3.42 a SOLUTION

$$y(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}$$

This satisfies the 2^{nd} order D.E.

[1] $xy'' + y' = y$ w/ ISP @ $x = \infty$

Let $y = e^{S(x)}$

$y' = S'e^S$

$y'' = S''e^S + (S')^2 e^S$

Plug into [1].

[2] $xS'' + x(S')^2 + S' - 1 = 0$

Assume,

[a1] $xS'' \ll x(S')^2$ as $x \to^{+} \infty$

$\Rightarrow$ $x(S')^2 + S' \sim 1$ as $x \to \infty$

or $S'(x) \sim [-1 \pm \sqrt{1+4x}]/2x$

For large x,

$S'(x) \sim \pm x^{-1/2}$ $x \to \infty$

Notice $y(x) = \sum_0^{\infty} \frac{x^n}{(n!)^2}$ is always (+) for x-large

So, only keep positive sign solution.

$S(x) \sim 2x^{1/2}$ $x \to \infty$

[3] or $S(x) = 2x^{1/2} + C(x)$ where $C(x) \ll 2x^{1/2}$ for $x \to \infty$

[a2] $C(x) \ll 2x^{1/2}$

Plug [3] into [2],

$$S' = x^{-1/2} + c'$$

$$S'' = -\tfrac{1}{2} x^{-3/2} + c''$$

$$\Rightarrow \quad -\tfrac{1}{2} x^{-1/2} + xc'' + 1 + 2x^{1/2}c' + x(c')^2 + x^{-1/2} + c' = 1$$

$$\underset{a}{\cancel{x}c''} + \underset{b}{x\cancel{(c')}^2} + (\underset{c}{2x^{1/2}} + \underset{d}{\cancel{1}})c' + \underset{e}{\tfrac{1}{2} x^{-1/2}} = 0$$

From [a2]: $c \ll 2x^{1/2}$

$$c' \ll x^{-1/2} \quad \Rightarrow \quad \underset{b}{x(c')^2} \ll \underset{c}{x^{1/2}c'} \;\;\&\;\; \underset{d}{c'} \ll \underset{e}{\tfrac{1}{2}x^{-1/2}}$$

$$c'' \ll -\tfrac{1}{2} x^{-3/2} \quad \Rightarrow \quad \underset{a}{xc''} \ll \underset{e}{-\tfrac{1}{2} x^{-1/2}}$$

So balance becomes,

$$[3] \quad 2x^{1/2}c' \sim -\tfrac{1}{2} x^{-1/2} \quad \text{or} \quad c' \sim -\tfrac{1}{4} x^{-1}$$

$$\Rightarrow \quad c(x) \sim -\tfrac{1}{4} \ln x \qquad x \to \infty$$

$$\text{or} \quad c(x) = -\tfrac{1}{4} \ln x + D(x)$$

where

$$[a3] \quad D(x) \ll -\tfrac{1}{4} \ln x$$

$$\therefore \quad y(x) = e^{S+c+D} = C\, x^{-1/4} e^{[2x^{1/2} + D(x)]}$$

Now look for $y(x) = e^{S+c+D}$ to find D.

$$y' = [S_0' + C_0' + D']\, e^{(S_0+C_0+D)}$$

$$y'' = [S_0'' + C_0'' + D'']\, e^{(S_0+C_0+D)} + (S_0' + C_0' + D')^2\, e^{(S_0+C_0+D)}$$

$$= [S_0'' + C_0'' + D'' + (S_0')^2 + (C_0')^2 + (D')^2 + 2S_0'C_0' + 2S_0'D' + 2C_0'D']\, e^{(S_0+C_0+D)}$$

Plug into [1];

$$x\cancelto{[3]}{S_0''} + xC_0'' + xD'' + x\cancelto{[2]}{(S_0')^2} + x(C_0')^2 + x(D')^2 + 2x\cancelto{[3]}{S_0'C_0'} + 2xS_0'D' + 2xC_0'D'$$

$$+ \cancelto{[2]}{S_0'} + C_0' + D' - \cancelto{[2]}{1} = 0$$

[2] Gives $\quad x(S_0')^2 + S_0' - 1 \sim 0$

[3] Gives $\quad C' \sim -\frac{1}{4}x^{-1} \quad$ or $\quad x(S_0'' + 2S_0C_0') = 0$

$$xC_0'' + xD'' + x(C_0')^2 + x(D')^2 + 2xS_0'D' + 2xC_0'D' + C_0' + D' = 0$$

$$S_0 = 2x^{1/2} \qquad C_0 = -\frac{1}{4}\ln x$$

$$S_0' = x^{-1/2} \qquad C_0' = -\frac{1}{4}x^{-1}$$

$$C_0'' = \frac{1}{4}x^{-2}$$

$$\Rightarrow \frac{1}{4}x^{-1} + xD'' + \frac{1}{16}x^{-1} + x(D')^2 + 2x\,x^{-1/2}D' + 2x\left(-\frac{1}{4}x^{-1}\right)D' + \left(-\frac{1}{4}x^{-1}\right) + D' = 0$$

$$xD'' + x(D')^2 + 2x^{1/2}D' + D' - \frac{1}{2}D' + \frac{1}{16}x^{-1} + \cancel{\frac{1}{4}x^{-1}} - \cancel{\frac{1}{4}x^{-1}} = 0$$

$$\underset{(a)}{xD''} + \underset{(b)}{x(D')^2} + (\underset{(c)}{2x^{1/2}} + \underset{(d)}{\tfrac{1}{2}})D' + \underset{(e)}{\frac{1}{16}x^{-1}} = 0$$

From $[a_8]$

$D \ll -\frac{1}{4}\ln x$

$D' \ll -\frac{1}{4}x^{-1} \longrightarrow xD'^2 \ll -\frac{1}{4}D' \rightarrow$ (b) $\ll$ (d)

$D'' \ll \frac{1}{4}x^{-2} \longrightarrow xD'' \ll x^{-1}$ or (a) $\ll$ (e)

$\frac{1}{2} \ll 2x^{1/2}$ for $x \to \infty$ or (d) $\ll$ (c)

Therefore,

{5} $$2x^{1/2} D' \sim -\frac{1}{16} x^{-1}$$

$$D' \sim -\frac{1}{32} x^{-3/2} \quad \therefore \quad \boxed{D \sim \frac{1}{16} x^{-1/2}}$$

or $$D = \frac{1}{16} x^{-1/2} + E(x) \qquad \text{where} \qquad E(x) \ll \frac{1}{16} x^{-1/2}$$

{a4} $$E(x) \ll \frac{1}{16} x^{-1/2}$$

Now let $S(x) = S_0 + C_0 + D_0 + E(x)$ $\qquad D_0 = \frac{1}{16} x^{-1/2}$

Thus, plugging into [1] gives,

$$\cancel{xS_0''}^{\{3\}} + \cancel{xC_0''}^{\{5\}} + xD_0'' + xE'' + \cancel{x(S_0')^2}^{\{2\}} + \cancel{x(C_0')^2}^{\{5\}} + x(D_0')^2 + x(E')^2$$
$$+ \cancel{2xS_0'C_0'}^{\{3\}} + \underbrace{\cancel{2xS_0'D_0'}}_{\{5\}} + 2xC_0'D_0' + 2xS_0'E' + 2xC_0'E' + 2xD_0'E' + \underbrace{\cancel{S_0'}}_{\{2\}} + \underbrace{\cancel{C_0'}}_{\{5\}} + D_0' + E' = \cancel{1}^{\{2\}}$$

Removing previous balances, including most recent balance from

{5} $\to$ $2x^{1/2}D' \sim -\frac{1}{16}x^{-1}$ or $x(C_0'' + (C_0')^2 + 2S_0'D_0') + C_0' \sim 0$

{2} $\to$ $x(S_0')^2 + S_0' - 1 \sim 0$

{3} $\to$ $c' + \frac{1}{4}x^{-1} \sim 0$

Results in,

$$xD_0'' + xE'' + x(D_0')^2 + x(E')^2 + 2xC_0'D_0' + 2xS_0'E' + 2xC_0'E' + 2xD_0'E' + D_0' + E' = 0$$

$$D_0' = -\frac{1}{32} x^{-3/2} \qquad C_0' = -\frac{1}{4} x^{-1} \qquad S_0' = x^{-1/2}$$

$$D_0'' = \frac{3}{64} x^{-5/2} \qquad C_0'' = \frac{1}{4} x^{-2}$$

$$\Rightarrow \frac{3}{64} x^{-3/2} + xE'' + \frac{1}{1024} x^{-2} + x(E')^2 + 2x\left(-\frac{1}{4}x^{-1}\right)\left(-\frac{1}{32}x^{-3/2}\right) + 2x \cdot x^{-1/2} E' + 2x\left(-\frac{1}{4}x^{-1}\right)E'$$
$$+ 2x\left(-\frac{1}{32}x^{-3/2}\right)E' + \left(-\frac{1}{32}x^{-3/2}\right) + E' = 0$$

$$\Rightarrow \quad \underset{(a)}{\cancel{x\epsilon''}} + \underset{(b)}{\cancel{x(\epsilon')^2}} + \underset{(c)}{2x^{1/2}\epsilon'} + \underset{(d)}{\cancel{\tfrac{1}{2}\epsilon'}} - \underset{(e)}{\tfrac{1}{16}x^{-1/2}\epsilon'} + \underset{(f)}{\cancel{\tfrac{1}{1024}x^{-2}}} + \underset{(g)}{\tfrac{1}{32}x^{-3/2}} = 0$$

From $[a_4]$:

$\epsilon << \frac{1}{16} x^{-1/2}$

$\epsilon' << -\frac{1}{32} x^{-3/2} \longrightarrow x(\epsilon')^2 << -\frac{1}{32} x^{-1/2}\epsilon'$ or $(b) << (e)$

$\longrightarrow \frac{1}{2}\epsilon' << \frac{1}{32} x^{-3/2}$ or $(d) << g$

$\epsilon'' << \frac{3}{64} x^{-5/2} \longrightarrow x\epsilon'' << \frac{3}{64} x^{-3/2}$ or $(a) << (g)$

$x^{-2} << x^{-3/2}$ or $(f) << (g)$

$2x^{1/2}\epsilon' >> \frac{1}{16} x^{-1/2}\epsilon'$ or $(c) >> (e)$

$$\Rightarrow \quad 2x^{1/2}\epsilon' + \frac{1}{32} x^{-3/2} \sim 0$$

or $\quad \epsilon' \sim -\frac{1}{64} x^{-2}$

Then $\boxed{\epsilon \sim \frac{1}{64} x^{-1}}$ or $\quad \epsilon = \frac{1}{64} x^{-1} + F(x)$

STOP HERE.

Finally,

$$y(x) = e^{S+C+D+\epsilon} = e^{2x^{1/2} + -\frac{1}{4}\ln x + \frac{1}{16}x^{-1/2} + \frac{1}{64}x^{-1} + F(x)}$$

or $\quad y(x) \sim c\, x^{-1/4} \exp\left[2x^{1/2} + \frac{1}{16}x^{-1/2} + \frac{1}{64}x^{-1}\right]$

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$$\frac{1}{16}x^{-\frac{3}{2}} + 2x^{\frac{1}{2}}E_0' - \frac{1}{32}x^{-\frac{3}{2}} + E' = 0$$

$$\frac{1}{32}x^{-\frac{3}{2}} + (2x^{\frac{1}{2}} + 1)E_0' = 0$$, And $1 \ll 2x^{\frac{1}{2}}$, $x \to \infty$, so

$$E_0' = \frac{-1}{64}x^{-2} \Rightarrow E_0 = \int^x E_0'(t)\,dt = \frac{1}{64}x^{-1}$$, which gives the approximation

$$\boxed{y(x) \sim c x^{-\frac{1}{4}} e^{2x^{1/2} + \frac{1}{16}x^{-1/2} + \frac{1}{64}x^{-1}}, \quad x \to \infty}$$

+8

the improvement in the numerical approximation to $y(x)$ can be seen in computing $y(x)$ for various x values, with $c = \frac{1}{2}(\pi)^{-1/2}$ for equivalence.

For $x = 10$; $y(10) = \sum_{n=0}^{\infty} \frac{10^n}{(n!)^2} \simeq 90.47595$

$y_1(10) = ce^{S_0(10) + C_0(10)} \simeq 88.53491$ (2.15% error)

$y_2(10) = ce^{S_0(10) + C_0(10) + D_0(10)} \simeq 90.30214$ (0.19% error)

$y_3(10) = ce^{S_0(10) + C_0(10) + D_0(10) + E_0(10)} \simeq 90.44335$ (0.04% error)

For $x = 10{,}000$, $y(10{,}000) = \sum_{n=0}^{\infty} \frac{(10{,}000)^n}{(n!)^2} \simeq 2.03968717 \times 10^{85}$

$y_1(10{,}000) \simeq 2.03840157 \times 10^{85}$ (6.26×10^{-2} % error)

$y_2(10{,}000) \simeq 2.03968397 \times 10^{85}$ (1.57×10^{-4} % error)

$y_3(10{,}000) \simeq 2.03968716 \times 10^{85}$ (8.20×10^{-7} % error)

+2