$\boxed{3.33\ c}$ $y'' = x^{\frac{1}{2}} y \quad x \to 0^+$

$$\left\{ S_0'' + S_1'' + \cdots \right\} + \left\{ (S_0')^2 + 2S_0'S_1' + \cdots \right\} \sim x^{\frac{1}{2}}$$

$\boxed{\text{Try 2-term balances}}$

$\boxed{1}$ $(S_0')^2 \sim x^{\frac{1}{2}} \quad x \to 0^+$; $S_0' \sim \pm x^{1/4}$;

$$S_0 \sim \pm \frac{4}{5} x^{5/4} + C_0$$

Check $(S_0')^2 >> S_0''$?

$$x^{\frac{1}{2}} >> \pm \frac{1}{4} x^{-3/4} \quad \text{NO} \quad \text{inconsistent}$$

$\boxed{2}$ $S_0'' \sim x^{\frac{1}{2}} \quad x \to 0^+$; $S_0' \sim \frac{2}{3} x^{\frac{3}{2}} + C_0$;

$$S_0 \sim \frac{2}{3} \cdot \frac{2}{5} x^{5/2} + C_0 x + C_1$$

Check $S_0'' >> (S_0')^2$?

$$x^{\frac{1}{2}} >> \frac{4}{9} x^3 + \frac{4}{3} C_0 x^{\frac{3}{2}} + C_0^2$$

NO inconsistent

$\boxed{3}$ $S_0'' \sim -(S_0')^2 \quad x \to 0^+$

Let $S_0' = z \Rightarrow z' = -z^2$; $\dfrac{dz}{z^2} = -dx$;

$-z^{-1} = -x + C \Rightarrow z = \dfrac{1}{x+D} \Rightarrow$

$S_0 \sim \ln|x+D| + E$

Check $(S_0')^2 \gg x^{\frac{1}{2}} \quad x \to 0^+$?

$\dfrac{1}{(x+D)^2} \gg x^{\frac{1}{2}} \quad x \to 0^+$

True for both $D=0$ and $D \neq 0$

So the controlling factor is

$y \sim \exp[S_0] \sim \exp[\ln|x+D| + E]$

$y \sim Ax + B$

(the 1st term in 2 linearly independent solutions)

$y_1 \sim Ax \quad y_2 \sim B$

Since the lowest order approximation has 2 coefficients A, B to be determined by boundary/initial conditions, from now on we set integration coefficients to zero.

$\boxed{\text{CASE 1}}$ $\quad y_1 \sim Ax \sim A\exp[\ln x]$, $S_0 \sim \ln x$

$$\{\cancel{S_0''} + S_1'' + \cdots\} + \{(\cancel{S_0'})^2 + 2S_0'S_1' + \cdots\} \sim x^{1/2}$$
$$x \to 0^+$$

Try $S_1'' + 2S_0'S_1' \sim x^{1/2}$, $\quad S_0' \sim \frac{1}{x}$

$$\Rightarrow \quad S_1'' + \frac{2}{x}S_1' \sim x^{1/2}$$

integrating factor $\mu(x) = \exp\left[\int \frac{2}{x}dx\right] = x^2$

$$\frac{d}{dx}\left[x^2 S_1'\right] \sim x^2 x^{1/2} \sim x^{5/2}$$

$$x^2 S_1' \sim \frac{2}{7}x^{7/2} \quad ; \quad S_1' \sim \frac{2}{7}x^{3/2}$$

$$\Rightarrow S_1 \sim \frac{2}{7}\frac{2}{5}x^{5/2} \sim \frac{4}{35}x^{5/2}$$

$$y_1 \sim A \exp\left[\ln x + \frac{4}{35} x^{5/2} + \cdots\right]$$

Check: $(S_1')^2 \ll x^{1/2} \quad x \to 0^+$?

$$x^3 \ll x^{1/2} \quad x \to 0^+ \quad \text{TRUE}$$

$\boxed{\text{CASE 2}}$ $\quad y_2 \sim B \sim \exp[C], \quad S_0 \sim C$ constant

$$\{S_0'' + S_1'' + \cdots\} + \{(S_0')^2 + 2S_0'S_1' + \cdots\} \sim x^{1/2}$$
$$x \to 0^+$$

Try $S_1'' + 2S_0'S_1' \sim x^{1/2} \Rightarrow S_1'' \sim x^{1/2}$

$$\Rightarrow S_1' \sim \frac{2}{3} x^{3/2} \Rightarrow S_1 \sim \frac{4}{15} x^{5/2}$$

with check $(S_1')^2 \ll x^{1/2}$ TRUE

$$y_2 \sim \exp\left[C + \frac{4}{15} x^{5/2} + \cdots\right]$$

$$y = y_1 + y_2 \sim Ax \exp\left[\frac{4}{35} x^{5/2} + \cdots\right] + B \exp\left[\frac{4}{15} x^{5/2} + \cdots\right]$$

$$\sim Ax \left\{1 + \frac{4}{35} x^{5/2} + \cdots\right\}$$

$$+ B \left\{1 + \frac{4}{15} x^{5/2} + \cdots\right\}$$

$y = y_1 + y_2$ ~

$$Ax \sum_{n=0}^{\infty} a_n x^{5n/2} + B \sum_{n=0}^{\infty} b_n x^{5n/2}$$

$a_0 = 1$, $a_1 = \frac{4}{35}$ $b_0 = 1$, $b_1 = \frac{4}{15}$

continue on to find a_n, $n \geq 2$ b_n, $n \geq 2$

Note: Balance $\boxed{2}$ would be consistent if we drop integration constants, but then we generate only solution y_2; use reduction of order to find y_1 ...