HW 2, NE 548, Spring 2017

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Contents

0.1 problem 3.27 (page 138)

Problem derive 3.4.28. Below is a screen shot from the book giving 3.4.28 at page 88, and the context it is used in before solving the problem

Example 5 *Local behavior of solutions near an irregular singular point of a general nth-order Schrödinger equation.* In this example we derive an extremely simple and important formula for the leading behavior of solutions to the nth-order Schrödinger equation

$$\frac{d^n y}{dx^n} = Q(x)y \tag{3.4.27}$$

near an irregular singular point at x_0.

The exponential substitution $y = e^S$ and the asymptotic approximations $d^kS/dx^k \ll (S')^k$ as $x \to x_0$ for $k = 2, 3, \ldots, n$ give the asymptotic differential equation $(S')^n \sim Q(x)$ $(x \to x_0)$. Thus, $S(x) \sim \omega \int^x [Q(t)]^{1/n}\, dt$ $(x \to x_0)$, where ω is an nth root of unity. This result determines the n possible controlling factors of $y(x)$.

The leading behavior of $y(x)$ is found in the usual way (see Prob. 3.27) to be

$$\boxed{y(x) \sim c[Q(x)]^{(1-n)/2n} \exp\left\{\omega \int^x [Q(t)]^{1/n}\, dt\right\}, \qquad x \to x_0.} \tag{3.4.28}$$

If $x_0 \neq \infty$, (3.4.28) is valid if $|(x - x_0)^n Q(x)| \to \infty$ as $x \to x_0$. If $x_0 = \infty$, then (3.4.28) is valid if $|x^n Q(x)| \to \infty$ as $x \to \infty$. This important formula forms the basis of WKB theory and will be rederived perturbatively and in much greater detail in Sec. 10.2. If $Q(x) < 0$, solutions to (3.4.27) oscillate as $x \to \infty$; the nature of asymptotic relations between oscillatory functions is discussed in Sec. 3.7.

Here are some examples of the application of (3.4.28):

(*a*) For $y'' = y/x^5$, $y(x) \sim cx^{5/4}e^{\pm 2x^{-3/2}/3}$ $(x \to 0+)$.
(*b*) For $y''' = xy$, $y(x) \sim cx^{-1/3}e^{3\omega x^{4/3}/4}$ $(x \to +\infty)$, where $\omega^3 = 1$.
(*c*) For $d^4y/dy^4 = (x^4 + \sin x)y$, $y(x) \sim cx^{-3/2}e^{\omega x^2/2}$ $(x \to +\infty)$, where $\omega = \pm 1, \pm i$.

Solution

For n^{th} order ODE, $S_0(x)$ is given by

$$S_0(x) \sim \omega \int^x Q(t)^{\frac{1}{n}}\, dt$$

And (page 497, textbook)

$$S_1(x) \sim \frac{1-n}{2n} \ln(Q(x)) + c \tag{10.2.11}$$

Therefore

$$\begin{aligned} y(x) &\sim \exp(S_0 + S_1) \\ &\sim \exp\left(\omega \int^x Q(t)^{\frac{1}{n}}\, dt + \frac{1-n}{2n} \ln(Q(x)) + c\right) \\ &\sim c\,[Q(x)]^{\frac{1-n}{2n}} \exp\left(\omega \int^x Q(t)^{\frac{1}{n}}\, dt\right) \end{aligned}$$

Note: I have tried other methods to proof this, such as a proof by induction. But was not able to after many hours trying. The above method uses a given formula which the book did not indicate how it was obtained. (see key solution)

0.2 Problem 3.33(b) (page 140)

Problem Find leading behavior as $x \to 0^+$ for $x^4y''' - 3x^2y' + 2y = 0$

Solution Let

$$
\begin{aligned}
y(x) &= e^{S(x)} \\
y' &= S'e^S \\
y'' &= S''e^S + (S')^2 e^S \\
y''' &= S'''e^S + S''S'e^S + 2S'S''e^S + (S')^3 e^S
\end{aligned}
$$

Hence the ODE becomes

$$
x^4 \left[S''' + 3S'S'' + (S')^3\right] - 3x^2 S' = -2 \tag{1}
$$

Now, we define $S(x)$ as sum of a number of leading terms, which we try to find

$$
S(x) = S_0(x) + S_1(x) + S_2(x) + \cdots
$$

Therefore (1) becomes (using only two terms for now $S = S_0 + S_1$)

$$
\begin{aligned}
\{S_0 + S_1\}''' + 3\left\{(S_0 + S_1)(S_0 + S_1)''\right\} + \left\{(S_0 + S_1)'\right\}^3 - \frac{3}{x^2}\{S_0 + S_1\}' &= -\frac{2}{x^4} \\
\left\{S_0''' + S_1'''\right\} + 3\left\{(S_0 + S_1)\left(S_0'' + S_1''\right)\right\} + \left\{S_0' + S_1'\right\}^3 - \frac{3}{x^2}\left(S_0' + S_1'\right) &= -\frac{2}{x^4} \\
\left\{S_0''' + S_1'''\right\} + 3\left\{S_0''S_0 + S_0''S_1 + S_1''S_0\right\} + \left\{\left(S_0'\right)^3 + 3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3}{x^2}\left\{S_0' + S_1'\right\} &= -\frac{2}{x^4}
\end{aligned} \tag{2}
$$

Assuming that $S_0' \ggg S_1', S_0''' \ggg S_1''', \left(S_0'\right)^3 \ggg 3\left(S_0'\right)^2 S_1'$ then equation (2) simplifies to

$$
S_0''' + 3S_0''S_0 + \left(S_0'\right)^3 - \frac{3}{x^2}S_0' \sim -\frac{2}{x^4}
$$

Assuming $\left(S_0'\right)^3 \ggg S_0''', \left(S_0'\right)^3 \ggg 3S_0''S_0, \left(S_0'\right)^3 \ggg \frac{3}{x^2}S_0'$ (which we need to verify later), then the above becomes

$$
\left(S_0'\right)^3 \sim -\frac{2}{x^4}
$$

Verification[1]

Since $S_0' \sim \left(\frac{-2}{x^4}\right)^{\frac{1}{3}} = \frac{1}{x^{\frac{4}{3}}}$ then $S_0'' \sim \frac{1}{x^{\frac{7}{3}}}$ and $S_0''' \sim \frac{1}{x^{\frac{10}{3}}}$. Now we need to verify the three assumptions made above, which we used to obtain S_0'.

$$
\begin{aligned}
\left(S_0'\right)^3 &\ggg S_0''' \\
\frac{1}{x^4} &\ggg \frac{1}{x^{\frac{10}{3}}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
\left(S_0'\right)^3 &\ggg 3S_0''S_0' \\
\frac{1}{x^4} &\ggg \left(\frac{1}{x^{\frac{7}{3}}}\right)\left(\frac{1}{x^{\frac{4}{3}}}\right) \\
\frac{1}{x^4} &\ggg \left(\frac{1}{x^{\frac{11}{3}}}\right)
\end{aligned}
$$

Yes.

$$
\begin{aligned}
\left(S_0'\right)^3 &\ggg \frac{3}{x^2}S_0' \\
\frac{1}{x^4} &\ggg \left(\frac{1}{x^2}\right)\frac{1}{x^{\frac{4}{3}}} \\
\frac{1}{x^4} &\ggg \frac{1}{x^{\frac{10}{3}}}
\end{aligned}
$$

[1]When carrying out verification, all constant multipliers and signs are automticaly simplified and removed going from one step to the next, as they do not affect the final result.

Yes. Assumed balance ise verified. Therefore

$$\left(S_0'\right)^3 \sim -\frac{2}{x^4}$$
$$S_0' \sim \omega x^{\frac{-4}{3}}$$

Where $\omega^3 = -2$. Integrating

$$\begin{aligned} S_0 &\sim \omega \int x^{\frac{-4}{3}} dx \\ &\sim \omega \int x^{-\frac{4}{3}} dx \\ &\sim -3\omega x^{\frac{-1}{3}} \end{aligned}$$

Where we ignored the constant of integration since subdominant. To find leading behavior, we go back to equation (2) and now solve for S_1.

$$\left\{S_0''' + S_1'''\right\} + 3\left\{S_0''S_0' + S_0''S_1' + S_1''S_0'\right\} + \left\{\left(S_0'\right)^3 + 3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3}{x^2}\left\{S_0' + S_1'\right\} = -\frac{2}{x^4}$$

Moving all known quantities (those which are made of S_0 and its derivatives) to the RHS and simplifying, gives

$$\left\{S_1'''\right\} + 3\left\{S_0''S_1' + S_1''S_0'\right\} + \left\{3\left(S_0'\right)^2 S_1' + 3S_0'\left(S_1'\right)^2\right\} - \frac{3S_1'}{x^2} \sim -S_0''' - 3S_0''S_0' + \frac{3}{x^2}S_0'$$

Now we assume the following (then will verify later)

$$\begin{aligned} 3\left(S_0'\right)^2 S_1' &\ggg 3S_0'\left(S_1'\right)^2 \\ 3\left(S_0'\right)^2 S_1' &\ggg S_1''' \\ 3\left(S_0'\right)^2 S_1' &\ggg S_1''S_0' \\ 3\left(S_0'\right)^2 S_1' &\ggg S_0''S_1' \\ 3\left(S_0'\right)^2 S_1' &\ggg S_1''S_0' \end{aligned}$$

Hence

$$3\left(S_0'\right)^2 S_1' - \frac{3S_1'}{x^2} \sim -S_0''' - 3S_0''S_0' + \frac{3S_0'}{x^2} \tag{3}$$

But

$$\begin{aligned} S_0' &\sim \omega x^{\frac{-4}{3}} \\ \left(S_0'\right)^2 &\sim \omega^2 x^{\frac{-8}{3}} \\ S_0'' &\sim -\frac{4}{3}\omega x^{\frac{-7}{3}} \\ S_0''' &\sim \frac{28}{9}\omega x^{\frac{-10}{3}} \end{aligned}$$

Hence (3) becomes

$$\begin{aligned} 3\left(\omega^2 x^{\frac{-8}{3}}\right) S_1' - \frac{3S_1'}{x^2} &\sim \frac{28}{9}\omega x^{\frac{-10}{3}} + 3\left(\frac{4}{3}\omega^2 x^{\frac{-7}{3}} x^{\frac{-4}{3}}\right) + \frac{3\omega x^{\frac{-4}{3}}}{x^2} \\ 3\omega^2 x^{\frac{-8}{3}} S_1' - 3x^{-2} S_1' &\sim \frac{28}{9}\omega x^{\frac{-10}{3}} + 4\omega^2 x^{\frac{-11}{3}} + 3\omega x^{\frac{-10}{3}} \end{aligned}$$

For small x, $x^{\frac{-8}{3}} S_1' \ggg x^{-2} S_1'$ and $x^{\frac{-11}{3}} \ggg x^{\frac{-10}{3}}$, then the above simplifies to

$$\begin{aligned} 3\omega^2 x^{\frac{-8}{3}} S_1' &\sim 4\omega^2 x^{\frac{-11}{3}} \\ S_1' &\sim \frac{4}{3}x^{-1} \end{aligned}$$

$$S_1 \sim \frac{4}{3}\ln x$$

Where constant of integration was dropped, since subdominant.

<u>Verification</u> Using $S_0' \backsim x^{\frac{-4}{3}}, \left(S_0'\right)^2 \backsim x^{\frac{-8}{3}}, S_0'' \backsim x^{\frac{-7}{3}}, S_1' \sim \frac{1}{x}, S_1'' \sim \frac{1}{x^2}, S_1''' \sim \frac{1}{x^3}$

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg 3S_0'\left(S_1'\right)^2 \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg x^{\frac{-4}{3}}\frac{1}{x^2} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1''' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^3} \\
\frac{1}{x^{\frac{8}{3}}} &\ggg \frac{1}{x^2}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1'' S_0' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^2}x^{\frac{-4}{3}} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_0'' S_1' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg x^{\frac{-7}{3}}\frac{1}{x} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes

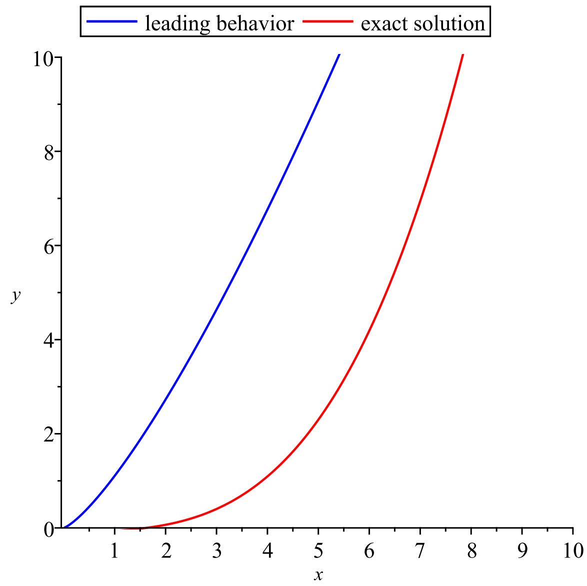
$$
\begin{aligned}
3\left(S_0'\right)^2 S_1' &\ggg S_1'' S_0' \\
x^{\frac{-8}{3}}\frac{1}{x} &\ggg \frac{1}{x^2}x^{\frac{-4}{3}} \\
x^{\frac{-8}{3}} &\ggg x^{\frac{-7}{3}}
\end{aligned}
$$

Yes. All verified. Leading behavior is

$$
\begin{aligned}
y(x) &\backsim e^{S_0(x)+S_1(x)} \\
&= \exp\left(c\omega x^{\frac{-1}{3}} + \frac{4}{3}\ln x\right) \\
&= x^{\frac{4}{3}} e^{c\omega x^{\frac{-1}{3}}}
\end{aligned}
$$

I now wanted to see how Maple solution to this problem compare with the leading behavior near $x = 0$. To obtain a solution from Maple, one have to give initial conditions a little bit removed from $x = 0$ else no solution could be generated. So using arbitrary initial conditions at $x = \frac{1}{100}$ a solution was obtained and compared to the above leading behavior. Another problem is how to select c in the above leading solution. By trial and error a constant was selected. Here is screen shot of the result. The exact solution generated by Maple is very complicated, in terms of hypergeom special functions.

```
ode:=x^4*diff(y(x),x$3)-3*x^2*diff(y(x),x)+2*y(x);
pt:=1/100:
ic:=y(pt)=500,D(y)(pt)=0,(D@@2)(y)(pt)=0:
sol:=dsolve({ode,ic},y(x)):
leading:=(x,c)->x^(4/3)*exp(c*x^(-1/3));
plot([leading(x,.1),rhs(sol)],x=pt..10,y=0..10,color=[blue,red],
legend=["leading behavior","exact solution"],legendstyle=[location=top]);
```



0.3 problem 3.33(c) (page 140)

Problem Find leading behavior as $x \to 0^+$ for $y'' = \sqrt{x}y$

Solution

Let $y(x) = e^{S(x)}$. Hence

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^{S_0} \\ y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\ &= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \end{aligned}$$

Substituting in the ODE gives

$$S_0'' + \left(S_0'\right)^2 = \sqrt{x} \tag{1}$$

Assuming $S_0'' \sim \left(S_0'\right)^2$ then (1) becomes

$$S_0'' \sim -\left(S_0'\right)^2$$

Let $S_0' = z$ then the above becomes $z' = -z^2$. Hence $\frac{dz}{dx}\frac{1}{z^2} = -1$ or $\frac{dz}{z^2} = -dx$. Integrating $-\frac{1}{z} = -x + c$ or $z = \frac{1}{x+c_1}$. Hence $S_0' = \frac{1}{x+c_1}$. Integrating again gives

$$S_0(x) \sim \ln|x + c_1| + c_2$$

Verification

$S_0' = \frac{1}{x+c_1}, \left(S_0'\right)^2 = \frac{1}{(x+c_1)^2}, S_0'' = \frac{-1}{(x+c_1)^2}$

$$\begin{aligned} S_0'' &\ggg x^{\frac{1}{2}} \\ \frac{1}{(x+c_1)^2} &\ggg x^{\frac{1}{2}} \end{aligned}$$

Yes for $x \to 0^+$.

$$\begin{aligned} \left(S_0'\right)^2 &\ggg x^{\frac{1}{2}} \\ \frac{1}{(x+c_1)^2} &\ggg x^{\frac{1}{2}} \end{aligned}$$

Yes for $x \to 0^+$. Verified. Controlling factor is

$$\begin{aligned} y(x) &\sim e^{S_0(x)} \\ &\sim e^{\ln|x+c_1|+c_2} \\ &\sim Ax + B \end{aligned}$$

0.4 problem 3.35

Problem: Obtain the full asymptotic behavior for small x of solutions to the equation

$$x^2 y'' + (2x+1) y' - x^2 \left(e^{\frac{2}{x}} + 1\right) y = 0$$

Solution

Let $y(x) = e^{S(x)}$. Hence

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^{S_0} \\ y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\ &= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \end{aligned}$$

Substituting in the ODE gives

$$\begin{aligned} x^2 \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} + (2x+1) S_0' e^{S_0} - x^2 \left(e^{\frac{2}{x}} + 1\right) e^{S_0(x)} &= 0 \\ x^2 \left(S_0'' + \left(S_0'\right)^2\right) + (2x+1) S_0' - x^2 \left(e^{\frac{2}{x}} + 1\right) &= 0 \\ x^2 \left(S_0'' + \left(S_0'\right)^2\right) + (2x+1) S_0' &= x^2 \left(e^{\frac{2}{x}} + 1\right) \\ S_0'' + \left(S_0'\right)^2 + \frac{(2x+1)}{x^2} S_0' &= e^{\frac{2}{x}} + 1 \end{aligned}$$

Assuming balance

$$\begin{aligned} \left(S_0'\right)^2 &\sim \left(e^{\frac{2}{x}} + 1\right) \\ \left(S_0'\right)^2 &\sim e^{\frac{2}{x}} \\ S_0' &\sim \pm e^{\frac{1}{x}} \end{aligned}$$

Where 1 was dropped since subdominant to $e^{\frac{1}{x}}$ for small x.

Verification Since $\left(S_0'\right)^2 \sim e^{\frac{2}{x}}$ then $S_0' \sim e^{\frac{1}{x}}$ and $S_0'' \sim -\frac{1}{x^2} e^{\frac{1}{x}}$, hence

$$\begin{aligned} \left(S_0'\right)^2 &\ggg S_0'' \\ e^{\frac{2}{x}} &\ggg \frac{1}{x^2} e^{\frac{1}{x}} \\ e^{\frac{1}{x}} &\ggg \frac{1}{x^2} \end{aligned}$$

Yes, As $x \to 0^+$

$$\begin{aligned} \left(S_0'\right)^2 &\ggg \frac{(2x+1) S_0'}{x^2} \\ e^{\frac{2}{x}} &\ggg \frac{(2x+1)}{x^2} e^{\frac{1}{x}} \\ e^{\frac{1}{x}} &\ggg \frac{(2x+1)}{x^2} \\ e^{\frac{1}{x}} &\ggg \frac{2}{x} + \frac{1}{x^2} \end{aligned}$$

Yes as $x \to 0^+$. Verified. Hence both assumptions used were verified OK. Hence

$$S_0' \sim \pm e^{\frac{2}{x}}$$
$$S_0 \sim \pm \int e^{\frac{1}{x}} dx$$

Since the integral do not have closed form, we will do asymptotic expansion on the integral. Rewriting $\int e^{\frac{1}{x}} dx$ as $\int \frac{e^{\frac{1}{x}}}{(-x^2)} \left(-x^2\right) dx$. Using $\int u dv = uv - \int v du$, where $u = -x^2, dv = \frac{-e^{\frac{1}{x}}}{x^2}$, gives $du = -2x$ and $v = e^{\frac{1}{x}}$, hence

$$\begin{aligned}\int e^{\frac{1}{x}} dx &= uv - \int v du \\ &= -x^2 e^{\frac{1}{x}} - \int -2x e^{\frac{1}{x}} dx \\ &= -x^2 e^{\frac{1}{x}} + 2 \int x e^{\frac{1}{x}} dx \qquad (1)\end{aligned}$$

Now we apply integration by parts on $\int x e^{\frac{1}{x}} dx = \int x \frac{e^{\frac{1}{x}}}{-x^3} \left(-x^3\right) du = \int \frac{e^{\frac{1}{x}}}{-x^2} \left(-x^3\right) du$, where $u = -x^3, dv = \frac{e^{\frac{1}{x}}}{-x^2}$, hence $du = -3x^2, v = e^{\frac{1}{x}}$, hence we have

$$\begin{aligned}\int x e^{\frac{1}{x}} dx &= uv - \int v du \\ &= -x^3 e^{\frac{1}{x}} + \int 3x^2 e^{\frac{1}{x}} dx\end{aligned}$$

Substituting this into (1) gives

$$\begin{aligned}\int e^{\frac{1}{x}} dx &= -x^2 e^{\frac{1}{x}} + 2\left(-x^3 e^{\frac{1}{x}} + \int 3x^2 e^{\frac{1}{x}} dx\right) \\ &= -x^2 e^{\frac{1}{x}} - 2x^3 e^{\frac{1}{x}} + 6 \int 3x^2 e^{\frac{1}{x}} dx\end{aligned}$$

And so on. The series will become

$$\begin{aligned}\int e^{\frac{1}{x}} dx &= -x^2 e^{\frac{1}{x}} - 2x^3 e^{\frac{1}{x}} + 6x^4 e^{\frac{1}{x}} + 24x^5 e^{\frac{1}{x}} + \cdots + n! x^{n+1} e^{\frac{1}{x}} + \cdots \\ &= -e^{\frac{1}{x}} \left(x^2 + 2x^3 + 6x^4 + \cdots + n! x^{n+1} + \cdots\right)\end{aligned}$$

Now as $x \to 0^+$, we can decide how many terms to keep in the RHS, If we keep one term, then we can say

$$\begin{aligned}S_0 &\sim \pm \int e^{\frac{1}{x}} dx \\ &\sim \pm x^2 e^{\frac{1}{x}}\end{aligned}$$

For two terms

$$S_0 \sim \pm \int e^{\frac{1}{x}} dx \sim \pm e^{\frac{1}{x}} \left(x^2 + 2x^3\right)$$

And so on. Let us use one term for now for the rest of the solution.

$$S_0 \sim \pm x^2 e^{\frac{1}{x}}$$

To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x) + S_1(x)}$ and hence now

$$\begin{aligned}y'(x) &= \left(S_0(x) + S_1(x)\right)' e^{S_0 + S_1} \\ y''(x) &= \left(\left(S_0 + S_1\right)'\right)^2 e^{S_0 + S_1} + \left(S_0 + S_1\right)'' e^{S_0 + S_1}\end{aligned}$$

Substituting into the given ODE gives

$$x^2\left[\left((S_0+S_1)'\right)^2+(S_0+S_1)''\right]+(2x+1)\left(S_0(x)+S_1(x)\right)'-x^2\left(e^{\frac{2}{x}}+1\right)=0$$
$$x^2\left[\left(S_0'+S_1'\right)^2+(S_0+S_1)''\right]+(2x+1)\left(S_0'(x)+S_1'(x)\right)-x^2\left(e^{\frac{2}{x}}+1\right)=0$$
$$x^2\left[\left(S_0'\right)^2+\left(S_1'\right)^2+2S_0'S_1'+\left(S_0''+S_1''\right)\right]+(2x+1)S_0'(x)+(2x+1)S_1'(x)-x^2\left(e^{\frac{2}{x}}+1\right)=0$$
$$x^2\left(S_0'\right)^2+x^2\left(S_1'\right)^2+2x^2S_0'S_1'+x^2S_0''+x^2S_1''+(2x+1)S_0'(x)+(2x+1)S_1'(x)=x^2\left(e^{\frac{2}{x}}+1\right)$$

But $x^2\left(S_0'\right)^2\sim x^2\left(e^{\frac{2}{x}}+1\right)$ since we found that $S_0'\sim e^{\frac{1}{x}}$. Hence the above simplifies to

$$x^2\left(S_1'\right)^2+2x^2S_0'S_1'+x^2S_0''+x^2S_1''+(2x+1)S_0'(x)+(2x+1)S_1'(x)=0$$
$$\left(S_1'\right)^2+2S_0'S_1'+S_0''+S_1''+\frac{(2x+1)}{x^2}S_0'(x)+\frac{(2x+1)}{x^2}S_1'(x)=0 \tag{2}$$

Now looking at $S_0''+\frac{(2x+1)}{x^2}S_0'(x)$ terms in the above. We can simplify this since we know $S_0'=e^{\frac{1}{x}}, S_0''=-\frac{1}{x^2}e^{\frac{1}{x}}$ This terms becomes

$$-\frac{1}{x^2}e^{\frac{1}{x}}+\frac{(2x+1)}{x^2}e^{\frac{1}{x}}=\frac{-e^{\frac{1}{x}}+2xe^{\frac{1}{x}}+e^{\frac{1}{x}}}{x^2}=\frac{2xe^{\frac{1}{x}}}{x^2}=\frac{2e^{\frac{1}{x}}}{x}$$

Therefore (2) becomes

$$\left(S_1'\right)^2+2S_0'S_1'+S_1''+\frac{(2x+1)}{x^2}S_1'(x)\sim\frac{-2e^{\frac{1}{x}}}{x}$$
$$\frac{\left(S_1'\right)^2}{S_0'}+2S_1'+\frac{S_1''}{S_0'}+\frac{(2x+1)}{x^2S_0'}S_1'(x)\sim\frac{-2e^{\frac{1}{x}}}{xS_0'}$$
$$\frac{\left(S_1'\right)^2}{e^{\frac{1}{x}}}+2S_1'+\frac{S_1''}{e^{\frac{1}{x}}}+\frac{(2x+1)}{x^2e^{\frac{1}{x}}}S_1'(x)\sim\frac{-2}{x}$$

Assuming the balance is

$$S_1'\sim\frac{-1}{x}$$

Hence

$$S_1(x)\sim-\ln(x)+c$$

Since c subdominant as $x\to0^+$ then

$$\boxed{S_1(x)\sim-\ln(x)}$$

<u>Verification</u>

$$S_1'\ggg\frac{\left(S_1'\right)^2}{e^{\frac{1}{x}}}$$
$$\frac{1}{x}\ggg\frac{1}{x^2}\frac{1}{e^{\frac{1}{x}}}$$
$$e^{\frac{1}{x}}\ggg\frac{1}{x}$$

<u>Yes</u>, for $x\to0^+$

$$S_1'\ggg\frac{S_1''}{e^{\frac{1}{x}}}$$
$$\frac{1}{x}\ggg\frac{1}{x^2}\frac{1}{e^{\frac{1}{x}}}$$

<u>Yes</u>.

$$S_1' \ggg \frac{(2x+1)}{x^2 e^{\frac{1}{x}}} S_1'(x)$$
$$\frac{1}{x} \ggg \frac{(2x+1)}{x^2 e^{\frac{1}{x}}} \frac{1}{x}$$
$$\frac{1}{x} \ggg \frac{1}{x^3 e^{\frac{1}{x}}}$$

Yes. All assumptions verified. Hence leading behavior is

$$\begin{aligned} y(x) &\sim \exp\left(S_0(x) + S_1(x)\right) \\ &\sim \exp\left(\pm x^2 e^{\frac{1}{x}} - \ln(x)\right) \\ &\sim \frac{1}{x}\left(\exp\left(x^2 e^{\frac{1}{x}}\right) + \exp\left(-x^2 e^{\frac{1}{x}}\right)\right) \end{aligned}$$

For small x, then we ignore $\exp\left(-x^2 e^{\frac{1}{x}}\right)$ since much smaller than $\exp\left(x^2 e^{\frac{1}{x}}\right)$. Therefore

$$y(x) \sim \frac{1}{x}\exp\left(x^2 e^{\frac{1}{x}}\right)$$

0.5 problem 3.39(h)

problem Find leading asymptotic behavior as $x \to \infty$ for $y'' = e^{-\frac{3}{x}} y$

solution Let $y(x) = e^{S(x)}$. Hence

$$\begin{aligned} y(x) &= e^{S_0(x)} \\ y'(x) &= S_0' e^{S_0} \\ y'' &= S_0'' e^{S_0} + \left(S_0'\right)^2 e^{S_0} \\ &= \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} \end{aligned}$$

Substituting in the ODE gives

$$\begin{aligned} \left(S_0'' + \left(S_0'\right)^2\right) e^{S_0} &= e^{-\frac{3}{x}} e^{S_0} \\ S_0'' + \left(S_0'\right)^2 &= e^{-\frac{3}{x}} \end{aligned}$$

Assuming $\left(S_0'\right)^2 \ggg S_0''$ the above becomes

$$\begin{aligned} \left(S_0'\right)^2 &\sim e^{-\frac{3}{x}} \\ S_0' &\sim \pm e^{-\frac{3}{2x}} \end{aligned}$$

Hence

$$S_0 \sim \pm \int e^{-\frac{3}{2x}} dx$$

Integration by parts. Since $\frac{d}{dx} e^{\frac{-3}{2x}} = \frac{3}{2x^2} e^{-\frac{3}{2x}}$, then we rewrite the integral above as

$$\int e^{-\frac{3}{2x}} dx = \int \frac{3}{2x^2} e^{-\frac{3}{2x}} \left(\frac{2x^2}{3}\right) dx$$

And now apply integration by parts. Let $dv = \frac{3}{2x^2} e^{-\frac{3}{2x}} \to v = e^{-\frac{3}{2x}}, u = \frac{2x^2}{3} \to du = \frac{4}{3}x$, hence

$$\begin{aligned} \int e^{-\frac{3}{2x}} dx &= [uv] - \int v du \\ &= \frac{2x^2}{3} e^{-\frac{3}{2x}} - \int \frac{4}{3} x e^{-\frac{3}{2x}} dx \end{aligned}$$

Ignoring higher terms, then we use

$$S_0 \sim \pm \frac{2x^2}{3} e^{-\frac{3}{2x}}$$

<u>Verification</u>

$$\left(S_0'\right)^2 \ggg S_0''$$
$$\left(e^{-\frac{3}{2x}}\right)^2 \ggg \frac{3}{2x^2}e^{-\frac{3}{2x}}$$
$$e^{-\frac{3}{x}} \ggg \frac{3}{2x^2}e^{-\frac{3}{2x}}$$

<u>Yes</u>, as $x \to \infty$. To find leading behavior, let

$$S(x) = S_0(x) + S_1(x)$$

Then $y(x) = e^{S_0(x)+S_1(x)}$ and hence now

$$y'(x) = (S_0(x) + S_1(x))' e^{S_0+S_1}$$
$$y''(x) = \left((S_0 + S_1)'\right)^2 e^{S_0+S_1} + (S_0 + S_1)'' e^{S_0+S_1}$$

Using the above, the ODE $y'' = e^{-\frac{3}{x}}y$ now becomes

$$\left((S_0 + S_1)'\right)^2 + (S_0 + S_1)'' = e^{-\frac{3}{x}}$$
$$\left(S_0' + S_1'\right)^2 + S_0'' + S_1'' = e^{-\frac{3}{x}}$$
$$\left(S_0'\right)^2 + \left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = e^{-\frac{3}{x}}$$

But $\left(S_0'\right)^2 \sim e^{-\frac{3}{x}}$ hence the above simplifies to

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' + S_1'' = 0$$

<u>Assuming</u> $\left(2S_0'S_1'\right) \ggg S_1''$ the above becomes

$$\left(S_1'\right)^2 + 2S_0'S_1' + S_0'' = 0$$

<u>Assuming</u> $2S_0'S_1' \ggg \left(S_1'\right)^2$

$$2S_0'S_1' + S_0'' = 0$$
$$S_1' \sim -\frac{S_0''}{2S_0'}$$
$$S_1 \sim -\frac{1}{2}\ln\left(S_0'\right)$$

But $S_0' \sim e^{-\frac{3}{2x}}$, hence the above becomes

$$S_1 \sim -\frac{1}{2}\ln\left(e^{-\frac{3}{2x}}\right) + c$$
$$\sim \frac{3}{4x} + c$$

<u>Verification</u>

$$\left(2S_0'S_1'\right) \ggg S_1''$$
$$\left(2e^{-\frac{3}{2x}}\frac{-3}{4x^2}\right) \ggg \frac{3}{2x^3}$$
$$\frac{3}{2}\frac{e^{-\frac{3}{2x}}}{x^2} \ggg \frac{3}{2x^3}$$

For large x the above simplifies to

$$\frac{1}{x^2} \ggg \frac{1}{x^3}$$

<u>Yes</u>.

$$\left(2S_0'S_1'\right) \ggg \left(S_1'\right)^2$$
$$\left(2e^{-\frac{3}{2x}}\frac{-3}{4x^2}\right) \ggg \left(\frac{-3}{4x^2}\right)^2$$
$$\frac{3}{2}\frac{e^{-\frac{3}{2x}}}{x^2} \ggg \frac{9}{16x^4}$$

For large x the above simplifies to

$$\frac{1}{x^2} \ggg \frac{1}{x^4}$$

Yes. All verified. Therefore, the leading behavior is

$$\begin{aligned} y(x) &\sim \exp\left(S_0(x) + S_1(x)\right) \\ &\sim \exp\left(\pm\frac{2x^2}{3}e^{-\frac{3}{2x}} - \frac{1}{2}\ln\left(e^{-\frac{3}{2x}}\right) + c\right) \\ &\sim ce^{\frac{3}{4x}}\exp\left(\pm\frac{2x^2}{3}e^{-\frac{3}{2x}}\right) \end{aligned} \tag{1}$$

Check if we can use 3.4.28 to verify:

$$\lim_{x\to\infty}|x^n Q(x)| = \lim_{x\to\infty}\left|x^2 e^{-\frac{3}{x}}\right| \to \infty$$

We can use it. Lets verify using 3.4.28

$$y(x) \sim c\left[Q(x)\right]^{\frac{1-n}{2n}}\exp\left(\omega\int^x Q[t]^{\frac{1}{n}}\,dt\right)$$

Where $\omega^2 = 1$. For $n = 2, Q(x) = e^{-\frac{3}{x}}$, the above gives

$$\begin{aligned} y(x) &\sim c\left[e^{-\frac{3}{x}}\right]^{\frac{1-2}{4}}\exp\left(\omega\int^x\left[e^{-\frac{3}{t}}\right]^{\frac{1}{2}}dt\right) \\ &\sim c\left[e^{-\frac{3}{x}}\right]^{\frac{-1}{4}}\exp\left(\omega\int^x e^{-\frac{3}{2t}}dt\right) \\ &\sim ce^{\frac{3}{4x}}\exp\left(\omega\int^x e^{-\frac{3}{2t}}dt\right) \end{aligned} \tag{2}$$

We see that (1,2) are the same. Verified OK. Notice that in (1), we use the approximation for the $e^{-\frac{3}{2x}}dx \approx \frac{2x^2}{3}e^{-\frac{3}{2x}}$ we found earlier. This was done, since there is no closed form solution for the integral.

QED.

0.6 problem 3.42(a)

Problem: Extend investigation of example 1 of section 3.5 (a) Obtain the next few corrections to the leading behavior (3.5.5) then see how including these terms improves the numerical approximation of $y(x)$ in 3.5.1.

Solution Example 1 at page 90 is $xy'' + y' = y$. The leading behavior is given by 3.5.5 as $(x \to \infty)$

$$y(x) \sim cx^{\frac{-1}{4}}e^{2x^{\frac{1}{2}}} \tag{3.5.5}$$

Where the book gives $c = \frac{1}{2}\pi^{\frac{-1}{2}}$ on page 91. And 3.5.1 is

$$y(x) = \sum_{n=0}^{\infty}\frac{x^n}{(n!)^2} \tag{3.5.1}$$

To see the improvement, the book method is followed. This is described at end of page 91. This is done by plotting the leading behavior as ratio to $y(x)$ as given in 3.5.1. Hence for the above leading behavior, we need to plot

$$\frac{\frac{1}{2}\pi^{\frac{-1}{2}}x^{\frac{-1}{4}}e^{2x^{\frac{1}{2}}}}{y(x)}$$

We are given $S_0(x), S_1(x)$ in the problem. They are

$$\begin{aligned} S_0(x) &= 2x^{\frac{1}{2}} \\ S_1(x) &= -\frac{1}{4}\ln x + c \end{aligned}$$

Hence

$$\begin{aligned} S_0'(x) &= x^{\frac{-1}{2}} \\ S_0'' &= \frac{-1}{2}x^{-\frac{3}{2}} \\ S_1'(x) &= -\frac{1}{4}\frac{1}{x} \\ S_1''(x) &= \frac{1}{4x^2} \end{aligned} \tag{1}$$

We need to find $S_2(x), S_3(x), \cdots$ to see that this will improve the solution $y(x) \sim \exp(S_0 + S_1 + S_2 + \cdots)$ as $x \to x_0$ compared to just using leading behavior $y(x) \sim \exp(S_0 + S_1)$. So now we need to find $S_2(x)$

Let $y(x) = e^S$, then the ODE becomes

$$x\left(S'' + (S')^2\right) + S' = 1$$

Replacing S by $S_0(x) + S_1(x) + S_2(x)$ in the above gives

$$\begin{aligned} (S_0 + S_1 + S_2)'' + \left[(S_0 + S_1 + S_2)'\right]^2 + \frac{1}{x}(S_0 + S_1 + S_2)' &\sim \frac{1}{x} \\ \left\{S_0'' + S_1'' + S_2''\right\} + \left[\left(S_0' + S_1' + S_2'\right)\right]^2 + \frac{1}{x}\left(S_0' + S_1' + S_2'\right) &\sim \frac{1}{x} \\ \left\{S_0'' + S_1'' + S_2''\right\} + \left\{\left[S_0'\right]^2 + 2S_0'S_1' + 2S_0'S_2' + \left[S_1'\right]^2 + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_0' + S_1' + S_2'\right\} &\sim \frac{1}{x} \end{aligned}$$

Moving all known quantities to the RHS, these are $S_0'', S_1'', \left[S_0'\right]^2, 2S_0'S_1', S_0', S_1', \left[S_1'\right]^2$ then the above reduces to

$$\left\{S_2''\right\} + \left\{+2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim \frac{1}{x} - S_0'' - S_1'' - \left[S_0'\right]^2 - 2S_0'S_1' - \frac{1}{x}S_0' - \frac{1}{x}S_1' - \left[S_1'\right]^2$$

Replacing known terms, by using (1) into the above gives

$$\begin{aligned} &\left\{S_2''\right\} + \left\{2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim \\ &\qquad \frac{1}{x} + \frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{4x^2} - \left[x^{\frac{-1}{2}}\right]^2 - 2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{4}\frac{1}{x}\right) - \frac{1}{x}\left(x^{\frac{-1}{2}}\right) - \frac{1}{x}\left(-\frac{1}{4}\frac{1}{x}\right) - \left(-\frac{1}{4}\frac{1}{x}\right)^2 \end{aligned}$$

Simplifying gives

$$\left\{S_2''\right\} + \left\{2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim \frac{1}{x} + \frac{1}{2}x^{-\frac{3}{2}} - \frac{1}{4x^2} - x^{-1} + \frac{1}{2}x^{\frac{-3}{2}} - x^{\frac{-3}{2}} + \frac{1}{4}\frac{1}{x^2} - \frac{1}{16}\frac{1}{x^2}$$

Hence

$$\left\{S_2''\right\} + \left\{2S_0'S_2' + 2S_1'S_2' + \left[S_2'\right]^2\right\} + \frac{1}{x}\left\{S_2'\right\} \sim -\frac{1}{16}\frac{1}{x^2}$$

Lets assume now that

$$2S_0'S_2' \sim -\frac{1}{16}\frac{1}{x^2} \tag{2}$$

Therefore

$$\begin{aligned} S_2' &\sim -\frac{1}{32}\frac{1}{S_0'x^2} \\ &\sim -\frac{1}{32}\frac{1}{\left(x^{\frac{-1}{2}}\right)x^2} \\ &\sim -\frac{1}{32}x^{\frac{-3}{2}} \end{aligned}$$

We can now verify this before solving the ODE. We need to check that (as $x \to \infty$)

$$\begin{aligned} 2S_0'S_2' &\ggg S_2'' \\ 2S_0'S_2' &\ggg 2S_1'S_2' \\ 2S_0'S_2' &\ggg \left[S_2'\right]^2 \\ 2S_0'S_2' &\ggg \frac{1}{x}S_2' \end{aligned}$$

Where $S_2'' \sim x^{\frac{-5}{2}}$, Hence

$$
\begin{aligned}
2S_0'S_2' &\ggg S_2'' \\
x^{\frac{-1}{2}}\left(x^{\frac{-3}{2}}\right) &\ggg x^{\frac{-5}{2}} \\
x^{-2} &\ggg x^{\frac{-5}{2}}
\end{aligned}
$$

Yes.

$$
\begin{aligned}
2S_0'S_2' &\ggg 2S_1'S_2' \\
x^{-2} &\ggg \left(\frac{1}{x}\right)\left(x^{\frac{-3}{2}}\right) \\
x^{-2} &\ggg x^{\frac{-5}{2}}
\end{aligned}
$$

Yes

$$
\begin{aligned}
2S_0'S_2' &\ggg \left[S_2'\right]^2 \\
x^{-2} &\ggg \left(x^{\frac{-3}{2}}\right)^2 \\
x^{-2} &\ggg x^{-3}
\end{aligned}
$$

Yes

$$
\begin{aligned}
2S_0'S_2' &\ggg \frac{1}{x}S_2' \\
x^{-2} &\ggg \frac{1}{x}x^{\frac{-3}{2}} \\
x^{-2} &\ggg x^{\frac{-5}{2}}
\end{aligned}
$$

Yes. All assumptions are verified. Therefore we can g ahead and solve for S_2 using (2)

$$
\begin{aligned}
2S_0'S_2' &\sim -\frac{1}{16}\frac{1}{x^2} \\
S_2' &\sim -\frac{1}{32}\frac{1}{x^2}\frac{1}{S_0'} \\
&\sim -\frac{1}{32}\frac{1}{x^2}\frac{1}{x^{\frac{-1}{2}}} \\
&\sim -\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}
\end{aligned}
$$

Hence

$$
\boxed{S_2 \sim \frac{1}{16}\frac{1}{\sqrt{x}}}
$$

The leading behavior now is

$$
\begin{aligned}
y(x) &\sim \exp\left(S_0 + S_1 + S_2\right) \\
&\sim \exp\left(2x^{\frac{1}{2}} - \frac{1}{4}\ln x + c + \frac{1}{16}\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Now we will find S_3. From

$$
x\left(S'' + (S')^2\right) + S' = 1
$$

Replacing S by $S_0 + S_1 + S_2 + S_3$ in the above gives

$$
\begin{aligned}
(S_0 + S_1 + S_2 + S_3)'' + \left[(S_0 + S_1 + S_2 + S_3)'\right]^2 + \frac{1}{x}(S_0 + S_1 + S_2 + S_3)' &\sim \frac{1}{x} \\
\left\{S_0'' + S_1'' + S_2'' + S_3''\right\} + \left[\left(S_0' + S_1' + S_2' + S_3'\right)\right]^2 + \frac{1}{x}\left(S_0' + S_1' + S_2' + S_3'\right) &\sim \frac{1}{x}
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\{S_0'' + S_1'' + S_2'' + S_3''\right\} + \\
&\quad\left\{\left[S_0'\right]^2 + 2S_0'S_1' + 2S_0'S_2' + 2S_1'S_2' + \left[S_1'\right]^2 + \left[S_2'\right]^2 + 2S_0'S_3' + 2S_1'S_3' + 2S_2'S_3' + \left[S_3'\right]^2\right\} \\
&\quad\quad + \frac{1}{x}\left\{S_0' + S_1' + S_2' + S_3'\right\} \sim \frac{1}{x}
\end{aligned}
$$

Moving all known quantities to the RHS gives

$$\begin{aligned}&\left\{S_3''\right\}+\left\{2S_0'S_3'+2S_1'S_3'+2S_2'S_3'+\left[S_3'\right]^2\right\}+\frac{1}{x}\left\{S_3'\right\}\\&\sim\frac{1}{x}-S_0''-S_1''-S_2''-\left[S_0'\right]^2-2S_0'S_1'-2S_0'S_2'-2S_1'S_2'-\left[S_1'\right]^2-\left[S_2'\right]^2-\frac{1}{x}S_0'-\frac{1}{x}S_1'-\frac{1}{x}S_2' \qquad (3)\end{aligned}$$

Now we will simplify the RHS, since it is all known. Using

$$\begin{aligned}S_0'(x)&=x^{\frac{-1}{2}}\\ \left[S_0'\right]^2&=x^{-1}\\ S_0''&=\frac{-1}{2}x^{-\frac{3}{2}}\\ S_1'(x)&=-\frac{1}{4}\frac{1}{x}\\ \left[S_1'(x)\right]^2&=\frac{1}{16}\frac{1}{x^2}\\ S_1''(x)&=\frac{1}{4x^2}\\ S_2'(x)&=-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\\ \left[S_2'(x)\right]^2&=\frac{1}{1024x^3}\\ S_2''(x)&=\frac{3}{64}\frac{1}{x^{\frac{5}{2}}}\\ 2S_0'S_1'&=2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{4}\frac{1}{x}\right)=-\frac{1}{2}\frac{1}{x^{\frac{3}{2}}}\\ 2S_0'S_2'&=2\left(x^{\frac{-1}{2}}\right)\left(-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\right)=-\frac{1}{16x^2}\\ 2S_1'S_2'&=2\left(-\frac{1}{4}\frac{1}{x}\right)\left(-\frac{1}{32}\frac{1}{x^{\frac{3}{2}}}\right)=\frac{1}{64x^{\frac{5}{2}}}\end{aligned}$$

Hence (3) becomes

$$\begin{aligned}&\left\{S_3''\right\}+\left\{2S_0'S_3'+2S_1'S_3'+2S_2'S_3'+\left[S_3'\right]^2\right\}+\frac{1}{x}\left\{S_3'\right\}\sim\\&\frac{1}{x}+\frac{1}{2x^{\frac{3}{2}}}-\frac{1}{4x^2}-\frac{3}{64}\frac{1}{x^{\frac{5}{2}}}-\frac{1}{x}+\frac{1}{2}\frac{1}{x^{\frac{3}{2}}}+\frac{1}{16x^2}-\frac{1}{64x^{\frac{5}{2}}}-\frac{1}{16}\frac{1}{x^2}-\frac{1}{1024x^3}-\frac{1}{x^{\frac{3}{2}}}+\frac{1}{4}\frac{1}{x^2}+\frac{1}{32}\frac{1}{x^{\frac{5}{2}}}\end{aligned}$$

Simplifying gives

$$\left\{S_3''\right\}+\left\{2S_0'S_3'+2S_1'S_3'+2S_2'S_3'+\left[S_3'\right]^2\right\}+\frac{1}{x}\left\{S_3'\right\}\sim-\left(\frac{32}{1024x^{\frac{5}{2}}}+\frac{1}{1024x^3}\right)$$

Let us now assume that

$$\begin{aligned}S_0'S_3'&\ggg S_1'S_3'\\ S_0'S_3'&\ggg S_2'S_3'\\ S_0'S_3'&\ggg\left[S_3'\right]^2\\ S_0'S_3'&\ggg\frac{1}{x}\left\{S_3'\right\}\\ S_0'S_3'&\ggg S_3''\end{aligned}$$

Therefore, we end up with the balance

$$\begin{aligned}
2S_0'S_3' &\sim -\left(\frac{32}{1024x^{\frac{5}{2}}}+\frac{1}{1024x^3}\right)\\
S_3' &\sim -\left(\frac{32}{1024x^{\frac{5}{2}}S_0'}+\frac{1}{1024x^3S_0'}\right)\\
&\sim -\left(\frac{32}{1024x^{\frac{5}{2}}\left(x^{\frac{-1}{2}}\right)}+\frac{1}{1024x^3\left(x^{\frac{-1}{2}}\right)}\right)\\
&\sim -\left(\frac{1}{32x^2}+\frac{1}{1024x^{\frac{5}{2}}}\right)
\end{aligned}$$

Hence

$$\boxed{S_3 \sim \frac{1}{1024}\left(\frac{2}{32x^2}+\frac{32}{x}\right)}$$

Where constant of integration was ignored. Let us now verify the assumptions made

$$\begin{aligned}
S_0'S_3' &\ggg S_1'S_3'\\
S_0' &\ggg S_1'
\end{aligned}$$

Yes.

$$\begin{aligned}
S_0'S_3' &\ggg S_2'S_3'\\
S_0' &\ggg S_2'
\end{aligned}$$

Yes

$$\begin{aligned}
S_0'S_3' &\ggg \left[S_3'\right]^2\\
S_0' &\ggg S_3'
\end{aligned}$$

Yes.

$$\begin{aligned}
S_0'S_3' &\ggg \frac{1}{x}\left\{S_3'\right\}\\
S_0' &\ggg \frac{1}{x}\\
x^{\frac{-1}{2}} &\ggg \frac{1}{x}
\end{aligned}$$

Yes, as $x \to \infty$, and finally

$$\begin{aligned}
S_0'S_3' &\ggg S_3''\\
x^{\frac{-1}{2}}\left(\frac{1}{32x^2}+\frac{1}{1024x^{\frac{5}{2}}}\right) &\ggg \left(\frac{5}{2048x^{\frac{7}{2}}}+\frac{1}{16x^3}\right)\\
\frac{\left(32\sqrt{x}+1\right)}{1024x^3} &\ggg \frac{128\sqrt{x}+5}{2048x^{\frac{7}{2}}}
\end{aligned}$$

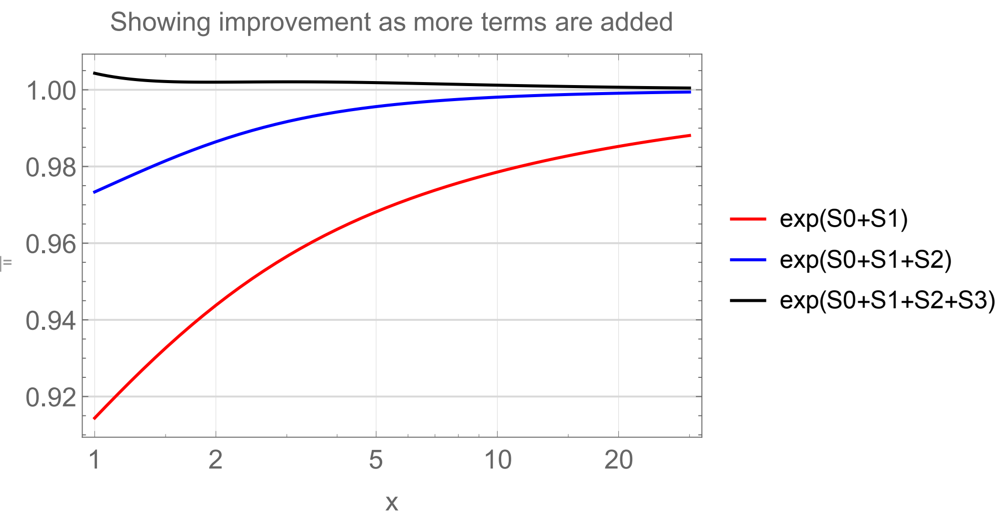
Yes, as $x \to \infty$. All assumptions verified. The leading behavior now is

$$\begin{aligned}
y(x) &\sim \exp\left(S_0+S_1+S_2+S_3\right)\\
&\sim \exp\left(2x^{\frac{1}{2}}-\frac{1}{4}\ln x+c+\frac{1}{16}\frac{1}{\sqrt{x}}+\frac{1}{1024}\left(\frac{2}{32x^2}+\frac{32}{x}\right)\right)\\
&\sim cx^{\frac{-1}{4}}\exp\left(2x^{\frac{1}{2}}+\frac{1}{16}\frac{1}{\sqrt{x}}+\frac{1}{1024}\left(\frac{2}{32x^2}+\frac{32}{x}\right)\right)
\end{aligned}$$

Now we will show how adding more terms to leading behavior improved the $y(x)$ solution for large x. When plotting the solutions, we see that $\frac{\exp(S_0+S_1+S_2+S_3)}{y(x)}$ approached the ratio 1 sooner than $\frac{\exp(S_0+S_1+S_2)}{y(x)}$ and this in turn approached the ratio 1 sooner than just using $\frac{\exp(S_0+S_1)}{y(x)}$. So the effect of adding more terms, is that the solution becomes more accurate for larger range of x values. Below is the code used and the plot generated.

```
ClearAll[y, x];
s0[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2)])/y[x, 300];
s0s1[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2) - 1/4 Log[x]])/y[x, 300];
s0s1s2[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2) - 1/4 Log[x] + 1/16 1/Sqrt[x]])/y[x, 300];
s0s1s2s3[x_] := ((1/(2 Pi^(1/2))) Exp[2 x^(1/2) - 1/4 Log[x] + 1/16 1/Sqrt[x] + 1/1024 (2/(32 x^2) + 32/x)])/y[x, 300];
y[x_, max_] := Sum[ x^n / (Factorial[n]^2), {n, 0, max}];
LogLinearPlot[Evaluate[{s0s1[x], s0s1s2[x], s0s1s2s3[x]}], {x, 1, 30},
 PlotRange → All, Frame → True, GridLines → Automatic, GridLinesStyle → LightGray,
 PlotLegends → {"exp(S0+S1)", "exp(S0+S1+S2)", "exp(S0+S1+S2+S3)"},
 FrameLabel → {{None, None}, {"x", "Showing improvement as more terms are added"}},
 PlotStyle → {Red, Blue, Black}, BaseStyle → 14]
```

Out[59]=



0.7 problem 3.49(c)

Problem Find the leading behavior as $x \to \infty$ of the general solution of $y'' + xy = x^5$

Solution This is non-homogenous ODE. We solve this by first finding the homogenous solution (asymptotic solution) and then finding particular solution. Hence we start with

$$y_h'' + xy_h = 0$$

$x = \infty$ is ISP point. Therefore, we assume $y_h(x) = e^{S(x)}$ and obtain

$$S'' + (S')^2 + x = 0 \tag{1}$$

Let

$$S(x) = S_0 + S_1 + \cdots$$

Therefore (1) becomes

$$\left(S_0'' + S_1'' + \cdots\right) + \left(S_0' + S_1' + \cdots\right)^2 = -x$$
$$\left(S_0'' + S_1'' + \cdots\right) + \left(\left[S_0'\right]^2 + 2S_0'S_1' + \left[S_1'\right]^2 + \cdots\right) = -x \tag{2}$$

Assuming $\left[S_0'\right]^2 \ggg S_0''$ we obtain

$$\left[S_0'\right]^2 \sim -x$$
$$S_0' \sim \omega\sqrt{x}$$

Where $\omega = \pm i$

Verification

$$\left[S_0'\right]^2 \ggg S_0''$$
$$x \ggg \frac{1}{2}\frac{1}{\sqrt{x}}$$

Yes, as $x \to \infty$. Hence

$$S_0 \sim \frac{3}{2}\omega x^{\frac{3}{2}}$$

Now we will find S_1. From (2), and moving all known terms to RHS

$$\left(S_1'' + \cdots\right) + \left(2S_0'S_1' + \left[S_1'\right]^2 + \cdots\right) \sim -x - S_0'' - \left[S_0'\right]^2 \tag{3}$$

Assuming

$$2S_0'S_1' \ggg S_1''$$
$$2S_0'S_1' \ggg \left[S_1'\right]^2$$

Then (3) becomes (where $S_0' \sim \omega\sqrt{x}, \left[S_0'\right]^2 \sim -x, S_0'' \sim \frac{1}{2}\omega\frac{1}{\sqrt{x}}$)

$$2S_0'S_1' \sim -x - S_0'' - \left[S_0'\right]^2$$
$$S_1' \sim \frac{-x - S_0'' - \left[S_0'\right]^2}{2S_0'}$$
$$\sim \frac{-x - \frac{1}{2}\omega\frac{1}{\sqrt{x}} - (-x)}{2\omega\sqrt{x}}$$
$$\sim -\frac{1}{4x}$$

Verification (where $S_1'' \sim \frac{1}{4}\frac{1}{x^2}$)

$$2S_0'S_1' \ggg S_1''$$
$$\sqrt{x}\left(\frac{1}{4x}\right) \ggg \frac{1}{4}\frac{1}{x^2}$$
$$\frac{1}{x^{\frac{1}{2}}} \ggg \frac{1}{x^2}$$

Yes, as $x \to \infty$

$$2S_0'S_1' \ggg \left[S_1'\right]^2$$
$$\frac{1}{x^{\frac{1}{2}}} \ggg \left(\frac{1}{4x}\right)^2$$
$$\frac{1}{x^{\frac{1}{2}}} \ggg \frac{1}{16x^2}$$

Yes, as $x \to \infty$. All validated. We solve for S_1

$$S_1' \sim -\frac{1}{4x}$$
$$S_1 \sim -\frac{1}{4}\ln x + c$$

y_h is found. It is given by

$$\begin{aligned} y_h(x) &\sim \exp\left(S_0(x) + S_1(x)\right) \\ &\sim \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}} - \frac{1}{4}\ln x + c\right) \\ &\sim c x^{\frac{-1}{4}} \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}}\right) \end{aligned}$$

Now that we have found y_h, we go back and look at

$$y'' + xy = x^5$$

And consider two cases (a) $y'' \sim x^5$ (b) $xy \sim x^5$. The case of $y'' \sim xy$ was covered above. This is what we did to find $y_h(x)$.

case (a)

$$y_p'' \sim x^5$$
$$y_p' \sim \frac{1}{5}x^4$$
$$y_p \sim \frac{1}{20}x^3$$

Where constants of integration are ignored since subdominant for $x \to \infty$. Now we check if this case is valid.

$$xy_p \lll x^5$$
$$x\frac{1}{20}x^3 \lll x^5$$
$$x^4 \lll x^5$$

No. Therefore case (a) did not work out. We try case (b) now

$$xy_p \sim x^5$$
$$y_p \sim x^4$$

Now we check is this case is valid.

$$y_p'' \lll x^5$$
$$12x^2 \lll x^5$$

Yes. Therefore, we found

$$y_p \sim x^4$$

Hence the complete asymptotic solution is

$$\begin{aligned} y(x) &\sim y_h(x) + y_p(x) \\ &\sim c x^{\frac{-1}{4}} \exp\left(\frac{3}{2}\omega x^{\frac{3}{2}}\right) + x^4 \end{aligned}$$