner solution approaches

$$y_{\text{in}}(x) \sim 1 + \left(\frac{1}{a} + \frac{b}{a^2} \right) \varepsilon + O(\varepsilon^2),$$

where I have cheated slightly.[2] Matching,

$$C_4 = \frac{1}{a} + \frac{b}{a^2},$$

and thus,

$$y_{\text{uniform}}(x) = e^{-bx/a} + \varepsilon \left(\left(\left(\frac{1}{a} + \frac{b}{a^2} \right) - \frac{x}{a} \right) e^{-bx/a} - \left(\frac{1}{a} + \frac{b}{a^2} \right) e^{-ax/\varepsilon} \right) + O(\varepsilon^2).$$

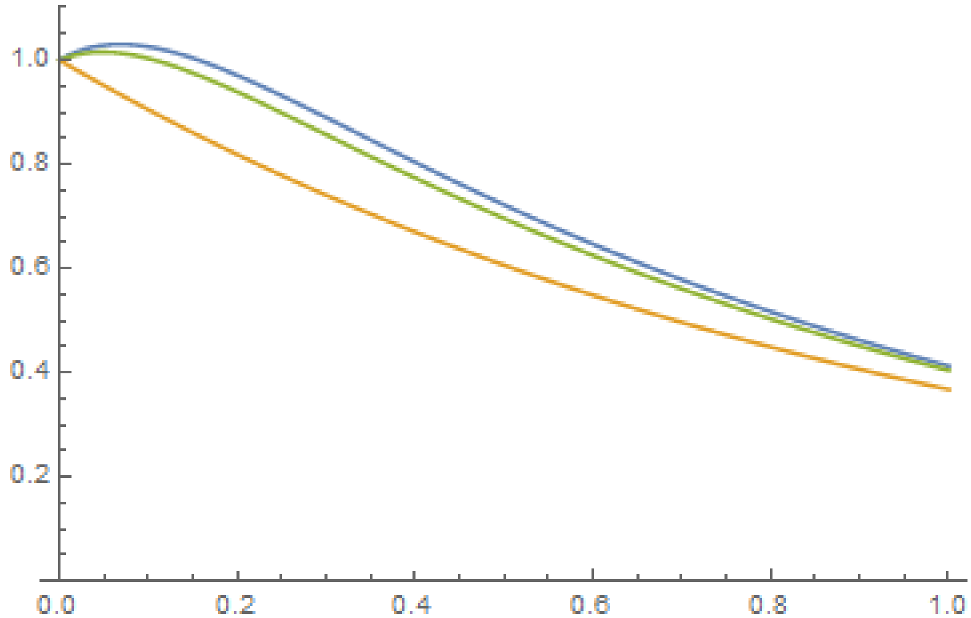
Let us now graphically compare the asymptotic solutions accurate to $O(1)$ and $O(\varepsilon)$ with the exact solution:

$$y_{\text{exact}}(x) = \frac{1}{2\sqrt{a^2 - 4b\varepsilon}} \Big(-a e^{-(a/\varepsilon + \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} - 2\varepsilon e^{-(a/\varepsilon + \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} + \sqrt{a^2 - 4b\varepsilon}\, e^{-(a/\varepsilon + \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2}$$
$$+ a e^{-(a/\varepsilon - \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} + 2\varepsilon e^{-(a/\varepsilon - \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} + \sqrt{a^2 - 4b\varepsilon}\, e^{-(a/\varepsilon - \sqrt{a^2-4b\varepsilon}/\varepsilon)x/2} \Big).$$

For simplicity, take $a = b = 1$.

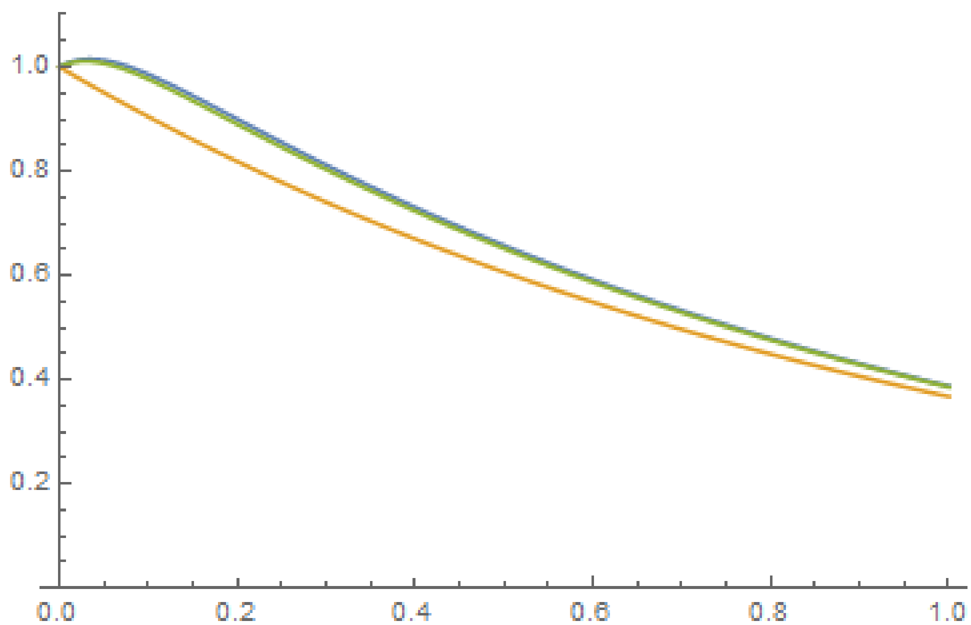
[2] I ignored the term which was linear in ξ, which blows up as $\xi \to \infty$. This term came from ignoring the outer expansion, which changes on the same order, and is called a secular term. This occurs because we ignored the outer expansion, as is conventional when working with the boundary layer, but the outer solution, when Taylor expanded about zero, matches this term exactly at order x, and so we do not have any problems. The outer solution changes at a slow rate compared to ξ, and so the appearance of two time scales in the boundary layer causes problems with the match (which I swept under the rug by cheating). This problem is therefore far better suited for multiscale methods, which form the subject matter of Chapter 11.

For the first graph, $\varepsilon = 0.1$:



The blue curve represents the exact solution, the orange curve the uniform solution to leading order, and the green curve the uniform solution accurate to $O(\varepsilon)$. We see that the differences between the curves remains small always, and that the higher order approximation is much closer to the exact solution. The leading order solution never differs by more than about $0.1 \approx \varepsilon$, while the next order solution differs from the exact solution by a much smaller amount (approximately $O(\varepsilon^2)$).

Now let $\varepsilon = 0.05$:



The color scheme is the same as before. We notice that the same qualitative observations from before hold for this graph as well. The difference between the orange and blue curves is even half as much, in good agreement with our $O(\varepsilon)$ error estimation. This also validates our conclusion that the boundary layer only appears at $O(\varepsilon)$. It is somewhat more difficult to verify pictorially that the error for the green curve is $O(\varepsilon^2)$, but it is also clear that the error in this plot is less than in the first plot, and by a greater factor than two.