HW 3, NE 548, Spring 2017

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Contents

0.1 problem 9.3 (page 479)

problem (a) show that if $a(x)<0$ for $0\leq x\leq 1$ then the solution to 9.1.7 has boundary layer at $x=1$.
(b) Find a uniform approximation with error $O(\varepsilon)$ to the solution 9.1.7 when $a(x)<0$ for $0\leq x\leq 1$
(c) Show that if $a(x)>0$ it is impossible to match to a boundary layer at $x=1$

solution

0.1.1 Part a

Equation 9.1.7 at page 422 is

$$\varepsilon y''+a(x)y'+b(x)y(x)=0 \tag{9.1.7}$$
$$y(0)=A$$
$$y(1)=B$$

For $0\leq x\leq 1$. Now we solve for $y_{in}(x)$, but first we introduce inner variable ξ. We assume boundary layer is at $x=0$, then show that this leads to inconsistency. Let $\xi=\frac{x}{\varepsilon^p}$ be the inner variable. We express the original ODE using this new variable. We also need to determine p. Since $\frac{dy}{dx}=\frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx}=\frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx}\equiv\varepsilon^{-p}\frac{d}{d\xi}$

$$\begin{aligned}\frac{d^2}{dx^2}&=\frac{d}{dx}\frac{d}{dx}\\&=\left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right)\\&=\varepsilon^{-2p}\frac{d^2}{d\xi^2}\end{aligned}$$

Therefore $\frac{d^2y}{dx^2}=\varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and (9.1.7) becomes

$$\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2}+a(x)\varepsilon^{-p}\frac{dy}{d\xi}+y=0$$
$$\varepsilon^{1-2p}\frac{d^2y}{d\xi^2}+a(x)\varepsilon^{-p}\frac{dy}{d\xi}+y=0$$

The largest terms are $\left\{\varepsilon^{1-2p},\varepsilon^{-p}\right\}$, therefore balance gives $1-2p=-p$ or $p=1$. The ODE now becomes

$$\varepsilon^{-1}\frac{d^2y}{d\xi^2}+a(x)\varepsilon^{-1}\frac{dy}{d\xi}+y=0 \tag{1}$$

Assuming that

$$y_{in}(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n=y_0+\varepsilon y_1+\varepsilon^2y_2+\cdots$$

And substituting the above into (1) gives

$$\varepsilon^{-1}\left(y_0''+\varepsilon y_1''+\cdots\right)+a(x)\varepsilon^{-1}\left(y_0'+\varepsilon y_1'+\cdots\right)+\left(y_0+\varepsilon y_1+\cdots\right)=0 \tag{1A}$$

Collecting powers of $O\left(\varepsilon^{-1}\right)$ terms, gives the ODE to solve for y_0^{in} as

$$y_0''\sim -a(x)y_0'$$

In the rapidly changing region, because the boundary layer is very thin, we can approximate $a(x)$ by $a(0)$. The above becomes

$$y_0'' \sim -a(0)\, y_0'$$

But we are told that $a(x) < 0$, so $a(0) < 0$, hence $-a(0)$ is positive. Let $-a(0) = n^2$, to make it more clear this is positive, then the ODE to solve is

$$y_0'' \sim n^2 y_0'$$

The solution to this ODE is

$$y_0(\xi) \sim \frac{C_1}{n^2} e^{n^2 \xi} + C_2$$

Using $y(0) = A$, then the above gives $A = \frac{C_1}{n^2} + C_2$ or $C_2 = A - \frac{C_1}{n^2}$ and the ODE becomes

$$\begin{aligned} y_0(\xi) &\sim \frac{C_1}{n^2} e^{n^2 \xi} + \left(A - \frac{C_1}{n^2}\right) \\ &\sim \frac{C_1}{n^2}\left(e^{n^2 \xi} - 1\right) + A \end{aligned}$$

We see from the above solution for the inner layer, that as ξ increases (meaning we are moving away from $x = 0$), then the solution $y_0(\xi)$ and its derivative is increasing and not decreasing since $y_0'(\xi) = C_1 e^{n^2 \xi}$ and $y_0''(\xi) = C_1 n^2 e^{n^2 \xi}$.

But this contradicts what we assumed that the boundary layer is at $x = 0$ since we expect the solution to change less rapidly as we move away from $x = 0$. Hence we conclude that if $a(x) < 0$, then the boundary layer can not be at $x = 0$.

Let us now see what happens by taking the boundary layer to be at $x = 1$. We repeat the same process as above, but now the inner variable as defined as

$$\xi = \frac{1 - x}{\varepsilon^p}$$

We express the original ODE using this new variable and determine p. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}(-\varepsilon^{-p})$. Hence $\frac{d}{dx} \equiv (-\varepsilon^{-p})\frac{d}{d\xi}$

$$\begin{aligned} \frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\ &= \left((-\varepsilon^{-p})\frac{d}{d\xi}\right)\left((-\varepsilon^{-p})\frac{d}{d\xi}\right) \\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2} \end{aligned}$$

Therefore $\frac{d^2 y}{dx^2} = \varepsilon^{-2p}\frac{d^2 y}{d\xi^2}$ and equation (9.1.7) becomes

$$\begin{aligned} \varepsilon\varepsilon^{-2p}\frac{d^2 y}{d\xi^2} - a(x)\,\varepsilon^{-p}\frac{dy}{d\xi} + y &= 0 \\ \varepsilon^{1-2p}\frac{d^2 y}{d\xi^2} - a(x)\,\varepsilon^{-p}\frac{dy}{d\xi} + y &= 0 \end{aligned}$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, therefore matching them gives $1-2p=-p$ or

$$\boxed{p=1}$$

The ODE now becomes

$$\varepsilon^{-1}\frac{d^2y}{d\xi^2} - a(x)\,\varepsilon^{-1}\frac{dy}{d\xi} + y = 0 \tag{2}$$

Assuming that

$$y_{in}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

And substituting the above into (2) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) - a(x)\,\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \cdots\right) + \left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{2A}$$

Collecting $O\left(\varepsilon^{-1}\right)$ terms, gives the ODE to solve for y_0^{in} as

$$y_0'' \sim a(x)\,y_0'$$

In the rapidly changing region, $\alpha = a(1)$, because the boundary layer is very thin, we approximated $a(x)$ by $a(1)$. The above becomes

$$y_0'' \sim \alpha y_0'$$

But we are told that $a(x) < 0$, so $\alpha < 0$, and the above becomes

$$y_0'' \sim \alpha y_0'$$

The solution to this ODE is

$$y_0(\xi) \sim \frac{C_1}{\alpha} e^{\alpha\xi} + C_2 \tag{3}$$

Using

$$\begin{aligned} y(x=1) &= y(\xi=0) \\ &= B \end{aligned}$$

Then (3) gives $B = \frac{C_1}{\alpha} + C_2$ or $C_2 = B - \frac{C_1}{\alpha}$ and (3) becomes

$$\begin{aligned} y_0(\xi) &\sim \frac{C_1}{\alpha} e^{\alpha\xi} + \left(B - \frac{C_1}{\alpha}\right) \\ &\sim \frac{C_1}{\alpha}\left(e^{\alpha\xi} - 1\right) + B \end{aligned} \tag{4}$$

From the above, $y_0'(\xi) = -C_1 e^{\alpha\xi}$ and $y_0''(\xi) = C_1 \alpha e^{\alpha\xi}$. We now see that as that as ξ increases (meaning we are moving away from $x=1$ towards the left), then the solution $y_0(\xi)$ is actually changing less rapidly. This is because $\alpha < 0$. The solution is changing less rapidly as we move away from the boundary layer as what we expect. Therefore, we conclude that if $a(x) < 0$ then the boundary layer can not be at $x=0$ and has to be at $x=1$.

0.1.2 Part b

To find uniform approximation, we need now to find $y^{out}(x)$ and then do the matching. Since from part(a) we concluded that y_{in} is near $x=1$, then we assume now that $y^{out}(x)$ is near $x=0$. Let

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (9.1.7) gives

$$\varepsilon\left(y_0''+\varepsilon y_1''+\varepsilon^2 y_2''+\cdots\right)+a(x)\left(y_0'+\varepsilon y_1'+\varepsilon^2 y_2'+\cdots\right)+b(x)\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right)=0$$

Collecting terms of $O(1)$ gives the ODE

$$a(x)\,y_0'+b(x)\,y_0=0$$

The solution to this ODE is

$$y_0(x)=C_2 e^{-\int_0^x \frac{b(s)}{a(s)}ds}$$

Applying $y(0)=A$ gives

$$\begin{aligned}A&=C_2 e^{-\int_0^1 \frac{b(s)}{a(s)}ds}\\&=C_2 E\end{aligned}$$

Where E is constant, which is the value of the definite integral $E=e^{-\int_0^1 \frac{b(s)}{a(s)}ds}$. Hence the solution $y^{out}(x)$ can now be written as

$$y_0^{out}(x)=\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}$$

We are now ready to do the matching.

$$\begin{aligned}\lim_{\xi\to\infty}y^{in}(\xi)&\sim\lim_{x\to 0}y^{out}(x)\\\lim_{\xi\to\infty}\frac{C_1}{\alpha}\left(e^{\alpha\xi}-1\right)+B&\sim\lim_{x\to 0}\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}\end{aligned}$$

But since $\alpha=a(1)<0$ then the above simplifies to

$$\begin{aligned}-\frac{C_1}{a(1)}+B&=\frac{A}{E}\\C_1&=-a(1)\left(\frac{A}{E}-B\right)\end{aligned}$$

Hence inner solution becomes

$$\begin{aligned}y_0^{in}(\xi)&\sim\frac{-a(1)\left(\frac{A}{E}-B\right)}{a(1)}\left(e^{a(1)\xi}-1\right)+B\\&\sim\left(B-\frac{A}{E}\right)\left(e^{a(1)\xi}-1\right)+B\\&\sim B\left(e^{a(1)\xi}-1\right)-\frac{A}{E}\left(e^{a(1)\xi}-1\right)+B\\&\sim\left(B-\frac{A}{E}\right)\left(e^{a(1)\xi}-1\right)+B\end{aligned}$$

The uniform solution is

$$\begin{aligned}y_{\text{uniform}}(x)&\sim y^{in}(\xi)+y^{out}(x)-y^{match}\\&\sim\overbrace{\left(B-\frac{A}{E}\right)\left(e^{a(1)\xi}-1\right)+B}^{y^{in}}+\overbrace{\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}}^{y^{out}}-\frac{A}{E}\end{aligned}$$

Or in terms of x only

$$y_{\text{uniform}}(x)\sim\left(B-\frac{A}{E}\right)\left(e^{a(1)\frac{1-x}{\varepsilon}}-1\right)+B+\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}-\frac{A}{E}$$

0.1.3 Part c

We now assume the boundary layer is at $x=1$ but $a(x)>0$. From part (a), we found that the solution for $y_0^{in}(\xi)$ where boundary layer at $x=1$ is

$$y_0(\xi) \sim \frac{C_1}{\alpha}\left(e^{\alpha\xi}-1\right)+B$$

But now $\alpha = a(1) > 0$ and not negative as before. We also found that $y_0^{out}(x)$ solution was

$$y_0^{out}(x) = \frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}$$

Lets now try to do the matching and see what happens

$$\lim_{\xi\to\infty} y^{in}(\xi) \sim \lim_{x\to 0} y^{out}(x)$$
$$\lim_{\xi\to\infty}\frac{C_1}{\alpha}\left(e^{\alpha\xi}-1\right)+B+O(\varepsilon) \sim \lim_{x\to 0}\frac{A}{E}e^{-\int_0^x \frac{b(s)}{a(s)}ds}+O(\varepsilon)$$
$$\lim_{\xi\to\infty} C_1\left(\frac{e^{\alpha\xi}}{\alpha}-\frac{1}{\alpha}\right) \sim \frac{A}{E}-B$$

Since now $\alpha>0$, then the term on the left blows up, while the term on the right is finite. Not possible to match, unless $C_1=0$. But this means the boundary layer solution is just a constant B and that $\frac{A}{E}=B$. So the matching does not work in general for arbitrary conditions. This means if $a(x)>0$, it is not possible to match boundary layer at $x=1$.

0.2 problem 9.4(b)

Problem Find the leading order uniform asymptotic approximation to the solution of

$$\varepsilon y'' + \left(1+x^2\right)y' - x^3 y(x) = 0 \tag{1}$$
$$y(0)=1$$
$$y(1)=1$$

For $0\le x\le 1$ in the limit as $\varepsilon\to 0$.

solution

Since $a(x)=\left(1+x^2\right)$ is positive, we expect the boundary layer to be near $x=0$. First we find $y^{out}(x)$, which is near $x=1$. Assuming

$$y_{out}(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n = y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots$$

And substituting this into (1) gives

$$\varepsilon\left(y_0''+\varepsilon y_1''+\varepsilon^2 y_2''+\cdots\right)+\left(1+x^2\right)\left(y_0'+\varepsilon y_1'+\varepsilon^2 y_2'+\cdots\right)-x^3\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right)=0$$

Collecting terms in $O(1)$ gives the ODE

$$\left(1+x^2\right)y_0' \sim x^3 y_0$$

The ODE becomes $y_0' \sim \frac{x^3}{\left(1+x^2\right)}y_0$ with integrating factor $\mu = e^{\int \frac{-x^3}{\left(1+x^2\right)}dx}$. To evaluate $\int \frac{-x^3}{\left(1+x^2\right)}dx$, let

$u = x^2$, hence $\frac{du}{dx} = 2x$ and the integral becomes

$$\int \frac{-x^3}{\left(1+x^2\right)} dx = -\int \frac{ux}{(1+u)} \frac{du}{2x} = -\frac{1}{2} \int \frac{u}{(1+u)} du$$

But

$$\begin{aligned}\int \frac{u}{(1+u)} du &= \int 1 - \frac{1}{(1+u)} du \\ &= u - \ln(1+u)\end{aligned}$$

But $u = x^2$, hence

$$\int \frac{-x^3}{\left(1+x^2\right)} dx = \frac{-1}{2}\left(x^2 - \ln\left(1+x^2\right)\right)$$

Therefore the integrating factor is $\mu = \exp\left(\frac{-1}{2}x^2 + \frac{1}{2}\ln\left(1+x^2\right)\right)$. The ODE becomes

$$\begin{aligned}\frac{d}{dx}\left(\mu y_0\right) &= 0 \\ \mu y_0 &\sim c \\ y_{out}(x) &\sim c \exp\left(\frac{1}{2}x^2 - \frac{1}{2}\ln\left(1+x^2\right)\right) \\ &\sim c e^{\frac{1}{2}x^2} e^{\ln\left(1+x^2\right)^{\frac{-1}{2}}} \\ &\sim c \frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\end{aligned}$$

To find c, using boundary conditions $y(1) = 1$ gives

$$\begin{aligned}1 &= c \frac{e^{\frac{1}{2}}}{\sqrt{2}} \\ c &= \sqrt{2} e^{-\frac{1}{2}}\end{aligned}$$

Hence

$$y_0^{out}(x) \sim \sqrt{2} \frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}}$$

Now we find $y^{in}(x)$ near $x = 0$. Let $\xi = \frac{x}{\varepsilon^p}$ be the inner variable. We express the original ODE using this new variable and determine p. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$\begin{aligned}\frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\ &= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right) \\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2}\end{aligned}$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and $\varepsilon y'' + \left(1+x^2\right)y' - x^3 y(x) = 0$ becomes

$$\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + \left(1 + (\xi\varepsilon^p)^2\right)\varepsilon^{-p}\frac{dy}{d\xi} - (\xi\varepsilon^p)^3 y = 0$$
$$\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(1 + \xi^2\varepsilon^{2p}\right)\varepsilon^{-p}\frac{dy}{d\xi} - \xi^3\varepsilon^{3p}y = 0$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, therefore matching them gives $1-2p=-p$ or $p=1$. The ODE now becomes

$$\varepsilon^{-1}\frac{d^2y}{d\xi^2} + \left(1 + \xi^2\varepsilon^2\right)\varepsilon^{-1}\frac{dy}{d\xi} - \xi^3\varepsilon^3 y = 0 \tag{2}$$

Assuming that

$$y_{in}(x) = \sum_{n=0}^{\infty}\varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

And substituting the above into (2) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(1+\xi^2\varepsilon^2\right)\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \cdots\right) - \xi^3\varepsilon^3\left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{2A}$$

Collecting terms in $O\left(\varepsilon^{-1}\right)$ gives the ODE

$$y_0''(\xi) \sim -y_0'(\xi)$$

The solution to this ODE is

$$y_0^{in}(\xi) \sim c_1 + c_2 e^{-\xi} \tag{3}$$

Applying $y_0^{in}(0) = 1$ gives

$$1 = c_1 + c_2$$
$$c_1 = 1 - c_2$$

Hence (3) becomes

$$\begin{aligned} y_0^{in}(\xi) &\sim (1-c_2) + c_2 e^{-\xi} \\ &\sim 1 + c_2\left(e^{-\xi} - 1\right) \end{aligned} \tag{4}$$

Now that we found y_{out} and y_{in}, we apply matching to find c_2 in the y_{in} solution.

$$\lim_{\xi\to\infty} y_0^{in}(\xi) \sim \lim_{x\to 0^+} y_0^{out}(x)$$
$$\lim_{\xi\to\infty} 1 + c_2\left(e^{-\xi}-1\right) \sim \lim_{x\to 0^+}\sqrt{2}\frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}}$$
$$\begin{aligned} 1 - c_2 &\sim \sqrt{\frac{2}{e}}\lim_{x\to 0^+}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} \\ &\sim \sqrt{\frac{2}{e}}\lim_{x\to 0^+}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} \\ &= \sqrt{\frac{2}{e}} \end{aligned}$$

Hence

$$c_2 = 1 - \sqrt{\frac{2}{e}}$$

Therefore the $y_0^{in}(\xi)$ becomes

$$
\begin{aligned}
y_0^{in}(\xi) &\sim 1+\left(1-\sqrt{\frac{2}{e}}\right)\left(e^{-\xi}-1\right)\\
&\sim 1+\left(e^{-\xi}-1\right)-\sqrt{\frac{2}{e}}\left(e^{-\xi}-1\right)\\
&\sim e^{-\xi}-\sqrt{\frac{2}{e}}\left(e^{-\xi}-1\right)\\
&\sim e^{-\xi}-\sqrt{\frac{2}{e}}e^{-\xi}+\sqrt{\frac{2}{e}}\\
&\sim e^{-\xi}\left(1-\sqrt{\frac{2}{e}}\right)+\sqrt{\frac{2}{e}}\\
&\sim 0.858+0.142e^{-\xi}
\end{aligned}
$$

Therefore, the uniform solution is

$$
y_{uniform} \sim y_{in}(x)+y_{out}(x)-y_{match}+O(\varepsilon) \tag{4}
$$

Where y_{match} is $y_{in}(x)$ at the boundary layer matching location. (or y_{out} at same matching location). Hence

$$
\begin{aligned}
y_{match} &\sim 1-c_2\\
&\sim 1-\left(1-\sqrt{\frac{2}{e}}\right)\\
&\sim \sqrt{\frac{2}{e}}
\end{aligned}
$$

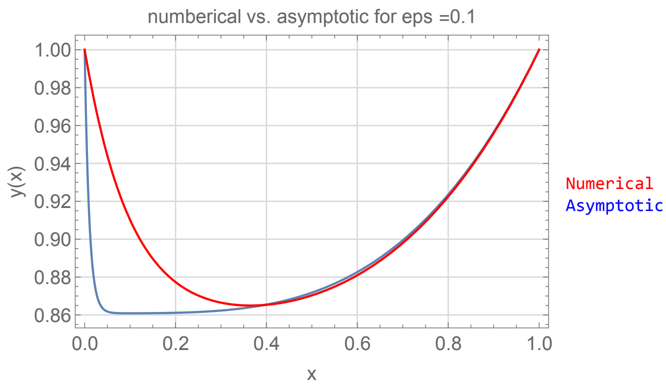
Hence (4) becomes

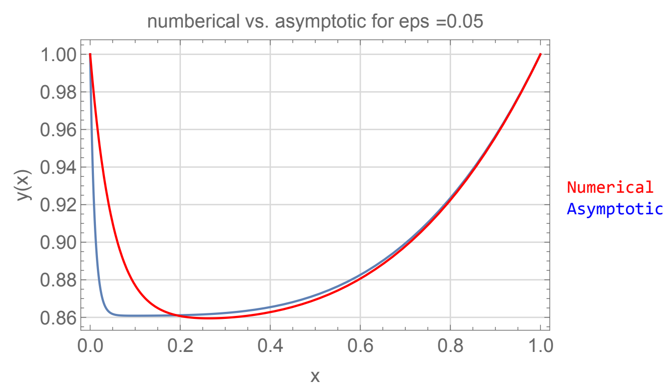
$$
\begin{aligned}
y_{uniform} &\sim \overbrace{e^{-\xi}\left(1-\sqrt{\frac{2}{e}}\right)+\sqrt{\frac{2}{e}}}^{y_{in}}+\overbrace{\sqrt{2}\frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}}}^{y_{out}}-\overbrace{\sqrt{\frac{2}{e}}}^{y_{match}}\\
&\sim e^{-\frac{x}{\varepsilon}}\left(1-\sqrt{\frac{2}{e}}\right)+\sqrt{2}\frac{e^{\frac{x^2-1}{2}}}{\sqrt{1+x^2}}+O(\varepsilon)
\end{aligned}
$$

This is the leading order uniform asymptotic approximation solution. To verify the result, the numerical solution was plotted against the above solution for $\varepsilon=\{0.1,0.05,0.01\}$. We see from these plots that as ε becomes smaller, the asymptotic solution becomes more accurate when compared to the numerical solution. This is because the error, which is $O(\varepsilon)$, becomes smaller. The code used to generate these plots is

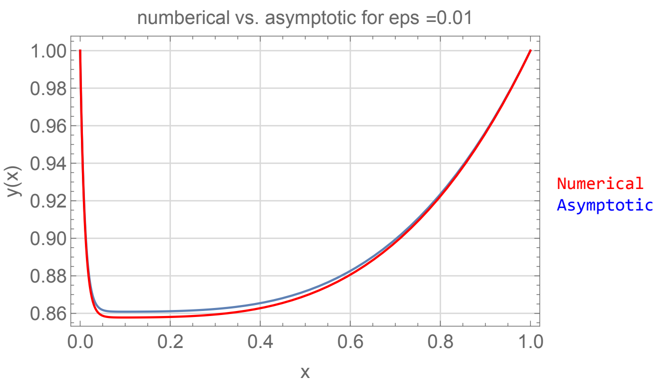
```
In[180]:= eps = 0.1;
sol = NDSolve[{1/100 y''[x] + (1 + x^2) y'[x] - x^3 y[x] == 0, y[0] == 1, y[1] == 1}, y, {x, 0, 1}];
p1 = Plot[Evaluate[y[x] /. sol], {x, 0, 1}, Frame → True,
   FrameLabel → {{"y(x)", None}, {"x", Row[{"numberical vs. asymptotic for eps =", eps}]}},
   GridLines → Automatic, GridLinesStyle → LightGray];
mysol[x_, eps_] := Exp[-x/eps] (1 - Sqrt[2/Exp[1]]) + (Sqrt[2] Exp[(x^2-1)/2])/(Sqrt[1 + x^2]);
p2 = Plot[mysol[x, eps], {x, 0, 1}, PlotRange → All, PlotStyle → Red];
Show[Legended[p1, Style["Numerical", Red]], Legended[p2, Style["Asymtotic", Blue]]]
```

The following are the three plots for each value of ε

Out[6]=


Out[12]=


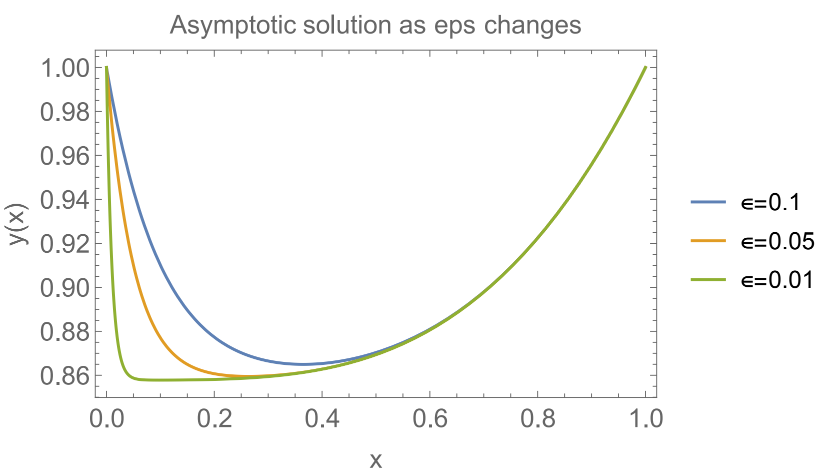
Out[17]=



To see the effect on changing ε on only the asymptotic approximation, the following plot gives the approximation solution only as ε changes. We see how the approximation converges to the numerical solution as ε becomes smaller.

In[236]:=
```
Plot[{mysol[x, .1], mysol[x, .05], mysol[x, .01]}, {x, 0, 1},
 PlotLegends → {"ϵ=0.1", "ϵ=0.05", "ϵ=0.01"}, Frame → True,
 FrameLabel → {{"y(x)", None}, {"x", "Asymptotic solution as eps changes"}},
 BaseStyle → 14]
```

Out[236]=



0.3 problem 9.6

Problem Consider initial value problem

$$y' = \left(1 + \frac{x^{-2}}{100}\right) y^2 - 2y + 1$$

With $y(1) = 1$ on the interval $0 \leq x \leq 1$. (a) Formulate this problem as perturbation problem by introducing a small parameter ε. (b) Find outer approximation correct to order ε with errors of order ε^2. Where does this approximation break down? (c) Introduce inner variable and find the inner solution valid to order 1 (with errors of order ε). By matching to the outer solution find a uniform valid solution to $y(x)$ on interval $0 \leq x \leq 1$. Estimate the accuracy of this approximation. (d) Find inner solution correct to order ε (with errors of order ε^2) and show that it matches to the outer solution correct to order ε.

solution

0.3.1 Part a

Since $\frac{1}{100}$ is relatively small compared to all other coefficients, we replace it with ε and the ODE becomes

$$y' - \left(1 + \frac{\varepsilon}{x^2}\right) y^2 + 2y = 1 \tag{1}$$

0.3.2 Part b

Assuming boundary layer is on the left side at $x = 0$. We now solve for $y_{out}(x)$, which is the solution near $x = 1$.

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (1) gives

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) - \left(1 + \frac{\varepsilon}{x^2}\right)\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right)^2 + 2\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 1$$

Expanding the above to see more clearly the terms gives

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) - \left(1 + \frac{\varepsilon}{x^2}\right)\left(y_0^2 + \varepsilon\left(2y_0 y_1\right) + \varepsilon^2\left(2y_0 y_2 + y_1^2\right) + \cdots\right) + 2\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 1 \tag{2}$$

The leading order are those terms of coefficient $O(1)$. This gives

$$y_0' - y_0^2 + 2y_0 \sim 1$$

With boundary conditions $y(1) = 1$.

$$\frac{dy_0}{dx} \sim y_0^2 - 2y_0 + 1$$

This is separable

$$\frac{dy_0}{y_0^2 - 2y_0 + 1} \sim dx$$

$$\frac{dy_0}{\left(y_0 - 1\right)^2} \sim dx$$

For $y_0 \neq 1$. Integrating

$$\int \frac{dy_0}{\left(y_0 - 1\right)^2} \sim \int dx$$

$$\frac{-1}{y_0 - 1} \sim x + C$$

$$\left(y_0 - 1\right)(x + C) \sim -1$$

$$y_0 \sim \frac{-1}{x + C} + 1 \tag{3}$$

To find C, from $y(1) = 1$, we find

$$1 \sim \frac{-1}{1+C} + 1$$
$$1 \sim \frac{C}{1+C}$$

This is only possible if $C = \infty$. Therefore from (2), we conclude that

$$\boxed{y_0(x) \sim 1}$$

The above is leading order for the outer solution. Now we repeat everything to find $y_1^{out}(x)$. From (2) above, we now keep all terms with $O(\varepsilon)$ which gives

$$y_1' - 2y_0y_1 + 2y_1 \sim \frac{1}{x^2}y_0^2$$

But we found $y_0(x) \sim 1$ from above, so the above ODE becomes

$$y_1' - 2y_1 + 2y_1 \sim \frac{1}{x^2}$$
$$y_1' \sim \frac{1}{x^2}$$

Integrating gives

$$y_1(x) \sim -\frac{1}{x} + C$$

The boundary condition now becomes $y_1(1) = 0$ (since we used $y(1) = 1$ earlier with y_0). This gives

$$0 = -\frac{1}{1} + C$$
$$C = 1$$

Therefore the solution becomes

$$\boxed{y_1(x) \sim 1 - \tfrac{1}{x}}$$

Therefore, the outer solution is

$$y_{out}(x) \sim y_0 + \varepsilon y_1$$

Or

$$\boxed{y(x) \sim 1 + \varepsilon\left(1 - \tfrac{1}{x}\right) + O\left(\varepsilon^2\right)}$$

Since the ODE is $y' - \left(1 + \frac{\varepsilon}{x^2}\right)y^2 + 2y = 1$, the approximation breaks down when $x < \sqrt{\varepsilon}$ or $x < \frac{1}{10}$. Because when $x < \sqrt{\varepsilon}$, the $\frac{\varepsilon}{x^2}$ will start to become large. The term $\frac{\varepsilon}{x^2}$ should remain small for the approximation to be accurate. The following are plots of the y_0 and $y_0 + \varepsilon y_1$ solutions (using $\varepsilon = \frac{1}{100}$) showing that with two terms the approximation has improved for the outer layer, compared to the full solution of the original ODE obtained using CAS. But the outer solution breaks down near $x = 0.1$ and smaller as can be seen in these plots. Here is the solution of the original ODE obtained using CAS

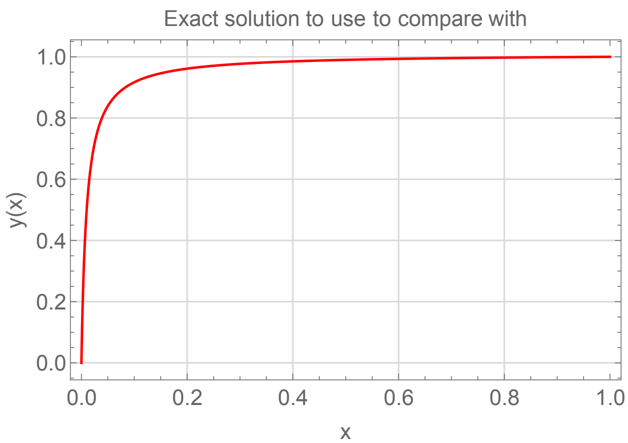
```
In[46]:= eps = 1/100;
ode = y'[x] == (1 + eps/x^2) y[x]^2 - 2 y[x] + 1;
sol = y[x] /. First@DSolve[{ode, y[1] == 1}, y[x], x]
```

Out[48]=

$$\frac{10\,x\left(-12-5\sqrt{6}-12\,x^{\frac{2\sqrt{6}}{5}}+5\sqrt{6}\,x^{\frac{2\sqrt{6}}{5}}\right)}{-\sqrt{6}-120\,x-50\sqrt{6}\,x+\sqrt{6}\,x^{\frac{2\sqrt{6}}{5}}-120\,x^{1+\frac{2\sqrt{6}}{5}}+50\sqrt{6}\,x^{1+\frac{2\sqrt{6}}{5}}}$$

```
In[53]:= Plot[sol, {x, 0, 1}, PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 FrameLabel → {{"y(x)", None}, {"x", "Exact solution to use to compare with"}}, BaseStyle → 14,
 PlotStyle → Red, ImageSize → 400]
```

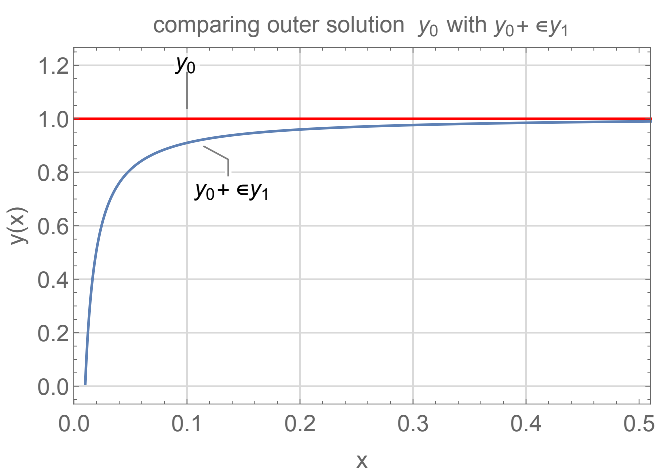
Out[53]=



In the following plot, the y_0 and the $y_0 + \varepsilon y_1$ solutions are superimposed on same figure, to show how the outer solution has improved when adding another term. But we also notice that the outer solution $y_0 + \varepsilon y_1$ only gives good approximation to the exact solution for about $x > 0.1$ and it breaks down quickly as x becomes smaller.

```
In[164]:= outer1 = 1;
outer2 = 1 + eps * (1 - 1 / x);
p2 = Plot[Callout[outer1, "y0", Scaled[0.1]], {x, 0, 1}, PlotRange → All, Frame → True,
   GridLines → Automatic, GridLinesStyle → LightGray,
   FrameLabel → {{"y(x)", None}, {"x", "comparing outer solution  y0 with y0+ ∈y1"}},
   BaseStyle → 14, PlotStyle → Red, ImageSize → 400];
p3 = Plot[Callout[outer2, "y0+ ∈y1", {Scaled[0.5], Below}], {x, 0.01, 1}, AxesOrigin → {0, 0}];
Show[p2, p3, PlotRange → {{0.01, .5}, {0, 1.2}}]
```

Out[168]=



0.3.3 Part c

Now we will obtain solution inside the boundary layer $y_{in}(\xi) = y_0^{in}(\xi) + O(\varepsilon)$. The first step is to always introduce new inner variable. Since the boundary layer is on the right side, then

$$\xi = \frac{x}{\varepsilon^p}$$

And then to express the original ODE using this new variable. We also need to determine p in the above expression. Since the original ODE is $y' - \left(1 + \varepsilon x^{-2}\right) y^2 + 2y = 1$, then $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx} = \frac{dy}{d\xi}(\varepsilon^{-p})$, then the ODE now becomes

$$\frac{dy}{d\xi}\varepsilon^{-p} - \left(1 + \frac{\varepsilon}{\xi\varepsilon^{p2}}\right) y^2 + 2y = 1$$

$$\frac{dy}{d\xi}\varepsilon^{-p} - \left(1 + \frac{\varepsilon^{1-2p}}{\xi^2}\right) y^2 + 2y = 1$$

Where in the above $y \equiv y(\xi)$. We see that we have $\left\{\varepsilon^{-p}, \varepsilon^{(1-2p)}\right\}$ as the two biggest terms to match. This means $-p = 1 - 2p$ or

$$p = 1$$

Hence the above ODE becomes

$$\frac{dy}{d\xi}\varepsilon^{-1} - \left(1 + \frac{\varepsilon^{-1}}{\xi^2}\right) y^2 + 2y = 1$$

We are now ready to replace $y(\xi)$ with $\sum_{n=0}^{\infty}\varepsilon^n y_n$ which gives

$$\left(y_0'+\varepsilon y_1'+\varepsilon^2 y_2'+\cdots\right)\varepsilon^{-1}-\left(1+\frac{\varepsilon^{-1}}{\xi^2}\right)\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right)^2+2\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right)=1$$
$$\left(y_0'+\varepsilon y_1'+\varepsilon^2 y_2'+\cdots\right)\varepsilon^{-1}-\left(1+\frac{\varepsilon^{-1}}{\xi^2}\right)\left(y_0^2+\varepsilon\left(2y_0y_1\right)+\cdots\right)+2\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\right)=1 \tag{3}$$

Collecting terms with $O\left(\varepsilon^{-1}\right)$ gives

$$y_0'\sim\frac{1}{\xi^2}y_0^2$$

This is separable

$$\begin{aligned}\int\frac{dy_0}{y_0^2}&\sim\int\frac{1}{\xi^2}d\xi\\ -y_0^{-1}&\sim-\xi^{-1}+C\\ \frac{1}{y_0}&\sim\frac{1}{\xi}-C\\ \frac{1}{y_0}&\sim\frac{1-C\xi}{\xi}\\ y_0^{in}&\sim\frac{\xi}{1-\xi C}\end{aligned}$$

Now we use matching with y_{out} to find C. We have found before that $y_0^{out}(x)\sim 1$ therefore

$$\begin{aligned}\lim_{\xi\to\infty}y_0^{in}(\xi)+O(\varepsilon)&=\lim_{x\to 0}y_0^{out}(x)+O(\varepsilon)\\ \lim_{\xi\to\infty}\frac{\xi}{1-\xi C}&=1+O(\varepsilon)\\ \lim_{\xi\to\infty}(-C)+O\left(\xi^{-1}\right)+O(\varepsilon)&=1+O(\varepsilon)\\ -C&=1\end{aligned}$$

Therefore

$$y_0^{in}(\xi)\sim\frac{\xi}{1+\xi} \tag{4}$$

Therefore,

$$\begin{aligned}y_{uniform}&=y_0^{in}+y_0^{out}-y_{match}\\ &=\overbrace{\frac{\xi}{1+\xi}}^{y_{in}}+\overbrace{1}^{y_{out}}-1\end{aligned}$$

Since $y_{match}=1$ (this is what $\lim_{\xi\to\infty}y_0^{in}$ is). Writing everything in x, using $\xi=\frac{x}{\varepsilon}$ the above becomes

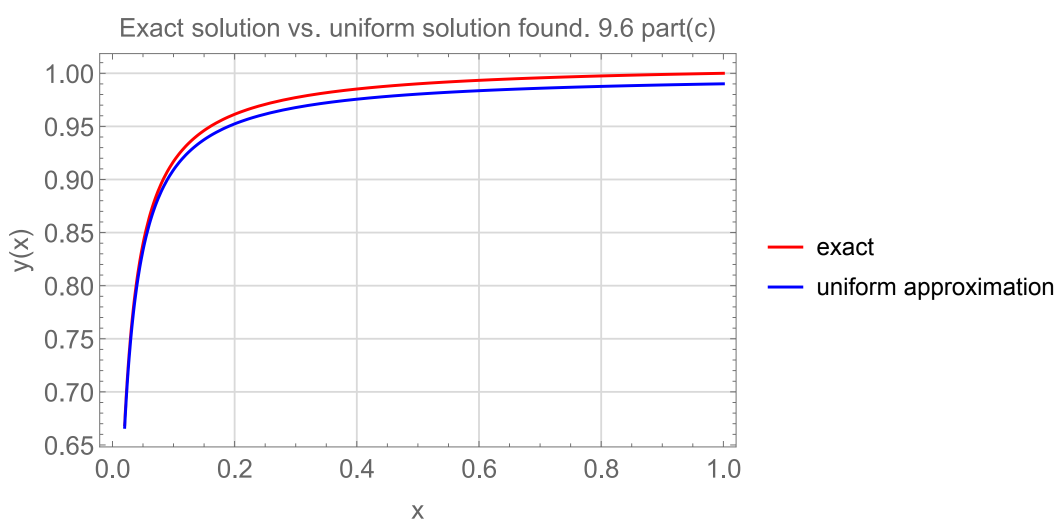
$$\begin{aligned}y_{\text{uniform}}&=\frac{\frac{x}{\varepsilon}}{1+\frac{x}{\varepsilon}}\\ &=\frac{x}{\varepsilon+x}\end{aligned}$$

The following is a plot of the above, using $\varepsilon=\frac{1}{100}$ to compare with the exact solution.,

In[189]:=
```
y[x_, eps_] := x/(x + eps)
p1 = Plot[{exactSol, y[x, 1 / 100]}, {x, 0.02, 1}, PlotRange → All,
  Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
  FrameLabel →
   {{"y(x)", None},
    {"x", "Exact solution vs. uniform solution found. 9.6 part(c)"}},
  BaseStyle → 14, PlotStyle → {Red, Blue}, ImageSize → 400,
  PlotLegends → {"exact", "uniform approximation"}]

Plot[y[x, 1 / 100], {x, 0, 1}, PlotRange → {{.05, 1}, Automatic}]
```

Out[190]=



0.3.4 Part (d)

Now we will obtain y_1^{in} solution inside the boundary layer. Using (3) we found in part (c), reproduced here

$$\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right)\varepsilon^{-1} - \left(1 + \frac{\varepsilon^{-1}}{\xi^2}\right)\left(y_0^2 + \varepsilon\left(2y_0y_1\right) + \cdots\right) + 2\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 1 \tag{3}$$

But now collecting all terms of order of $O(1)$, this results in

$$y_1' - y_0^2 - \frac{2}{\xi^2}y_0y_1 + 2y_0 \sim 1$$

Using y_0^{in} found in part (c) into the above gives

$$\begin{aligned} y_1' - \frac{2}{\xi}\left(\frac{1}{1+\xi}\right)y_1 &\sim 1 - 2\left(\frac{\xi}{1+\xi}\right) + \left(\frac{\xi}{1+\xi}\right)^2 \\ y_1' - \left(\frac{2}{\xi(1+\xi)}\right)y_1 &\sim \frac{1}{(\xi+1)^2} \end{aligned}$$

This can be solved using integrating factor $\mu = e^{\int \frac{-2}{\xi+\xi^2}d\xi}$ using partial fractions gives $\mu = \exp(-2\ln\xi + 2\ln(1+\xi))$ or $\mu = \frac{1}{\xi^2}(1+\xi)^2$. Hence we obtain

$$\begin{aligned} \frac{d}{dx}\left(\mu y_1\right) &\sim \mu\frac{1}{(\xi+1)^2} \\ \frac{d}{dx}\left(\frac{1}{\xi^2}(1+\xi)^2 y_1\right) &\sim \frac{1}{\xi^2} \end{aligned}$$

Integrating

$$\begin{aligned} \frac{1}{\xi^2}(1+\xi)^2 y_1 &\sim \int \frac{1}{\xi^2}d\xi \\ \frac{1}{\xi^2}(1+\xi)^2 y_1 &\sim \frac{-1}{\xi} + C_2 \\ (1+\xi)^2 y_1 &\sim -\xi + \xi^2 C_2 \\ y_1 &\sim \frac{-\xi + \xi^2 C_2}{(1+\xi)^2} \end{aligned}$$

Therefore, the inner solution becomes

$$\begin{aligned} y^{in}(\xi) &= y_0 + \varepsilon y_1 \\ &= \frac{\xi}{1+\xi C_1} + \varepsilon\frac{\xi^2 C_2 - \xi}{(1+\xi)^2} \end{aligned}$$

To find C_1, C_2 we do matching with with y^{out} that we found in part (a) which is $y_{out}(x) \sim 1 + \varepsilon\left(1 - \frac{1}{x}\right)$

$$\lim_{\xi\to\infty}\left(\frac{\xi}{1+\xi C_1} + \varepsilon\frac{\xi^2 C_2 - \xi}{(1+\xi)^2}\right) \sim \lim_{x\to 0} 1 + \varepsilon\left(1 - \frac{1}{x}\right)$$

Doing long division $\frac{\xi}{1+\xi C_1} = \frac{1}{C_1} - \frac{1}{\xi C_1^2} + \frac{1}{\xi^2 C_1^3} + \cdots$ and $\frac{\xi^2 C_2 - \xi}{(1+\xi)^2} = C_2 - \frac{2C_2+1}{\xi} - \cdots$, hence the above becomes

$$\lim_{\xi\to\infty}\left(\left(\frac{1}{C_1} - \frac{1}{\xi C_1^2} + \frac{1}{\xi^2 C_1^3} + \cdots\right) + \left(\varepsilon C_2 - \varepsilon\frac{2C_2+1}{\xi} + \cdots\right)\right) \sim \lim_{x\to 0} 1 + \varepsilon\left(1 - \frac{1}{x}\right)$$
$$\frac{1}{C_1} + \varepsilon C_2 \sim \lim_{x\to 0} 1 + \varepsilon\left(1 - \frac{1}{x}\right)$$

Using $x = \xi\varepsilon$ on the RHS, the above simplifies to

$$\begin{aligned}\frac{1}{C_1} + \varepsilon C_2 &\sim \lim_{\xi\to\infty} 1 + \varepsilon\left(1 - \frac{1}{\xi\varepsilon}\right)\\ &\sim \lim_{\xi\to\infty} 1 + \left(\varepsilon - \frac{1}{\xi}\right)\\ &\sim 1 + \varepsilon\end{aligned}$$

Therefore, $C_1 = 1$ and $C_2 = 1$. Hence the inner solution is

$$\begin{aligned}y^{in}(\xi) &= y_0 + \varepsilon y_1\\ &= \frac{\xi}{1+\xi} + \varepsilon\frac{\xi^2 - \xi}{(1+\xi)^2}\end{aligned}$$

Therefore

$$\begin{aligned}y_{\text{uniform}} &= y_{in} + y_{out} - y_{match}\\ &= \overbrace{\frac{\xi}{1+\xi} + \varepsilon\frac{\xi^2-\xi}{(1+\xi)^2}}^{y_{in}} + \overbrace{1 + \varepsilon\left(1 - \frac{1}{x}\right)}^{y_{out}} - (1+\varepsilon)\end{aligned}$$

Writing everything in x, using $\xi = \frac{x}{\varepsilon}$ the above becomes

$$\begin{aligned}y_{\text{uniform}} &= \frac{\frac{x}{\varepsilon}}{1+\frac{x}{\varepsilon}} + \varepsilon\frac{\frac{x^2}{\varepsilon^2} - \frac{x}{\varepsilon}}{\left(1+\frac{x}{\varepsilon}\right)^2} + 1 + \varepsilon\left(1 - \frac{1}{x}\right) - (1+\varepsilon)\\ &= \frac{x}{\varepsilon + x} + \frac{x^2 - x\varepsilon}{\varepsilon\left(1+\frac{x}{\varepsilon}\right)^2} + 1 + \varepsilon - \frac{\varepsilon}{x} - 1 - \varepsilon\\ &= \frac{x}{\varepsilon + x} + \frac{x^2 - x\varepsilon}{\varepsilon\left(1+\frac{x}{\varepsilon}\right)^2} - \frac{\varepsilon}{x} + O\left(\varepsilon^2\right)\end{aligned}$$

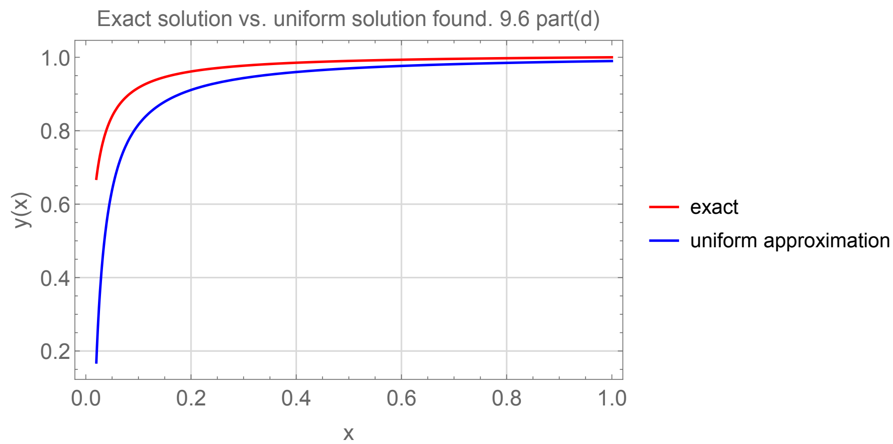
The following is a plot of the above, using $\varepsilon = \frac{1}{100}$ to compare with the exact solution.

```
In[192]:= y[x_, eps_] := x/(x + eps) + (x^2 - x eps)/(eps (1 + x/eps)^2) - eps/x

p1 = Plot[{exactSol, y[x, 1 / 100]}, {x, 0.02, 1}, PlotRange → All, Frame → True,
  GridLines → Automatic, GridLinesStyle → LightGray,
  FrameLabel → {{"y(x)", None}, {"x", "Exact solution vs. uniform solution found. 9.6 part(d)"}},
  BaseStyle → 14, PlotStyle → {Red, Blue}, ImageSize → 400,
  PlotLegends → {"exact", "uniform approximation"}]

Plot[y[x, 1 / 100], {x, 0, 1}, PlotRange → {{.05, 1}, Automatic}]
```

Out[193]=



Let us check if $y_{\text{uniform}}(x)$ satisfies $y(1) = 1$ or not.

$$
\begin{aligned}
y_{\text{uniform}}(1) &= \frac{1}{\varepsilon+1} + \frac{1-\varepsilon}{\varepsilon\left(1+\frac{1}{\varepsilon}\right)^2} - \varepsilon + O\left(\varepsilon^2\right) \\
&= \frac{1-\varepsilon^3-3\varepsilon^2+\varepsilon}{(\varepsilon+1)^2}
\end{aligned}
$$

Taking the limit $\varepsilon \to 0$ gives 1. Therefore $y_{\text{uniform}}(x)$ satisfies $y(1) = 1$.

0.4 problem 9.9

problem Use boundary layer methods to find an approximate solution to initial value problem

$$
\begin{aligned}
\varepsilon y'' + a(x) y' + b(x) y &= 0 \qquad (1) \\
y(0) &= 1 \\
y'(0) &= 1
\end{aligned}
$$

And $a > 0$. Show that leading order uniform approximation satisfies $y(0) = 1$ but not $y'(0) = 1$ for arbitrary b. Compare leading order uniform approximation with the exact solution to the problem when $a(x), b(x)$ are constants.

Solution

Since $a(x) > 0$ then we expect the boundary layer to be at $x = 0$. We start by finding $y_{out}(x)$.

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (1) gives

$$\varepsilon \left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms with $O(1)$ results in

$$\begin{aligned} a y_0' &\sim -b y_0 \\ \frac{dy_0}{dx} &\sim -\frac{b}{a} y_0 \end{aligned}$$

This is separable

$$\begin{aligned} \int \frac{dy_0}{y_0} &\sim -\int \frac{b(x)}{a(x)} dx \\ \ln|y_0| &\sim -\int \frac{b(x)}{a(x)} dx + C \\ y_0 &\sim C e^{-\int_1^x \frac{b(s)}{a(s)} ds} \end{aligned}$$

Now we find y_{in}. First we introduce interval variable

$$\xi = \frac{x}{\varepsilon^p}$$

And transform the ODE. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$\begin{aligned} \frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\ &= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right) \\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2} \end{aligned}$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and the ODE becomes

$$\begin{aligned} \varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + a(\xi)\frac{dy}{d\xi}\varepsilon^{-p} + b(\xi)y &= 0 \\ \varepsilon^{1-2p}y'' + a\varepsilon^{-p}y' + by &= 0 \end{aligned}$$

Balancing $1 - 2p$ with $-p$ shows that

$$\boxed{p = 1}$$

Hence

$$\varepsilon^{-1}y'' + a\varepsilon^{-1}y' + by = 0$$

Substituting $y_{in} = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$ in the above gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms with order $O\left(\varepsilon^{-1}\right)$ gives

$$y_0'' \sim -a y_0'$$

Assuming $z = y_0'$ then the above becomes $z' \sim -az$ or $\frac{dz}{d\xi} \sim -az$. This is separable. The solution is $\frac{dz}{z} \sim -ad\xi$ or

$$\ln|z| \sim -\int_0^\xi a(s)\,ds + C_1$$
$$z \sim C_1 e^{-\int_0^\xi a(s)ds}$$

Hence

$$\frac{dy_0}{d\xi} \sim C_1 e^{-\int_0^\xi a(s)ds}$$
$$dy_0 \sim \left(C_1 e^{-\int_0^\xi a(s)ds}\right) d\xi$$

Integrating again

$$y_0^{in} \sim \int_0^\xi \left(C_1 e^{-\int_0^\eta a(s)ds}\right) d\eta + C_2$$

Applying initial conditions at $y(0)$ since this is where the y_{in} exist. Using $y_{in}(0) = 1$ then the above becomes

$$1 = C_2$$

Hence the solution becomes

$$y_0^{in} \sim \int_0^\xi \left(C_1 e^{-\int_0^\eta a(s)ds}\right) d\eta + 1$$

To apply the second initial condition, which is $y'(0) = 1$, we first take derivative of the above w.r.t. ξ

$$y_0' \sim C_1 e^{-\int_0^\xi a(s)ds}$$

Applying $y_0'(0) = 1$ gives

$$1 = C_1$$

Hence

$$y_0^{in} \sim 1 + \int_0^\xi e^{-\int_0^\eta a(s)ds} d\eta$$

Now to find constant of integration for y^{out} from earlier, we need to do matching.

$$\lim_{\xi\to\infty} y_0^{in} \sim \lim_{x\to 0} y_0^{out}$$
$$\lim_{\xi\to\infty} 1 + \int_0^\xi e^{-\int_0^\eta a(s)ds} d\eta \sim \lim_{x\to 0} Ce^{-\int_1^x \frac{b(s)}{a(s)}ds}$$

On the LHS the integral $\int_0^\xi e^{-\int_0^\eta a(s)ds} d\eta$ since $a > 0$ and negative power on the exponential. So as $\xi \to \infty$ the integral value is zero. So we have now

$$1 \sim \lim_{x\to 0} Ce^{-\int_1^x \frac{b(s)}{a(s)}ds}$$

Let $\lim_{x\to 0} e^{-\int_1^x \frac{b(s)}{a(s)}ds} \to E$, where E is the value of the definite integral $Ce^{-\int_1^0 \frac{b(s)}{a(s)}ds}$. Another constant, which if we know $a(x), b(x)$ we can evaluate. Hence the above gives the value of C as

$$C = \frac{1}{E}$$

The uniform solution can now be written as

$$\begin{aligned} y_{\text{uniform}} &= y_{in} + y_{out} - y_{\text{match}} \\ &= 1 + \int_0^{\xi} e^{-\int_0^{\eta} a(s)ds} d\eta + \frac{1}{E} e^{-\int_1^x \frac{b(s)}{a(s)} ds} - 1 \\ &= \int_0^{\xi} e^{-\int_0^{\eta} a(s)ds} d\eta + \frac{1}{E} e^{-\int_1^x \frac{b(s)}{a(s)} ds} \end{aligned} \tag{2}$$

Finally, we need to show that $y_{\text{uniform}}(0) = 1$ but not $y'_{\text{uniform}}(0) = 1$. From (2), at $x = 0$ which also means $\xi = 0$, since boundary layer at left side, equation (2) becomes

$$y_{\text{uniform}}(0) = 0 + \frac{1}{E} \lim_{x \to 0} e^{-\int_1^x \frac{b(s)}{a(s)} ds}$$

But we said that $\lim_{x \to 0} e^{-\int_1^x \frac{b(s)}{a(s)} ds} = E$, therefore

$$y_{\text{uniform}}(0) = 1$$

Now we take derivative of (2) w.r.t. x and obtain

$$\begin{aligned} y'_{\text{uniform}}(x) &= \frac{d}{dx}\left(\int_0^{\frac{x}{\varepsilon}} e^{-\int_0^{\eta} a(s)ds} d\eta\right) + \frac{1}{E}\frac{d}{dx}\left(e^{-\int_1^x \frac{b(s)}{a(s)} ds}\right) \\ &= e^{-\int_0^{\frac{x}{\varepsilon}} a(s)ds} - \frac{1}{E}\frac{b(x)}{a(x)} e^{-\int_1^x \frac{b(s)}{a(s)} ds} \end{aligned}$$

And at $x = 0$ the above becomes

$$y'_{\text{uniform}}(0) = 1 - \frac{1}{E}\frac{b(0)}{a(0)}$$

The above is zero only if $b(0) = 0$ (since we know $a(0) > 0$). Therefore, we see that $y'_{\text{uniform}}(0) \neq 1$ for any arbitrary $b(x)$. Which is what we are asked to show.

Will now solve the whole problem again, <u>when a, b are constants</u>.

$$\begin{aligned} \varepsilon y'' + ay' + by &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned} \tag{1A}$$

And $a > 0$. And compare leading order uniform approximation with the exact solution to the problem when $a(x), b(x)$ are constants. Since $a > 0$ then boundary layer will occur at $x = 0$. We start by finding $y_{out}(x)$.

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (1) gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms with $O(1)$ results in

$$\begin{aligned} a y_0' &\sim -b y_0 \\ \frac{dy_0}{dx} &\sim -\frac{b}{a} y_0 \end{aligned}$$

This is separable

$$\int \frac{dy_0}{y_0} \sim -\frac{b}{a}dx$$
$$\ln |y_0| \sim -\frac{b}{a}x + C$$
$$y_0^{out} \sim C_1 e^{-\frac{b}{a}x}$$

Now we find y_{in}. First we introduce internal variable $\xi = \frac{x}{\varepsilon^p}$ and transform the ODE as we did above. This results in

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + a\varepsilon^{-1}\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) + b\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0$$

Collecting terms with order $O\left(\varepsilon^{-1}\right)$ gives

$$y_0'' \sim -a y_0'$$

Assuming $z = y_0'$ then the above becomes $z' \sim -az$ or $\frac{dz}{d\xi} \sim -az$. This is separable. The solution is $\frac{dz}{z} \sim -a d\xi$ or

$$\ln |z| \sim -a\xi + E_1$$
$$z \sim E_1 e^{-a\xi}$$

Hence

$$\frac{dy_0}{d\xi} \sim E_1 e^{-a\xi}$$
$$dy_0 \sim E_1 e^{-a\xi} d\xi$$

Integrating again

$$y_0^{in} \sim E_1 \left(\frac{-1}{a}\right) e^{-a\xi} + E_2$$

Applying initial conditions at $y(0)$ since this is where the y_{in} exist. Using $y_{in}(0) = 1$ then the above becomes

$$1 = E_1 \left(\frac{-1}{a}\right) + E_2$$
$$a(E_2 - 1) = E_1$$

Hence the solution becomes

$$y_0^{in} \sim (1 - E_2) e^{-a\xi} + E_2 \tag{1B}$$

To apply the second initial condition, which is $y'(0) = 1$, we first take derivative of the above w.r.t. ξ

$$y_0' \sim -a(1 - E_2) e^{-a\xi}$$

Hence $y'(0) = 1$ gives

$$
\begin{aligned}
1 &= -a\left(1-E_2\right)\\
1 &= -a + aE_2\\
E_2 &= \frac{1+a}{a}
\end{aligned}
$$

And the solution y_{in} in (1B) becomes

$$
\begin{aligned}
y_0^{in} &\sim \left(1-\frac{1+a}{a}\right)e^{-a\xi}+\frac{1+a}{a}\\
&\sim \left(\frac{-1}{a}\right)e^{-a\xi}+\frac{1+a}{a}\\
&\sim \frac{(1+a)-e^{-a\xi}}{a}
\end{aligned}
$$

Now to find constant of integration for $y^{out}(x)$ from earlier, we need to do matching.

$$
\begin{aligned}
\lim_{\xi\to\infty} y_0^{in} &\sim \lim_{x\to 0} y_0^{out}\\
\lim_{\xi\to\infty} \frac{(1+a)-e^{-a\xi}}{a} &\sim \lim_{x\to 0} C_1 e^{-\frac{b}{a}x}\\
\frac{1+a}{a} &\sim C_1
\end{aligned}
$$

Hence now the uniform solution can be written as

$$
\begin{aligned}
y_{\text{uniform}}(x) &\sim y_{in}+y_{out}-y_{\text{match}}\\
&\sim \overbrace{\frac{(1+a)-e^{-a\frac{x}{\varepsilon}}}{a}}^{y_{in}}+\overbrace{\frac{1+a}{a}e^{-\frac{b}{a}x}}^{y_{out}}-\frac{1+a}{a}\\
&\sim \frac{(1+a)}{a}-\frac{e^{-a\frac{x}{\varepsilon}}}{a}+\frac{1+a}{a}e^{-\frac{b}{a}x}-\frac{1+a}{a}\\
&\sim -\frac{e^{-a\frac{x}{\varepsilon}}}{a}+\frac{1+a}{a}e^{-\frac{b}{a}x}\\
&\sim \frac{1}{a}\left((1+a)\,e^{-\frac{b}{a}x}-e^{-a\frac{x}{\varepsilon}}\right) \qquad \text{(2A)}
\end{aligned}
$$

Now we compare the above, which is the leading order uniform approximation, to the exact solution. Since now a,b are constants, then the exact solution is

$$
y_{exact}(x) = Ae^{\lambda_1 x}+Be^{\lambda_2 x} \qquad (3)
$$

Where $\lambda_{1,2}$ are roots of characteristic equation of $\varepsilon y''+ay'+by=0$. These are $\lambda=\frac{-a}{2\varepsilon}\pm\frac{1}{2\varepsilon}\sqrt{a^2-4\varepsilon b}$. Hence

$$
\begin{aligned}
\lambda_1 &= \frac{-a}{2}+\frac{1}{2}\sqrt{a^2-4\varepsilon b}\\
\lambda_2 &= \frac{-a}{2}-\frac{1}{2}\sqrt{a^2-4\varepsilon b}
\end{aligned}
$$

Applying initial conditions to (3). $y(0)=1$ gives

$$
\begin{aligned}
1 &= A+B\\
B &= 1-A
\end{aligned}
$$

And solution becomes $y_{exact}(x) = Ae^{\lambda_1 x} + (1-A)e^{\lambda_2 x}$. Taking derivatives gives

$$y'_{exact}(x) = A\lambda_1 e^{\lambda_1 x} + (1-A)\lambda_2 e^{\lambda_2 x}$$

Using $y'(0) = 1$ gives

$$\begin{aligned} 1 &= A\lambda_1 + (1-A)\lambda_2 \\ 1 &= A(\lambda_1 - \lambda_2) + \lambda_2 \\ A &= \frac{1-\lambda_2}{\lambda_1 - \lambda_2} \end{aligned}$$

Therefore, $B = 1 - \frac{1-\lambda_2}{\lambda_1-\lambda_2}$ and the exact solution becomes

$$\begin{aligned} y_{exact}(x) &= \frac{1-\lambda_2}{\lambda_1-\lambda_2} e^{\lambda_1 x} + \left(1 - \frac{1-\lambda_2}{\lambda_1-\lambda_2}\right) e^{\lambda_2 x} \\ &= \frac{1-\lambda_2}{\lambda_1-\lambda_2} e^{\lambda_1 x} + \left(\frac{(\lambda_1-\lambda_2)-(1-\lambda_2)}{\lambda_1-\lambda_2}\right) e^{\lambda_2 x} \\ &= \frac{1-\lambda_2}{\lambda_1-\lambda_2} e^{\lambda_1 x} + \left(\frac{\lambda_1 - 1}{\lambda_1-\lambda_2}\right) e^{\lambda_2 x} \end{aligned} \tag{4}$$

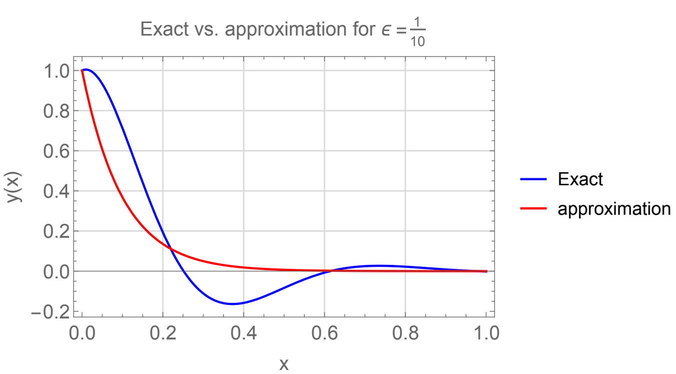
While the uniform solution above was found to be $\frac{1}{a}\left((1+a)e^{-\frac{b}{a}x} - e^{-a\frac{x}{\varepsilon}}\right)$. Here is a plot of the exact solution above, for $\varepsilon = \{1/10, 1/50, 1/100\}$, and for some values for a, b such as $a = 1, b = 10$ in order to compare with the uniform solution. Note that the uniform solution is $O(\varepsilon)$. As ε becomes smaller, the leading order uniform solution will better approximate the exact solution. At $\varepsilon = 0.01$ the uniform approximation gives very good approximation. This is using only leading term approximation.

```
In[134]:= ClearAll[x, y]
eps = 1 / 10; a = 1; b = 10;
mySol = 1 / a ((1 + a) * Exp[-b / a x] - Exp[-a x / eps]);
sol = y[x] /. First@DSolve[{eps y''[x] + a y'[x] + b y[x] == 0, y[0] == 1, y'[0] == 1}, y[x], x]
```

Out[137]= $\frac{1}{5} e^{-5x} \left(5 \operatorname{Cos}\left[5\sqrt{3}\, x\right] + 2\sqrt{3} \operatorname{Sin}\left[5\sqrt{3}\, x\right]\right)$

```
In[125]:= Plot[{sol, mySol}, {x, 0, 1}, PlotRange → All, PlotStyle → {Blue, Red}, PlotLegends → {"Exact", "approximation"},
  Frame → True, FrameLabel → {{"y(x)", None}, {"x", Row[{"Exact vs. approximation for ϵ =", eps}]}},
  BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray]
```

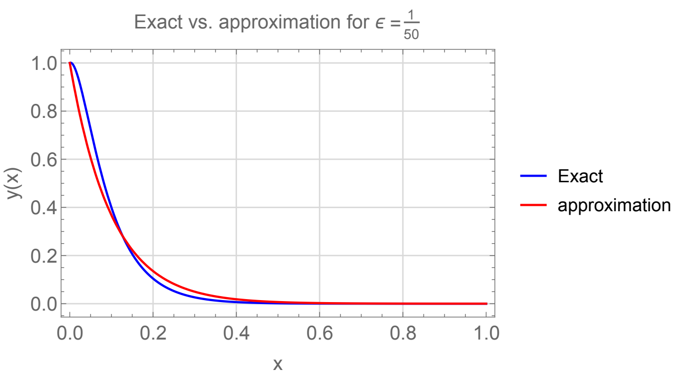
Out[125]=



In[126]:=
```
ClearAll[x, y]
eps = 1 / 50; a = 1; b = 10;
sol = y[x] /. First@DSolve[{eps y''[x] + a y'[x] + b y[x] == 0, y[0] == 1, y'[0] == 1}, y[x], x]
Plot[{sol, mySol}, {x, 0, 1}, PlotRange → All, PlotStyle → {Blue, Red}, PlotLegends → {"Exact", "approximation"},
 Frame → True, FrameLabel → {{"y(x)", None}, {"x", Row[{"Exact vs. approximation for ϵ =", eps}]}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[128]= $\frac{1}{50}\left(25\, e^{\left(-25-5\sqrt{5}\right)x} - 26\sqrt{5}\, e^{\left(-25-5\sqrt{5}\right)x} + 25\, e^{\left(-25+5\sqrt{5}\right)x} + 26\sqrt{5}\, e^{\left(-25+5\sqrt{5}\right)x}\right)$

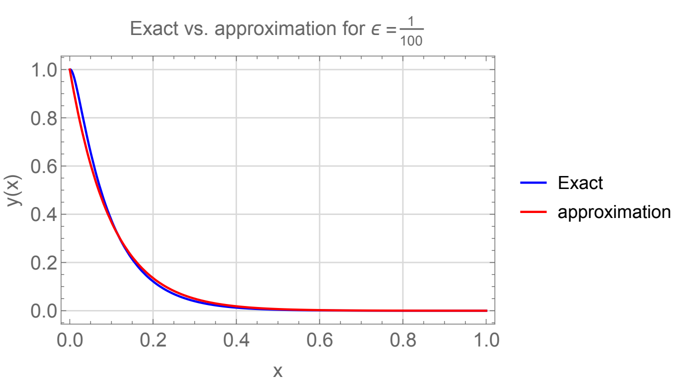
Out[129]=



In[130]:=
```
ClearAll[x, y]
eps = 1 / 100; a = 1; b = 10;
sol = y[x] /. First@DSolve[{eps y''[x] + a y'[x] + b y[x] == 0, y[0] == 1, y'[0] == 1}, y[x], x]
Plot[{sol, mySol}, {x, 0, 1}, PlotRange → All, PlotStyle → {Blue, Red}, PlotLegends → {"Exact", "approximation"},
 Frame → True, FrameLabel → {{"y(x)", None}, {"x", Row[{"Exact vs. approximation for ϵ =", eps}]}},
 BaseStyle → 14, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[132]= $\frac{1}{100}\left(50\, e^{\left(-50-10\sqrt{15}\right)x} - 17\sqrt{15}\, e^{\left(-50-10\sqrt{15}\right)x} + 50\, e^{\left(-50+10\sqrt{15}\right)x} + 17\sqrt{15}\, e^{\left(-50+10\sqrt{15}\right)x}\right)$

Out[133]=



0.5 problem 9.15(b)

Problem Find first order uniform approximation valid as $\varepsilon \to 0^+$ for $0 \le x \le 1$

$$\varepsilon y'' + \left(x^2+1\right) y' - x^3 y = 0 \tag{1}$$
$$y(0) = 1$$
$$y(1) = 1$$

Solution

Since $a(x) = \left(x^2+1\right)$ is positive for $0 \le x \le 1$, therefore we expect the boundary layer to be on the left side at $x = 0$. Assuming this is the case for now (if it is not, then we expect not to be able to do the matching). We start by finding $y_{out}(x)$.

$$y_{out}(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

Substituting this into (1) gives

$$\varepsilon\left(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots\right) + \left(x^2+1\right)\left(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots\right) - x^3\left(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots\right) = 0 \qquad (2)$$

Collecting terms with $O(1)$ results in

$$\left(x^2+1\right) y_0' \sim x^3 y_0$$
$$\frac{dy_0}{dx} \sim \frac{x^3}{\left(x^2+1\right)} y_0$$

This is separable.

$$\int \frac{dy_0}{y_0} \sim \int \frac{x^3}{\left(x^2+1\right)} dx$$
$$\ln\left|y_0\right| \sim \int x - \frac{x}{1+x^2} dx$$
$$\sim \frac{x^2}{2} - \frac{1}{2}\ln\left(1+x^2\right) + C$$

Hence

$$y_0 \sim e^{\frac{x^2}{2} - \frac{1}{2}\ln\left(1+x^2\right) + C}$$
$$\sim \frac{Ce^{\frac{x^2}{2}}}{\sqrt{1+x^2}}$$

Applying $y_0^{out}(1) = 1$ to the above (since this is where the outer solution is), we solve for C

$$1 \sim \frac{Ce^{\frac{1}{2}}}{\sqrt{2}}$$
$$C \sim \sqrt{2}e^{\frac{-1}{2}}$$

Therefore

$$y_0^{out} \sim \frac{\sqrt{2}e^{\frac{-1}{2}} e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}$$
$$\sim \sqrt{\frac{2}{e}} \frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}$$

Now we need to find y_1^{out}. From (2), but now collecting terms in $O(\varepsilon)$ gives

$$y_0'' + \left(x^2+1\right) y_1' \backsim x^3 y_1 \qquad (3)$$

In the above y_0'' is known.

$$y_0'(x) = \sqrt{\frac{2}{e}}\frac{d}{dx}\left(\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\right)$$

$$= \sqrt{\frac{2}{e}}\frac{x^3 e^{\frac{x^2}{2}}}{\left(1+x^2\right)^{\frac{3}{2}}}$$

And

$$y_0''(x) = \sqrt{\frac{2}{e}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{5}{2}}}$$

Hence (3) becomes

$$\left(x^2+1\right)y_1' \sim x^3 y_1 - y_0''$$

$$\left(x^2+1\right)y_1' \sim x^3 y_1 - \sqrt{\frac{2}{e}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{5}{2}}}$$

$$y_1' - \frac{x^3}{\left(x^2+1\right)}y_1 \sim -\sqrt{\frac{2}{e}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{7}{2}}}$$

Integrating factor is $\mu = e^{-\int \frac{x^3}{\left(x^2+1\right)}dx} = e^{-\frac{x^2}{2}+\frac{1}{2}\ln\left(1+x^2\right)} = \left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}}$, hence the above becomes

$$\frac{d}{dx}\left(\left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}} y_1\right) \sim -\sqrt{\frac{2}{e}}\left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}}\frac{x^2 e^{\frac{x^2}{2}}\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{\frac{7}{2}}}$$

$$\sim -\sqrt{\frac{2}{e}}\frac{x^2\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{3}}$$

Integrating gives (with help from CAS)

$$\left(1+x^2\right)^{\frac{1}{2}} e^{\frac{-x^2}{2}} y_1(x) \sim -\sqrt{\frac{2}{e}}\int \frac{x^2\left(x^4+x^2+3\right)}{\left(1+x^2\right)^{3}}dx$$

$$\sim -\sqrt{\frac{2}{e}}\int 1 - \frac{3}{\left(1+x^2\right)^{3}} + \frac{4}{\left(1+x^2\right)^{2}} - \frac{2}{1+x^2}dx$$

$$\sim -\sqrt{\frac{2}{e}}\left(x - \frac{3x}{4\left(1+x^2\right)^{2}} + \frac{7x}{8\left(1+x^2\right)} - 9\frac{\arctan(x)}{8}\right) + C_1$$

Hence

$$y_1^{out}(x) \sim -\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\left(1+x^2\right)^{\frac{1}{2}}}\left(x - \frac{3x}{4\left(1+x^2\right)^{2}} + \frac{7x}{8\left(1+x^2\right)} - 9\frac{\arctan(x)}{8}\right) + C_1\frac{e^{\frac{x^2}{2}}}{\left(1+x^2\right)^{\frac{1}{2}}}$$

Now we find C_1 from boundary conditions $y_1(1) = 0$. (notice the BC now is $y_1(1) = 0$ and not $y_1(1) = 1$, since we used $y_1(1) = 1$ already).

$$\sqrt{\frac{2}{e}}\frac{e^{\frac{1}{2}}}{(1+1)^{\frac{1}{2}}}\left(1-\frac{3}{4(1+1)^2}+\frac{7}{8(1+1)}-9\frac{\arctan(1)}{8}\right)=C_1\frac{e^{\frac{1}{2}}}{(1+1)^{\frac{1}{2}}}$$

$$\sqrt{\frac{2}{e}}\frac{e^{\frac{1}{2}}}{\sqrt{2}}\left(1-\frac{3}{16}+\frac{7}{16}-\frac{9}{8}\arctan(1)\right)=C_1\frac{e^{\frac{1}{2}}}{\sqrt{2}}$$

Simplifying

$$1-\frac{3}{16}+\frac{7}{16}-\frac{9}{8}\arctan(1)=C_1\frac{e^{\frac{1}{2}}}{\sqrt{2}}$$

$$C_1=\sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{8}\arctan(1)\right)$$

$$C_1=\sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{32}\pi\right)$$

$$=0.31431$$

Hence

$$y_1^{out}\sim-\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{(1+x^2)}}\left(x-\frac{3x}{4\left(1+x^2\right)^2}+\frac{7x}{8\left(1+x^2\right)}-\frac{9}{8}\arctan(x)\right)+\sqrt{\frac{2}{e}}\left(\frac{5}{4}-\frac{9}{32}\pi\right)\frac{e^{\frac{x^2}{2}}}{\sqrt{(1+x^2)}}$$

$$\sim\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{(1+x^2)}}\left(\left(\frac{5}{4}-\frac{9}{32}\pi\right)-\left(x-\frac{3x}{4\left(1+x^2\right)^2}+\frac{7x}{8\left(1+x^2\right)}-\frac{9}{8}\arctan(x)\right)\right)$$

$$\sim\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{(1+x^2)}}\left(\frac{5}{4}-\frac{9}{32}\pi-x+\frac{3x}{4\left(1+x^2\right)^2}-\frac{7x}{8\left(1+x^2\right)}+\frac{9}{8}\arctan(x)\right)$$

Hence

$$y^{out}(x)\sim y_0^{out}+\varepsilon y_1^{out}$$

$$\sim\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}+\varepsilon\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{(1+x^2)}}\left(\frac{5}{4}-\frac{9}{32}\pi-x+\frac{3x}{4\left(1+x^2\right)^2}-\frac{7x}{8\left(1+x^2\right)}+\frac{9}{8}\arctan(x)\right)+O\left(\varepsilon^2\right)$$

$$\sim\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1+\varepsilon\left(\frac{5}{4}-\frac{9}{32}\pi-x+\frac{3x}{4\left(1+x^2\right)^2}-\frac{7x}{8\left(1+x^2\right)}+\frac{9}{8}\arctan(x)\right)\right)+O\left(\varepsilon^2\right)\qquad\text{(3A)}$$

Now that we found $y^{out}(x)$, we need to find $y^{in}(x)$ and then do the matching and the find uniform approximation. Since the boundary layer at $x=0$, we introduce inner variable $\xi=\frac{x}{\varepsilon^p}$ and then express the original ODE using this new variable. We also need to determine p in the above expression. Since

$\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$\begin{aligned}\frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx}\\ &= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right)\\ &= \varepsilon^{-2p}\frac{d^2}{d\xi^2}\end{aligned}$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and the ODE $\varepsilon y'' + \left(x^2+1\right)y' - x^3y = 0$ now becomes

$$\begin{aligned}\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + \left((\xi\varepsilon^p)^2+1\right)\varepsilon^{-p}\frac{dy}{d\xi} - (\xi\varepsilon^p)^3\,y = 0\\ \varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(\xi^2\varepsilon^p + \varepsilon^{-p}\right)\frac{dy}{d\xi} - \xi^3\varepsilon^{3p}y = 0\end{aligned}$$

The largest terms are $\left\{\varepsilon^{1-2p}, \varepsilon^{-p}\right\}$, therefore matching them gives $p = 1$. The ODE now becomes

$$\varepsilon^{-1}\frac{d^2y}{d\xi^2} + \left(\xi^2\varepsilon + \varepsilon^{-1}\right)\frac{dy}{d\xi} - \xi^3\varepsilon^3y = 0 \tag{4}$$

Assuming that

$$y_{in}(x) = \sum_{n=0}^{\infty}\varepsilon^n y_n = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots$$

And substituting the above into (4) gives

$$\varepsilon^{-1}\left(y_0'' + \varepsilon y_1'' + \cdots\right) + \left(\xi^2\varepsilon + \varepsilon^{-1}\right)\left(y_0' + \varepsilon y_1' + \cdots\right) - \xi^3\varepsilon^3\left(y_0 + \varepsilon y_1 + \cdots\right) = 0 \tag{4A}$$

Collecting terms in $O\left(\varepsilon^{-1}\right)$ gives

$$y_0'' \backsim -y_0'$$

Letting $z = y_0'$, the above becomes

$$\begin{aligned}\frac{dz}{d\xi} &\backsim -z\\ \frac{dz}{z} &\backsim -d\xi\\ \ln|z| &\backsim -\xi + C_1\\ z &\backsim C_1e^{-\xi}\end{aligned}$$

Hence

$$\begin{aligned}\frac{dy_0}{d\xi} &\backsim C_1e^{-\xi}\\ y_0 &\backsim C_1\int e^{-\xi}d\xi + C_2\\ &\backsim -C_1e^{-\xi} + C_2\end{aligned} \tag{5}$$

Applying <u>boundary conditions $y_0^{in}(0) = 1$</u> gives

$$\begin{aligned}1 &= -C_1 + C_2\\ C_2 &= 1 + C_1\end{aligned}$$

And (5) becomes

$$\begin{aligned} y_0^{in}(\xi) &\backsim -C_1 e^{-\xi} + (1 + C_1) \\ &\backsim 1 + C_1\left(1 - e^{-\xi}\right) \end{aligned} \tag{6}$$

We now find y_1^{in}. Going back to (4) and collecting terms in $O(1)$ gives the ODE

$$y_1'' \backsim y_1'$$

This is the same ODE we solved above. But it will have different B.C. Hence

$$y_1^{in} \backsim -C_3 e^{-\xi} + C_4$$

Applying boundary conditions $y_1^{in}(0) = 0$ gives

$$\begin{aligned} 0 &= -C_3 + C_4 \\ C_3 &= C_4 \end{aligned}$$

Therefore

$$\begin{aligned} y_1^{in} &\backsim -C_3 e^{-\xi} + C_3 \\ &\backsim C_3\left(1 - e^{-\xi}\right) \end{aligned}$$

Now we have the leading order y^{in}

$$\begin{aligned} y^{in}(\xi) &= y_0^{in} + \varepsilon y_1^{in} \\ &= 1 + C_1\left(1 - e^{-\xi}\right) + \varepsilon C_3\left(1 - e^{-\xi}\right) + O\left(\varepsilon^2\right) \end{aligned} \tag{7}$$

Now we are ready to do the matching between (7) and (3A)

$$\lim_{\xi\to\infty} 1 + C_1\left(1 - e^{-\xi}\right) + \varepsilon C_3\left(1 - e^{-\xi}\right) \sim$$

$$\lim_{x\to 0}\sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1 + \varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right)$$

Or

$$1 + C_1 + \varepsilon C_3 \sim \sqrt{\frac{2}{e}}\lim_{x\to 0}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} + \sqrt{\frac{2}{e}}\varepsilon\lim_{x\to 0}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)$$

But $\lim_{x\to 0}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}} \to 1$, $\lim_{x\to 0}\frac{3x}{4\left(1+x^2\right)^2} \to 0$ $\lim_{x\to 0}\frac{7x}{8\left(1+x^2\right)} \to 0$ therefore the above becomes

$$1 + C_1 + \varepsilon C_3 \sim \sqrt{\frac{2}{e}} + \sqrt{\frac{2}{e}}\varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi\right)$$

Hence

$$\begin{aligned} 1 + C_1 &= \sqrt{\frac{2}{e}} \\ C_1 &= \sqrt{\frac{2}{e}} - 1 \\ C_3 &= \sqrt{\frac{2}{e}}\left(\frac{5}{4} - \frac{9}{32}\pi\right) \end{aligned}$$

This means that

$$\begin{aligned} y^{in}(\xi) &\sim 1 + C_1\left(1 - e^{-\xi}\right) + \varepsilon C_3\left(1 - e^{-\xi}\right) \\ &\sim 1 + \left(\sqrt{\frac{2}{e}} - 1\right)\left(1 - e^{-\xi}\right) + \varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4} - \frac{9}{32}\pi\right)\left(1 - e^{-\xi}\right) \end{aligned}$$

Therefore

$$\begin{aligned} y_{\text{uniform}(x)} &\sim y^{in}(\xi) + y^{out}(x) - y_{match} \\ &\sim 1 + \left(\sqrt{\frac{2}{e}} - 1\right)\left(1 - e^{-\xi}\right) + \varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4} - \frac{9}{32}\pi\right)\left(1 - e^{-\xi}\right) \\ &\quad + \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1 + \varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right) \\ &\quad - \left(\sqrt{\frac{2}{e}} + \sqrt{\frac{2}{e}}\varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi\right)\right) \end{aligned}$$

Or (replacing ξ by $\frac{x}{\varepsilon}$ and simplifying)

$$\begin{aligned} y_{\text{uniform}(x)} &\sim 1 + \left(\sqrt{\frac{2}{e}} - 1\right)\left(1 - e^{-\xi}\right) - e^{-\xi}\varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4} - \frac{9}{32}\pi\right) \\ &\quad + \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1 + \varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right) \\ &\quad - \sqrt{\frac{2}{e}} \end{aligned}$$

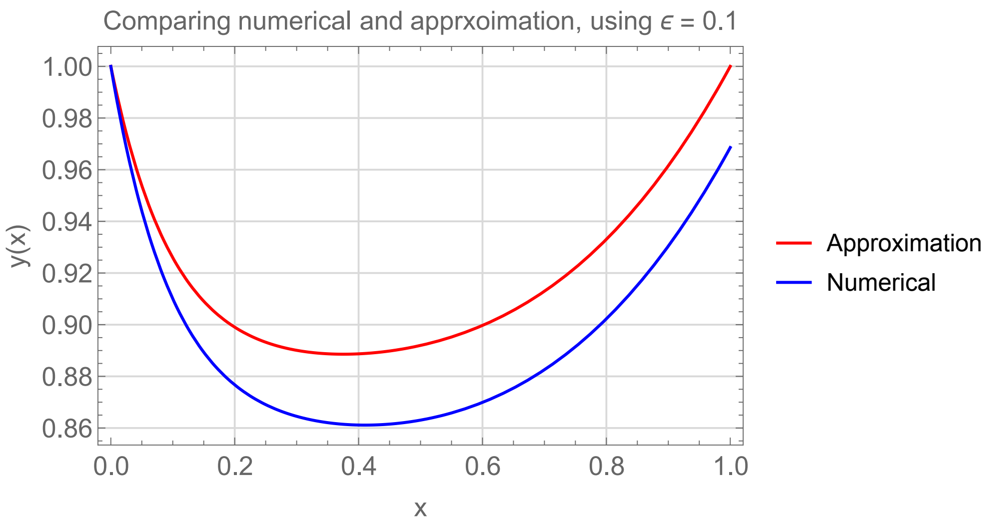
Or

$$\begin{aligned} y_{\text{uniform}(x)} \sim -\sqrt{\frac{2}{e}}e^{-\xi} + e^{-\xi} - e^{-\xi}\varepsilon\sqrt{\frac{2}{e}}\left(\frac{5}{4} - \frac{9}{32}\pi\right) \\ + \sqrt{\frac{2}{e}}\frac{e^{\frac{x^2}{2}}}{\sqrt{1+x^2}}\left(1 + \varepsilon\left(\frac{5}{4} - \frac{9}{32}\pi - x + \frac{3x}{4\left(1+x^2\right)^2} - \frac{7x}{8\left(1+x^2\right)} + \frac{9}{8}\arctan(x)\right)\right) \end{aligned}$$

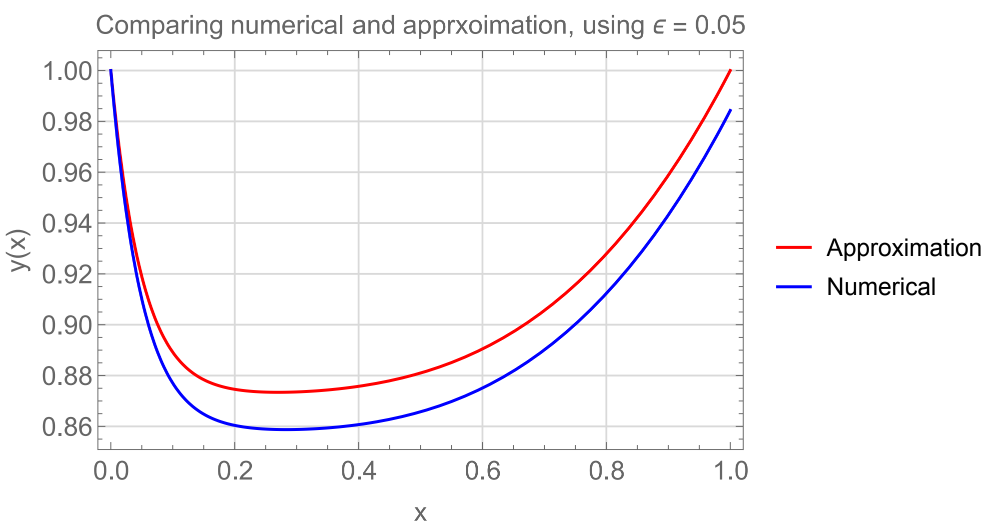
To check validity of the above solution, the approximate solution is plotted against the numerical solution for different values of $\varepsilon = \{0.1, 0.05, 0.01\}$. This shows very good agreement with the numerical solution. At $\varepsilon = 0.01$ the solutions are almost the same.

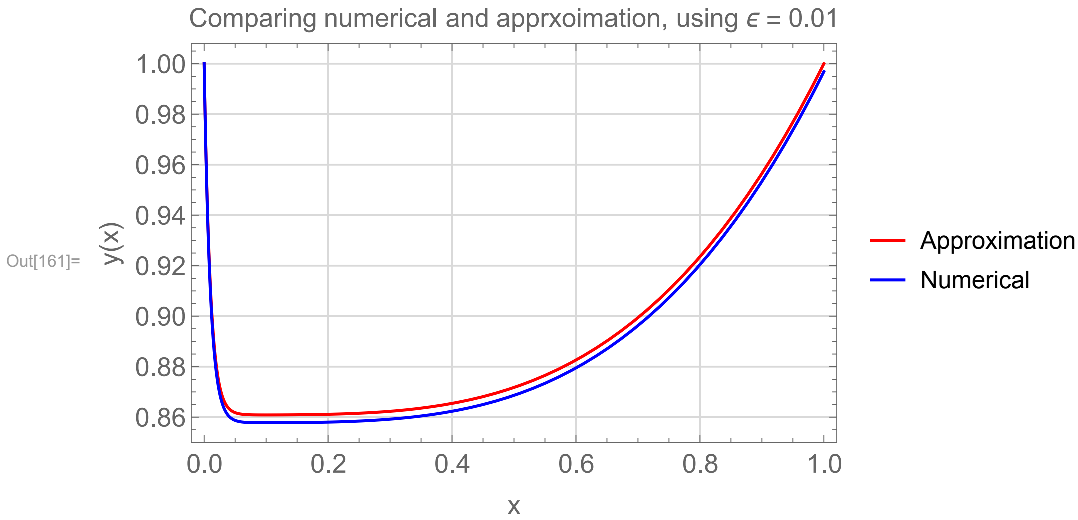
```
ClearAll[x, y, ϵ];
ϵ = 0.01;
r = Sqrt[2/Exp[1]];
ode = ϵ y''[x] + (x^2 + 1) y'[x] - x^3 y[x] == 0;
sol = First@NDSolve[{ode, y[0] == 1, y[1] == 1}, y, {x, 0, 1}];
p1 = Plot[Evaluate[y[x] /. sol], {x, 0, 1}];
mysol[x_, ϵ_] := -r Exp[-x/ϵ] + Exp[-x/ϵ] - ϵ r (5/4 - 9/32 π) + r Exp[x^2/2]/Sqrt[1 + x^2] (1 + ϵ (5/4 - 9/32 π - x + (3 x)/(4 (1 + x^2)^2) - (7 x)/(8 (1 + x^2)) + 9/8 ArcTan[x]));
p2 = Plot[{Evaluate[y[x] /. sol], mysol[x, ϵ]}, {x, 0, 1}, Frame → True, PlotStyle → {Red, Blue},
  FrameLabel → {{"y(x)", None}, {"x", Row[{"Comparing numerical and apprxoimation, using ϵ = ", N@ϵ}]}}, GridLines → Automatic,
  GridLinesStyle → LightGray, PlotLegends → {"Approximation", "Numerical"}, BaseStyle → 14, ImageSize → 400]
```

Out[145]=



Out[153]=





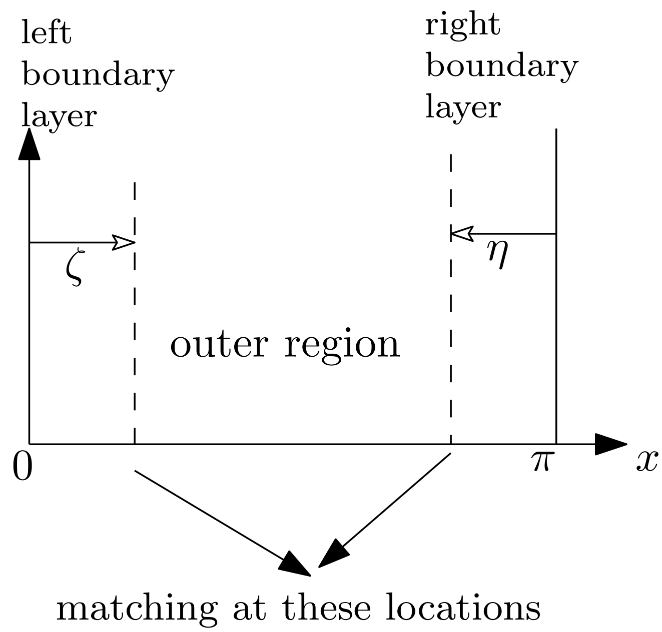
0.6 problem 9.19

Problem Find lowest order uniform approximation to boundary value problem

$$\varepsilon y'' + (\sin x)\, y' + y \sin(2x) = 0$$
$$y(0) = \pi$$
$$y(\pi) = 0$$

Solution

We expect a boundary layer at left end at $x = 0$. Therefore, we need to find $y^{in}(\xi), y^{out}(x)$, where ξ is an inner variable defined by $\xi = \frac{x}{\varepsilon^p}$.



<u>Finding $y^{in}(\xi)$</u>

At $x = 0$, we introduce inner variable $\xi = \frac{x}{\varepsilon^p}$ and then express the original ODE using this new variable. We also need to determine p in the above expression. Since $\frac{dy}{dx} = \frac{dy}{d\xi}\frac{d\xi}{dx}$ then $\frac{dy}{dx} = \frac{dy}{d\xi}\varepsilon^{-p}$. Hence $\frac{d}{dx} \equiv \varepsilon^{-p}\frac{d}{d\xi}$

$$
\begin{aligned}
\frac{d^2}{dx^2} &= \frac{d}{dx}\frac{d}{dx} \\
&= \left(\varepsilon^{-p}\frac{d}{d\xi}\right)\left(\varepsilon^{-p}\frac{d}{d\xi}\right) \\
&= \varepsilon^{-2p}\frac{d^2}{d\xi^2}
\end{aligned}
$$

Therefore $\frac{d^2y}{dx^2} = \varepsilon^{-2p}\frac{d^2y}{d\xi^2}$ and the ODE $\varepsilon y'' + (\sin x)\,y' + y\sin(2x) = 0$ now becomes

$$
\begin{aligned}
\varepsilon\varepsilon^{-2p}\frac{d^2y}{d\xi^2} + (\sin(\xi\varepsilon^p))\,\varepsilon^{-p}\frac{dy}{d\xi} + \sin(2\xi\varepsilon^p)\,y = 0 \\
\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + (\sin(\xi\varepsilon^p))\,\varepsilon^{-p}\frac{dy}{d\xi} + \sin(2\xi\varepsilon^p)\,y = 0
\end{aligned}
$$

Expanding the sin terms in the above, in Taylor series around zero, $\sin(x) = x - \frac{x^3}{3!} + \cdots$ gives

$$
\begin{aligned}
\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(\xi\varepsilon^p - \frac{(\xi\varepsilon^p)^3}{3!} + \cdots\right)\varepsilon^{-p}\frac{dy}{d\xi} + \left(2\xi\varepsilon^p - \frac{(2\xi\varepsilon^p)^3}{3!} + \cdots\right)y = 0 \\
\varepsilon^{1-2p}\frac{d^2y}{d\xi^2} + \left(\xi - \frac{\xi^3\varepsilon^{2p}}{3!} + \cdots\right)\frac{dy}{d\xi} + \left(2\xi\varepsilon^p - \frac{(2\xi\varepsilon^p)^3}{3!} + \cdots\right)y = 0
\end{aligned}
$$

Then the largest terms are $\left\{\varepsilon^{1-2p},1\right\}$, therefore $1-2p=0$ or

$$\boxed{p=\frac{1}{2}}$$

The ODE now becomes

$$y''+\left(\xi-\frac{\xi^3\varepsilon}{3!}+\cdots\right)y'+\left(2\xi\sqrt{\varepsilon}-\frac{\left(2\xi\sqrt{\varepsilon}\right)^3}{3!}+\cdots\right)y=0 \tag{1}$$

Assuming that

$$y^{left}(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n=y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots$$

Then (1) becomes

$$\left(y_0''+\varepsilon y_1''+\cdots\right)+\left(\xi-\frac{\xi^3\varepsilon}{3!}+\cdots\right)\left(y_0'+\varepsilon y_1'+\cdots\right)+\left(2\xi\sqrt{\varepsilon}-\frac{\left(2\xi\sqrt{\varepsilon}\right)^3}{3!}+\cdots\right)\left(y_0+\varepsilon y_1+\cdots\right)=0$$

Collecting terms in $O(1)$ gives the balance

$$\begin{aligned} y_0''(\xi)&\sim-\xi y_0'(\xi)\\ y_0(0)&=\pi \end{aligned}$$

Assuming $z=y_0'$, then

$$\begin{aligned} z'&\sim-\xi z\\ \frac{dz}{z}&\sim-\xi\\ \ln|z|&\sim-\frac{\xi^2}{2}+C_1\\ z&\sim C_1e^{\frac{-\xi^2}{2}} \end{aligned}$$

Therefore $y_0'\sim C_1e^{\frac{-\xi^2}{2}}$. Hence

$$y_0(\xi)\sim C_1\int_0^{\xi}e^{\frac{-s^2}{2}}ds+C_2$$

With boundary conditions $y(0)=\pi$. Hence

$$\pi=C_2$$

And the solution becomes

$$y_0^{in}(\xi)\sim C_1\int_0^{\xi}e^{\frac{-s^2}{2}}ds+\pi \tag{2}$$

Now we need to find $y^{out}(x)$. Assuming that

$$y^{out}(x)=\sum_{n=0}^{\infty}\varepsilon^n y_n=y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots$$

Then $\varepsilon y''+(\sin x)y'+y\sin(2x)=0$ becomes

$$\varepsilon\left(y_0''+\varepsilon y_1''+\cdots\right)+\sin(x)\left(y_0'+\varepsilon y_1'+\cdots\right)+\sin(2x)\left(y_0+\varepsilon y_1+\cdots\right)=0$$

Collecting terms in $O(1)$ gives the balance

$$\begin{aligned} \sin(x)\, y_0'(x) &\sim -\sin(2x)\, y_0(x) \\ \frac{dy_0}{y_0} &\sim -\frac{\sin(2x)}{\sin(x)} dx \\ \ln|y_0| &\sim -\int \frac{\sin(2x)}{\sin(x)} dx \\ &\sim -\int \frac{2\sin x \cos x}{\sin(x)} dx \\ &\sim -\int 2\cos x dx \\ &\sim -2\sin x + C_5 \end{aligned}$$

Hence

$$\begin{aligned} y_0^{out}(x) &\sim Ae^{-2\sin x} \\ y_0(\pi) &= 0 \end{aligned}$$

Therefore $A = 0$ and $y_0^{out}(x) = 0$.Now that we found all solutions, we can do the matching. The matching on the left side gives

$$\begin{aligned} \lim_{\xi\to\infty} y^{in}(\xi) &= \lim_{x\to 0} y^{out}(x) \\ \lim_{\xi\to\infty} C_1 \int_0^{\xi} e^{\frac{-s^2}{2}} ds + \pi &= \lim_{x\to 0} C_5 e^{-2\sin x} \\ \lim_{\xi\to\infty} C_1 \int_0^{\xi} e^{\frac{-s^2}{2}} ds + \pi &= 0 \end{aligned} \tag{3}$$

But

$$\int_0^{\xi} e^{\frac{-s^2}{2}} ds = \sqrt{\frac{\pi}{2}} \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)$$

And $\lim_{\xi\to\infty} \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right) = 1$, hence (3) becomes

$$\begin{aligned} C_1\sqrt{\frac{\pi}{2}} + \pi &= 0 \\ C_1 &= -\pi\sqrt{\frac{2}{\pi}} \\ &= -\sqrt{2\pi} \end{aligned} \tag{4}$$

Therefore from (2)

$$y_0^{in}(\xi) \sim -\sqrt{2\pi} \int_0^{\xi} e^{\frac{-s^2}{2}} ds + \pi \tag{5}$$

Near $x = \pi$, using $\eta = \frac{\pi - x}{\varepsilon^p}$. Expansion $y^{in}(\eta) \sim y_0(\eta) + \varepsilon y_1(\eta) + O(\varepsilon^2)$ gives $p = \frac{1}{2}$. Hence $O(1)$ terms gives

$$\begin{aligned} y_0''(\eta) &\sim \eta y_0'(\eta) \\ y_0^{in}(0) &= 0 \\ y_0^{in}(\eta) &\sim D \int_0^{\eta} e^{\frac{-s^2}{2}} ds \end{aligned}$$

And matching on the right side gives

$$\lim_{\eta\to\infty} y^{in}\left(\eta\right) = \lim_{x\to\pi} y^{out}\left(x\right)$$
$$\lim_{\eta\to\infty} D\int_0^{\eta} e^{\frac{-s^2}{2}}\,ds = 0$$
$$D = 0 \tag{6}$$

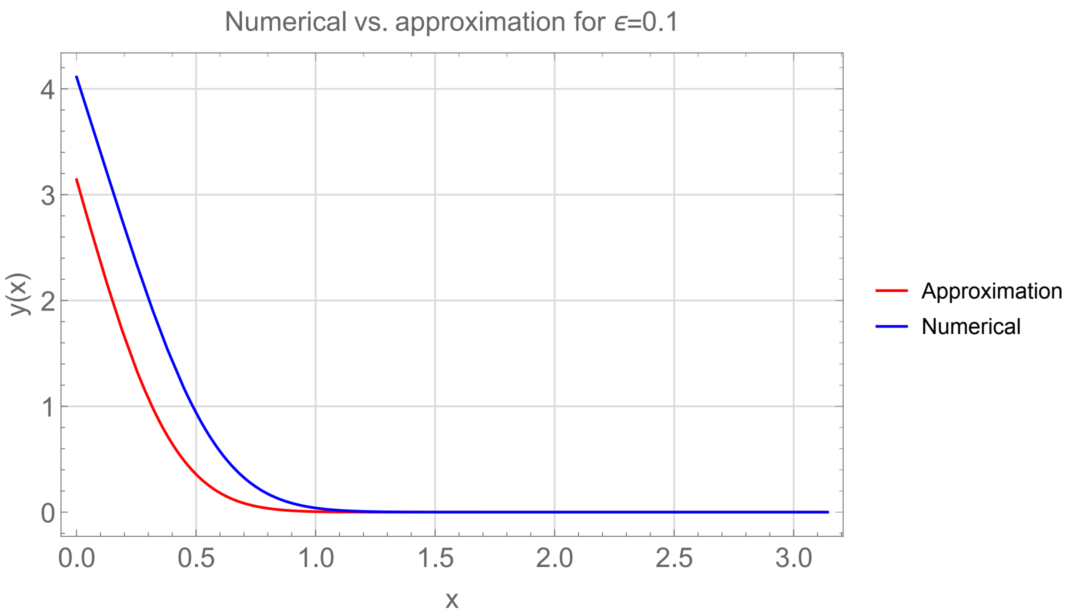
Therefore the solution is

$$\begin{aligned} y\left(x\right) &\sim y^{in}\left(\xi\right) + y^{in}\left(\eta\right) + y^{out}\left(x\right) - y^{match} \\ &\sim -\sqrt{2\pi}\int_0^{\xi} e^{\frac{-s^2}{2}}\,ds + \pi + 0 \\ &\sim -\sqrt{2\pi}\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right) + \pi \\ &\sim -\sqrt{2\pi}\sqrt{\frac{\pi}{2}}\operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) + \pi \\ &\sim \pi - \pi\operatorname{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) \end{aligned} \tag{7}$$

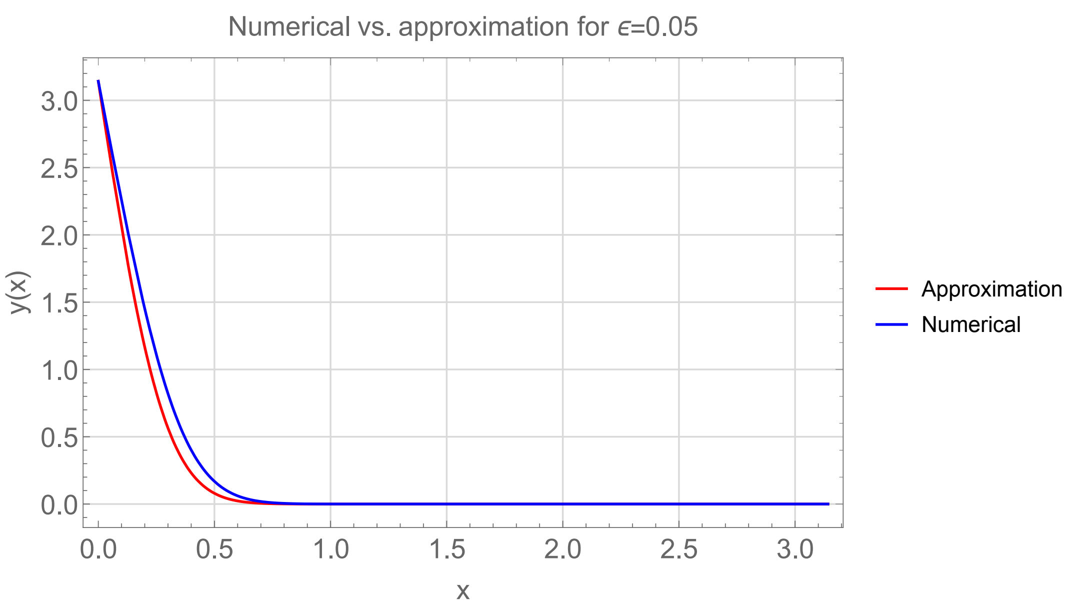
The following plot compares exact solution with (7) for $\varepsilon = 0.1, 0.05$. We see from these results, that as ε decreased, the approximation solution improved.

```
In[37]:= ClearAll[x, ϵ, y]
mySol[x_, ϵ_] := Pi - Pi Erf[x/Sqrt[2 ϵ]];
ϵ = 0.1;
ode = ϵ y''[x] + Sin[x] y'[x] + y[x] Sin[2 x] == 0;
sol = NDSolve[{ode, y[0] == π, y'[π] == 0}, y, {x, 0, π}];
Plot[{mySol[x, ϵ], Evaluate[y[x] /. sol]}, {x, 0, Pi}, Frame → True,
 FrameLabel → {{"y(x)", None}, {"x", Row[{"Numerical vs. approximation for ϵ=", ϵ}]}}, GridLines → Automatic,
 GridLinesStyle → LightGray, BaseStyle → 16, ImageSize → 500, PlotLegends → {"Approximation", "Numerical"},
 PlotStyle → {Red, Blue}, PlotRange → All]
```

Out[42]=


Numerical vs. approximation for ϵ=0.1
y(x)
x
Approximation
Numerical

Out[54]=


Numerical vs. approximation for ϵ=0.05
y(x)
x
Approximation
Numerical