HW 1, EP 548, Spring 2017

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0.1 problem 3.3 (page 138)

0.1.1 Part c

problem Classify all the singular points (finite and infinite) of the following

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0$$

Answer

Writing the DE in standard form

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y'' + \frac{c-(a+b+1)x}{x(1-x)}y' - \frac{ab}{x(1-x)}y &= 0 \\ y'' + \overbrace{\left(\frac{c}{x(1-x)} - \frac{(a+b+1)}{(1-x)}\right)}^{p(x)}y' - \overbrace{\frac{ab}{x(1-x)}}^{q(x)}y &= 0 \end{aligned}$$

$x = 0, 1$ are singular points for $p(x)$ as well as for $q(x)$. Now we classify what type of singularity each point is. For $p(x)$

$$\begin{aligned} \lim_{x\to 0} xp(x) &= \lim_{x\to 0} x\left(\frac{c}{x(1-x)} - \frac{(a+b+1)}{(1-x)}\right) \\ &= \lim_{x\to 0}\left(\frac{c}{(1-x)} - \frac{x(a+b+1)}{(1-x)}\right) \\ &= c \end{aligned}$$

Hence $xp(x)$ is analytic at $x = 0$. Therefore $x = 0$ is regular singularity point. Now we check for $q(x)$

$$\begin{aligned} \lim_{x\to 0} x^2 q(x) &= \lim_{x\to 0} x^2 \frac{-ab}{x(1-x)} \\ &= \lim_{x\to 0} \frac{-xab}{(1-x)} \\ &= 0 \end{aligned}$$

Hence $x^2 q(x)$ is also analytic at $x = 0$. Therefore $x = 0$ is regular singularity point. Now we look at $x = 1$ and classify it. For $p(x)$

$$\begin{aligned} \lim_{x\to 1}(x-1)p(x) &= \lim_{x\to 1}(x-1)\left(\frac{c}{x(1-x)} - \frac{(a+b+1)}{(1-x)}\right) \\ &= \lim_{x\to 1}(x-1)\left(\frac{-c}{x(x-1)} + \frac{(a+b+1)}{(x-1)}\right) \\ &= \lim_{x\to 1}\left(\frac{-c}{x} + (a+b+1)\right) \\ &= -c + (a+b+1) \end{aligned}$$

Hence $(x-1)p(x)$ is analytic at $x = 1$. Therefore $x = 1$ is regular singularity point. Now we check for $q(x)$

$$\begin{aligned} \lim_{x\to 1}(x-1)^2 q(x) &= \lim_{x\to 1}(x-1)^2\left(\frac{-ab}{x(1-x)}\right) \\ &= \lim_{x\to 1}(x-1)^2\left(\frac{ab}{x(x-1)}\right) \\ &= \lim_{x\to 1}(x-1)\left(\frac{ab}{x}\right) \\ &= 0 \end{aligned}$$

Hence $(x-1)^2 q(x)$ is analytic also at $x = 1$. Therefore $x = 1$ is regular singularity point. Therefore $x = 0, 1$ are regular singular points for the ODE. Now we check for x at ∞. To check the type of singularity, if any, at $x = \infty$, the DE is first transformed using

$$x = \frac{1}{t} \tag{1}$$

This transformation will always results in [1] new ODE in t of this form

$$\frac{d^2y}{dt^2}+\frac{\left(-p(t)+2t\right)}{t^2}+\frac{q(t)}{t^4}y=0 \tag{2}$$

But

$$\begin{aligned}p(t)=p(x)\big|_{x=\frac{1}{t}}&=\left(\frac{c-(a+b+1)x}{x(1-x)}\right)_{x=\frac{1}{t}}\\&=\left(\frac{c-(a+b+1)\frac{1}{t}}{\frac{1}{t}\left(1-\frac{1}{t}\right)}\right)\\&=\frac{ct^2-t(a+b+1)}{(t-1)}\end{aligned} \tag{3}$$

And

$$\begin{aligned}q(t)=q(x)\big|_{x=\frac{1}{t}}&=\left(-\frac{ab}{x(1-x)}\right)_{x=\frac{1}{t}}\\&=-\frac{ab}{\frac{1}{t}\left(1-\frac{1}{t}\right)}\\&=-\frac{abt^2}{(t-1)}\end{aligned} \tag{4}$$

Substituting equations (3,4) into (2) gives

$$y''+\frac{\left(-\left(\frac{ct^2-t(a+b+1)}{(t-1)}\right)+2t\right)}{t^2}+\frac{\left(-\frac{abt^2}{(t-1)}\right)}{t^4}y=0$$

$$y''+\frac{\left(2t(t-1)-t^2c+(a+b+1)t\right)}{t^2(t-1)}y'-\frac{ab}{t^2(t-1)}y=0$$

Expanding

$$y''+\overbrace{\left(\frac{2t-1-tc+a+b}{t(t-1)}\right)}^{p(t)}y'-\overbrace{\frac{ab}{t^2(t-1)}}^{q(t)}y=0$$

We see that $t=0$ (this means $x=\infty$) is singular point for both $p(x),q(x)$. Now we check

[1] Let $x=\frac{1}{t}$, then

$$\frac{d}{dx}=\frac{d}{dt}\frac{dt}{dx}=-t^2\frac{d}{dt}$$

And

$$\begin{aligned}\frac{d^2}{dx^2}=\frac{d}{dx}\left(\frac{d}{dx}\right)&=\left(-t^2\frac{d}{dt}\right)\left(-t^2\frac{d}{dt}\right)=-t^2\frac{d}{dt}\left(-t^2\frac{d}{dt}\right)\\&=-t^2\left(-2t\frac{d}{dt}-t^2\frac{d^2}{dt^2}\right)\\&=2t^3\frac{d}{dt}+t^4\frac{d^2}{dt^2}\end{aligned}$$

The original ODE becomes

$$\begin{aligned}\left(2t^3\frac{d}{dt}+t^4\frac{d^2}{dt^2}\right)y+p(x)\big|_{x=\frac{1}{2}}\left(-t^2\frac{d}{dt}\right)y+q(x)\big|_{x=\frac{1}{t}}y&=0\\\left(2t^3y'+t^4y''\right)-t^2p(t)y'+q(t)y&=0\\t^4y''+\left(-t^2p(t)+2t^3\right)y'+q(t)y&=0\\y''+\frac{\left(-p(t)+2t\right)}{t^2}y'+\frac{q(t)}{t^4}y&=0\end{aligned}$$

what type it is. For $p(t)$

$$\begin{aligned}\lim_{t\to 0} tp(t) &= \lim_{t\to 0} t\left(\frac{2}{t} - \frac{c}{(t-1)} + \frac{(a+b+1)}{t(t-1)}\right)\\ &= \lim_{t\to 0}\left(2 - \frac{tc}{(t-1)} + \frac{(a+b+1)}{(t-1)}\right)\\ &= 1-a-b\end{aligned}$$

$tp(t)$ is therefore analytic at $t=0$. Hence $t=0$ is regular singular point. Now we check for $q(t)$

$$\begin{aligned}\lim_{t\to 0} t^2 q(t) &= \lim_{t\to 0} t^2\left(-\frac{ab}{t^2(t-1)}\right)\\ &= \lim_{t\to 0}\left(\frac{-ab}{(t-1)}\right)\\ &= ab\end{aligned}$$

$t^2 q(t)$ is therefore analytic at $t=0$. Hence $t=0$ is regular singular point for $q(t)$. Therefore $t=0$ is regular singular point which mean that $x\to\infty$ is a regular singular point for the ODE.

Summary

Singular points are $x=0,1$. Both are regular singular points. Also $x=\infty$ is regular singular point.

0.1.2 Part (d)

Problem Classify all the singular points (finite and infinite) of the following

$$xy'' + (b-x)y' - ay = 0$$

solution

Writing the ODE in standard for

$$y'' + \frac{(b-x)}{x}y' - \frac{a}{x}y = 0$$

We see that $x=0$ is singularity point for both $p(x)$ and $q(x)$. Now we check its type. For $p(x)$

$$\begin{aligned}\lim_{x\to 0} xp(x) &= \lim_{x\to 0} x\frac{(b-x)}{x}\\ &= b\end{aligned}$$

Hence $xp(x)$ is analytic at $x=0$. Therefore $x=0$ is regular singularity point for $p(x)$. For $q(x)$

$$\begin{aligned}\lim_{x\to 0} x^2 q(x) &= \lim_{x\to 0} x^2\left(\frac{-a}{x}\right)\\ &= \lim_{x\to 0}(-ax)\\ &= 0\end{aligned}$$

Hence $x^2 q(x)$ is analytic at $x=0$. Therefore $x=0$ is regular singularity point for $q(x)$.

Now we check for x at ∞. To check the type of singularity, if any, at $x=\infty$, the DE is first transformed using

$$x = \frac{1}{t} \tag{1}$$

This results in (as was done in above part)

$$\frac{d^2y}{dt^2} + \frac{\left(-p(t)+2t\right)}{t^2} + \frac{q(t)}{t^4}y = 0$$

Where

$$\begin{aligned} p(t) &= \left.\frac{(b-x)}{x}\right|_{x=\frac{1}{t}} \\ &= \frac{\left(b-\frac{1}{t}\right)}{\frac{1}{t}} \\ &= (bt-1) \end{aligned}$$

And $q(t) = -\frac{a}{x} = -at$ Hence the new ODE is

$$\begin{aligned} \frac{d^2y}{dt^2} + \frac{(-(bt-1)+2t)}{t^2} - \frac{at}{t^4}y = 0 \\ \frac{d^2y}{dt^2} + \frac{-bt+1+2t}{t^2} - \frac{a}{t^3}y = 0 \end{aligned}$$

Therefore $t=0$ (or $x=\infty$) is singular point. Now we will find the singularity type

$$\begin{aligned} \lim_{t\to 0} tp(t) &= \lim_{t\to 0} t\left(\frac{-bt+1+2t}{t^2}\right) \\ &= \lim_{t\to 0}\left(\frac{-bt+1+2t}{t}\right) \\ &= \infty \end{aligned}$$

Hence $tp(t)$ is not analytic, since the limit do not exist, which means $t=0$ is an irregular singular point for $p(t)$. We can stop here, but will also check for $q(t)$

$$\begin{aligned} \lim_{t\to 0} t^2 q(t) &= \lim_{t\to 0} t^2\left(-\frac{a}{t^3}\right) \\ &= \lim_{t\to 0}\left(-\frac{a}{t}\right) \\ &= \infty \end{aligned}$$

Therefore, $t=0$ is irregular singular point, which means $x=\infty$ is an irregular singular point.

Summary

$x=0$ is regular singular point. $x=\infty$ is an irregular singular point.

0.2 problem 3.4

0.2.1 part d

problem Classify $x=0$ and $x=\infty$ of the following

$$x^2y'' = ye^{\frac{1}{x}}$$

Answer

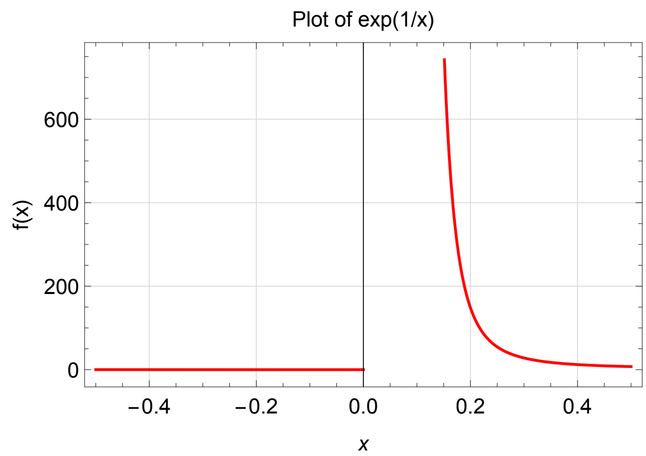
In standard form

$$y'' - \frac{e^{\frac{1}{x}}}{x^2}y = 0$$

Hence $p(x)=0, q(x) = -\frac{e^{\frac{1}{x}}}{x^2}$. The singularity is $x=0$. We need to check on $q(x)$ only.

$$\begin{aligned} \lim_{x\to 0} x^2 q(x) &= \lim_{x\to 0} x^2\left(-\frac{e^{\frac{1}{x}}}{x^2}\right) \\ &= \lim_{x\to 0}\left(-e^{\frac{1}{x}}\right) \end{aligned}$$

The above is not analytic, since $\lim_{x\to 0^+}\left(-e^{\frac{1}{x}}\right) = \infty$ while $\lim_{x\to 0^-}\left(-e^{\frac{1}{x}}\right) = 0$. This means $e^{\frac{1}{x}}$ is not differentiable at $x=0$. Here is plot $e^{\frac{1}{x}}$ near $x=0$



Therefore $x=0$ is an irregular singular point. We now convert the ODE using $x=\frac{1}{t}$ in order to check what happens at $x=\infty$. This results in (as was done in above part)

$$\frac{d^2y}{dt^2}+\frac{q(t)}{t^4}y=0$$

But

$$\begin{aligned} q(t) &= q(x)\Big|_{x=\frac{1}{t}} \\ &= \left(-\frac{e^{\frac{1}{x}}}{x^2}\right)_{x=\frac{1}{t}} \\ &= -t^2e^t \end{aligned}$$

Hence the ODE becomes

$$\begin{aligned} y''-\frac{-t^2e^t}{t^4}y &= 0 \\ y''+\frac{e^t}{t^2}y &= 0 \end{aligned}$$

We now check $q(t)$.

$$\begin{aligned} \lim_{t\to 0} t^2 q(t) &= \lim_{t\to 0} t^2\frac{e^t}{t^2} \\ &= \lim_{t\to 0} e^t \\ &= 1 \end{aligned}$$

This is analytic. Hence $t=0$ is regular singular point, which means $x=\infty$ is regular singular point.

Summary $x=0$ is irregular singular point, $x=\infty$ is regular singular point.

0.2.2 Part e

problem

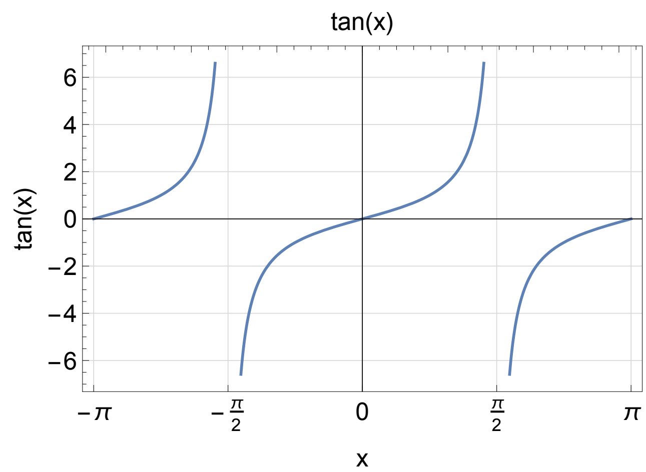
Classify $x=0$ and $x=\infty$ of the following

$$(\tan x)\,y'=y$$

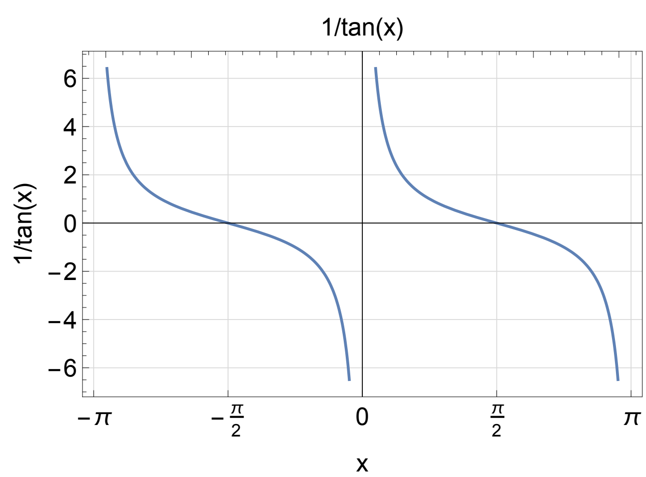
Answer

$$y'-\frac{1}{\tan x}y=0$$

The function $\tan x$ looks like



Therefore, $\tan(x)$ is not analytic at $x = \left(n - \frac{1}{2}\right)\pi$ for $n \in \mathbb{Z}$. Hence the function $\frac{1}{\tan(x)}$ is not analytic at $x = n\pi$ as seen in this plot



Hence singular points are $x = \{\cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots\}$. Looking at $x = 0$

$$\begin{aligned}
\lim_{x\to 0} xp(x) &= \lim_{x\to 0}\left(x\frac{1}{\tan(x)}\right) \\
&= \lim_{x\to 0}\left(\frac{\frac{dx}{dx}}{\frac{d\tan(x)}{dx}}\right) \\
&= \lim_{x\to 0}\frac{1}{\sec^2 x} \\
&= \lim_{x\to 0}\cos^2 x \\
&= 1
\end{aligned}$$

Therefore the point $x = 0$ is regular singular point. To classify $x = \infty$, we use $x = \frac{1}{t}$ substitution. $\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = -t^2\frac{d}{dt}$ and the ODE becomes

$$\begin{aligned}
\left(-t^2\frac{d}{dt}\right)y - \frac{1}{\tan\left(\frac{1}{t}\right)}y &= 0 \\
-t^2 y' - \frac{1}{\tan\left(\frac{1}{t}\right)}y &= 0 \\
y' + \frac{1}{t^2\tan\left(\frac{1}{t}\right)}y &= 0
\end{aligned}$$

Hence

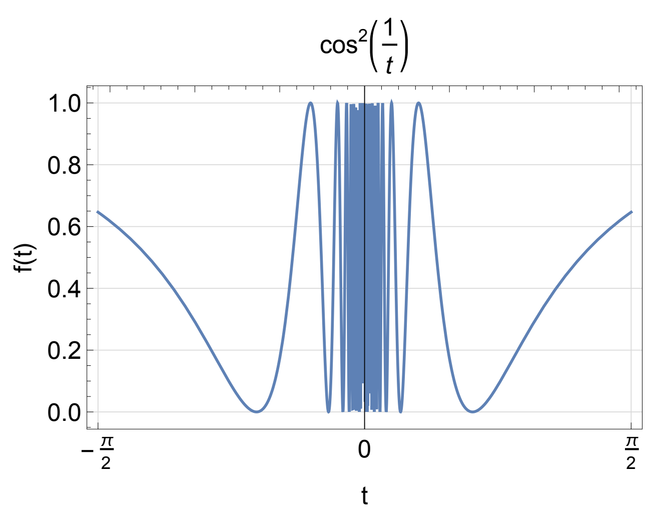
$$p(t) = \frac{1}{t^2\tan\left(\frac{1}{t}\right)}$$

This function has singularity at $t = 0$ and at $t = \frac{1}{n\pi}$ for $n \in \mathbb{Z}$. We just need to consider

$t=0$ since this maps to $x=\infty$. Hence

$$\lim_{t\to 0} tp(t) = \lim_{t\to 0}\left(t\frac{1}{t^2\tan\left(\frac{1}{t}\right)}\right)$$
$$= \lim_{t\to 0}\left(\frac{1}{t\tan\left(\frac{1}{t}\right)}\right)$$
$$= \lim_{t\to 0}\left(\frac{\frac{1}{t}}{\tan\left(\frac{1}{t}\right)}\right) \underset{\text{L'hopitals}}{\rightarrow} \lim_{t\to 0}\left(\frac{-\frac{1}{t^2}}{-\frac{\sec^2\left(\frac{1}{t}\right)}{t^2}}\right)$$
$$= \lim_{t\to 0}\left(\frac{1}{\sec^2\left(\frac{1}{t}\right)}\right)$$
$$= \lim_{t\to 0}\left(\cos^2\left(\frac{1}{t}\right)\right)$$

The following is a plot of $\cos^2\left(\frac{1}{t}\right)$ as t goes to zero.



This is the same as asking for $\lim_{x\to\infty}\cos^2(x)$ which does not exist, since $\cos(x)$ keeps oscillating, hence it has no limit. Therefore, we conclude that $tp(t)$ is not analytic at $t=0$, hence t is irregular singular point, which means $x=\infty$ is an irregular singular point.

Summary

$x=0$ is regular singular point and $x=\infty$ is an irregular singular point

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0.3 Problem 3.6

0.3.1 part b

Problem Find the Taylor series expansion about $x=0$ of the solution to the initial value problem

$$y'' - 2xy' + 8y = 0$$
$$y(0) = 0$$
$$y'(0) = 4$$

solution

Since $x=0$ is ordinary point, then we can use power series solution

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Hence

$$\begin{aligned}y'(x) &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \\ y''(x) &= \sum_{n=0}^{\infty} n(n+1) a_{n+1} x^{n-1} = \sum_{n=1}^{\infty} n(n+1) a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n\end{aligned}$$

Therefore the ODE becomes

$$\begin{aligned}\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - 2x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 8 \sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} 2(n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} 8 a_n x^n &= 0 \\ \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} 8 a_n x^n &= 0\end{aligned}$$

Hence, for $n = 0$ we obtain

$$\begin{aligned}(n+1)(n+2) a_{n+2} x^n + 8 a_n x^n &= 0 \\ 2a_2 + 8a_0 &= 0 \\ a_2 &= -4a_0\end{aligned}$$

For $n \geq 1$

$$\begin{aligned}(n+1)(n+2) a_{n+2} - 2n a_n + 8 a_n &= 0 \\ a_{n+2} &= \frac{2n a_n - 8 a_n}{(n+1)(n+2)} \\ &= \frac{a_n (2n-8)}{(n+1)(n+2)}\end{aligned}$$

Hence for $n = 1$

$$a_3 = \frac{a_1 (2-8)}{(1+1)(1+2)} = -a_1$$

For $n = 2$

$$a_4 = \frac{a_2 (2(2)-8)}{(2+1)(2+2)} = -\frac{1}{3} a_2 = -\frac{1}{3}(-4a_0) = \frac{4}{3} a_0$$

For $n = 3$

$$a_5 = \frac{a_3 (2(3)-8)}{(3+1)(3+2)} = -\frac{1}{10} a_3 = -\frac{1}{10}(-a_1) = \frac{1}{10} a_1$$

For $n = 4$

$$a_6 = \frac{a_4 (2(4)-8)}{(4+1)(4+2)} = 0$$

For $n = 5$

$$a_7 = \frac{a_5 (2(5)-8)}{(5+1)(5+2)} = \frac{1}{21} a_5 = \frac{1}{21}\left(\frac{1}{10} a_1\right) = \frac{1}{210} a_1$$

For $n = 6$

$$a_8 = \frac{a_6 (2(6)-8)}{(6+1)(6+2)} = \frac{1}{14} a_6 = 0$$

For $n = 7$

$$a_9 = \frac{a_7 (2(7)-8)}{(7+1)(7+2)} = \frac{1}{12} a_7 = \frac{1}{12}\left(\frac{1}{210} a_1\right) = \frac{1}{2520} a_1$$

Writing now few terms

$$\begin{aligned}y(x) &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + a_9 x^9 + \cdots \\ &= a_0 + a_1 x + (-4a_0) x^2 + (-a_1) x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{10} a_1 x^5 + 0 + \frac{1}{210} a_1 x^7 + 0 + \frac{1}{2520} a_1 x^9 + \cdots \\ &= a_0 + a_1 x - 4a_0 x^2 - a_1 x^3 + \frac{4}{3} a_0 x^4 + \frac{1}{10} a_1 x^5 + \frac{1}{210} a_1 x^7 + \frac{1}{2520} a_1 x^9 + \cdots \\ &= a_0 \left(1 - 4x^2 + \frac{4}{3} x^4\right) + a_1 \left(x - x^3 + \frac{1}{10} x^5 + \frac{1}{210} x^7 + \frac{1}{2520} x^9 + \cdots\right) \qquad (1)\end{aligned}$$

We notice that a_0 terms terminates at $\frac{4}{3}x^4$ but the a_1 terms do not terminate. Now we need to find a_0, a_1 from initial conditions. At $x = 0, y(0) = 0$. Hence from (1)

$$0 = a_0$$

Hence the solution becomes

$$y(x) = a_1 \left(x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \frac{1}{2520}x^9 + \cdots \right) \tag{2}$$

Taking derivative of (2), term by term

$$y'(x) = a_1 \left(1 - 3x^2 + \frac{5}{10}x^4 + \frac{7}{210}x^6 + \frac{9}{2520}x^8 + \cdots \right)$$

Using $y'(0) = 4$ the above becomes

$$4 = a_1$$

Hence the solution is

$$y(x) = 4 \left(x - x^3 + \frac{1}{10}x^5 + \frac{1}{210}x^7 + \frac{1}{2520}x^9 + \cdots \right)$$

Or

$$\boxed{y(x) = 4x - 4x^3 + \frac{2}{5}x^5 + \frac{2}{105}x^7 + \frac{1}{630}x^9 + \cdots}$$

The above is the Taylor series of the solution expanded around $x = 0$.

0.4 Problem 3.7

Problem: Estimate the number of terms in the Taylor series (3.2.1) and (3.2.2) at page 68 of the text, that are necessary to compute

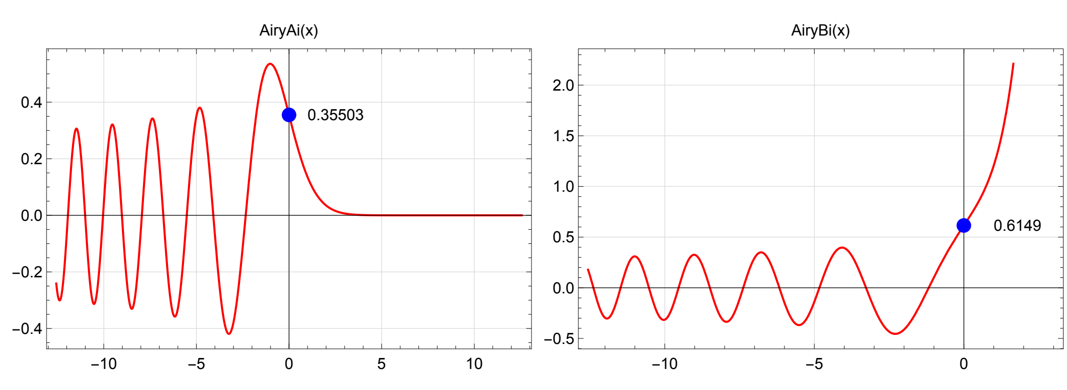
Ai(x) and Bi(x) correct to three decimal places at $x = \pm 1, \pm 100, \pm 10000$

Answer:

$$\operatorname{Ai}(x) = 3^{\frac{-2}{3}} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma\left(n + \frac{2}{3}\right)} - 3^{\frac{-4}{3}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma\left(n + \frac{4}{3}\right)} \tag{3.2.1}$$

$$\operatorname{Bi}(x) = 3^{\frac{-1}{6}} \sum_{n=0}^{\infty} \frac{x^{3n}}{9^n n! \Gamma\left(n + \frac{2}{3}\right)} - 3^{\frac{-5}{6}} \sum_{n=0}^{\infty} \frac{x^{3n+1}}{9^n n! \Gamma\left(n + \frac{4}{3}\right)} \tag{3.2.2}$$

The radius of convergence of these series extends from $x = 0$ to $\pm\infty$ so we know this will converge to the correct value of $\operatorname{Ai}(x), \operatorname{Bi}(x)$ for all x, even though we might need large number of terms to achieve this, as will be shown below. The following is a plot of $\operatorname{Ai}(x)$ and $\operatorname{Bi}(x)$



0.4.1 AiryAI series

For $\mathrm{Ai}(x)$ looking at the $\frac{(N+1)^{th}}{N^{th}}$ term

$$\Delta = \frac{3^{\frac{-2}{3}} \frac{x^{3(N+1)}}{9^{N+1}(N+1)!\Gamma\left(N+1+\frac{2}{3}\right)} - 3^{\frac{-4}{3}} \frac{x^{3N+2}}{9^{N+1}(N+1)!\Gamma\left(N+1+\frac{4}{3}\right)}}{3^{\frac{-2}{3}} \frac{x^{3N}}{9^{N}N!\Gamma\left(N+\frac{2}{3}\right)} - 3^{\frac{-4}{3}} \frac{x^{3N+1}}{9^{N}N!\Gamma\left(N+\frac{4}{3}\right)}}$$

This can be simplified using $\Gamma(N+1) = N\Gamma(N)$ giving

$$\Gamma\left(\left(N+\frac{2}{3}\right)+1\right) = \left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)$$
$$\Gamma\left(\left(N+\frac{4}{3}\right)+1\right) = \left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)$$

Hence Δ becomes

$$\Delta = \frac{\frac{3^{\frac{-2}{3}} x^{3(N+1)}}{9^{N+1}(N+1)!\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}} x^{3N+2}}{9^{N+1}(N+1)!\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}{\frac{3^{\frac{-2}{3}} x^{3N}}{9^{N}N!\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}} x^{3N+1}}{9^{N}N!\Gamma\left(N+\frac{4}{3}\right)}}$$

Or

$$\Delta = \frac{\frac{3^{\frac{-2}{3}} x^{3(N+1)}}{9(N+1)\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}} x^{3N+2}}{9(N+1)\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}{\frac{3^{\frac{-2}{3}} x^{3N}}{\Gamma\left(N+\frac{2}{3}\right)} - \frac{3^{\frac{-4}{3}} x^{3N+1}}{\Gamma\left(N+\frac{4}{3}\right)}}$$

Or

$$\Delta = \frac{\frac{3^{\frac{-2}{3}} x^{3(N+1)}\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N+2}\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)}{9(N+1)\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}{\frac{3^{\frac{-2}{3}} x^{3N}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N+1}\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{4}{3}\right)}}$$

Or

$$\Delta = \frac{\frac{3^{\frac{-2}{3}} x^{3(N+1)}\left(N+\frac{4}{3}\right)\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N+2}\left(N+\frac{2}{3}\right)\Gamma\left(N+\frac{2}{3}\right)}{9(N+1)\left(N+\frac{2}{3}\right)\left(N+\frac{4}{3}\right)}}{3^{\frac{-2}{3}} x^{3N}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N+1}\Gamma\left(N+\frac{2}{3}\right)}$$

For large N, we can approximate $\left(N+\frac{2}{3}\right), \left(N+\frac{4}{3}\right), (N+1)$ to just $N+1$ and the above becomes

$$\Delta = \frac{\frac{3^{\frac{-2}{3}} x^{3(N+1)}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N+2}\Gamma\left(N+\frac{2}{3}\right)}{9(N+1)^2}}{3^{\frac{-2}{3}} x^{3N}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N+1}\Gamma\left(N+\frac{2}{3}\right)}$$

Or

$$\Delta = \frac{3^{\frac{-2}{3}} x^{3N}x^{3}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N}x^{2}\Gamma\left(N+\frac{2}{3}\right)}{9(N+1)^2\left(3^{\frac{-2}{3}} x^{3N}\Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-4}{3}} x^{3N}x\Gamma\left(N+\frac{2}{3}\right)\right)}$$

Or

$$\Delta = \frac{0.488x^{3N}x^2\left(x\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)\right)}{(0.488)\,9\,(N+1)^2\,x^{3N}\left(\Gamma\left(N+\frac{4}{3}\right)-0.488x\Gamma\left(N+\frac{2}{3}\right)\right)}$$
$$= \frac{x^2}{9\,(N+1)^2}\left(\frac{x\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488x\Gamma\left(N+\frac{2}{3}\right)}\right)$$

We want to solve for N s.t. $\left|\frac{x^2}{9(N+1)^2}\left(\frac{x\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488x\Gamma\left(N+\frac{2}{3}\right)}\right)\right| < 0.001.$

<u>For $x=1$</u>

$$\frac{1}{9\,(N+1)^2}\left(\frac{\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}\right) < 0.001$$
$$\frac{1}{9\,(N+1)^2} < 0.001$$
$$9\,(N+1)^2 > 1000$$
$$(N+1)^2 > \frac{1000}{9}$$
$$N+1 > \sqrt{\frac{1000}{9}}$$
$$N > 9.541$$
$$N = 10$$

<u>For $x=100$</u>

$$\left|\frac{100^2}{9\,(N+1)^2}\left(\frac{100\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488\,(100)\,\Gamma\left(N+\frac{2}{3}\right)}\right)\right| < 0.001$$

I could not simplify away the Gamma terms above any more. Is there a way? So wrote small function (in Mathematica, which can compute this) which increments N and evaluate the above, until the value became smaller than 0.001. At <u>$N = 11,500$</u> this was achieved.

<u>For $x=10000$</u>

$$\left|\frac{10000^2}{9\,(N+1)^2}\left(\frac{100\Gamma\left(N+\frac{4}{3}\right)-0.488\Gamma\left(N+\frac{2}{3}\right)}{\Gamma\left(N+\frac{4}{3}\right)-0.488\,(10000)\,\Gamma\left(N+\frac{2}{3}\right)}\right)\right| < 0.001$$

Using the same program, found that $N = 11,500,000$ was needed to obtain the result below 0.001.

For $= -1$ also $N = 10$. For $x = -100$, $N = 10,030$, which is little less than $x = +100$. For $x = -10000$, $N = 10,030,000$

<u>Summary table for AiryAI(x)</u>

x	N
1	10
-1	10
100	11,500
-100	10,030
10000	11,500,000
-10000	10,030,000

The Mathematica function which did the estimate is the following

```
estimateAiryAi[x_, n_] := (x^2/(9*(n + 1)^2))*((x*Gamma[n + 4/3] -
0.488*Gamma[n + 2/3])/(Gamma[n + 4/3] - 0.488*x*Gamma[n + 2/3]))
```

```
estimateAiryAi[-10000, 10030000] // N
-0.0009995904
estimateAiryAi[100, 11500] // N
0.000928983646707407
estimateAiryAi[10000, 11500000] // N
0.000929156800155198
```

0.4.2 AiryBI series

For $\text{Bi}(x)$ the difference is the coefficients. Hence using the result from above, and just replace the coefficients

$$\begin{aligned}\Delta &= \frac{3^{\frac{-1}{6}} x^{3N} x^3 \Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-5}{8}} x^{3N} x^2 \Gamma\left(N+\frac{2}{3}\right)}{9(N+1)^2 \left(3^{\frac{-1}{6}} x^{3N} \Gamma\left(N+\frac{4}{3}\right) - 3^{\frac{-5}{8}} x^{3N} x \Gamma\left(N+\frac{2}{3}\right)\right)} \\ &= \frac{x^2\left(x\Gamma\left(N+\frac{4}{3}\right) - 0.60439\Gamma\left(N+\frac{2}{3}\right)\right)}{9(N+1)^2\left(\Gamma\left(N+\frac{4}{3}\right) - 0.60439 x\Gamma\left(N+\frac{2}{3}\right)\right)}\end{aligned}$$

Hence for $x=1$, using the above reduces to

$$\frac{1}{9(N+1)^2} < 0.001$$

Which is the same as AiryAI, therefore $N=10$. For $x=100$, using the same small function in Mathematica to calculate the above, here are the result.

<u>Summary table for AiryBI(x)</u>

x	N
1	10
−1	10
100	11,290
−100	9,900
10000	10,900,000
−10000	9,950,000

The result between AiryAi and AiryBi are similar. AiryBi needs a slightly less number of terms in the series to obtain same accuracy.

The Mathematica function which did the estimate for the larger N value for the above table is the following

```
estimateAiryBI[x_, n_] :=  Abs[(x^2/(9*(n + 1)^2))*((x*Gamma[n + 4/3] -
0.60439*Gamma[n + 2/3])/(Gamma[n + 4/3] - 0.6439*x*Gamma[n + 2/3]))]
```

0.5 Problem 3.8

<u>Problem</u> How many terms in the Taylor series solution to $y''' = x^3 y$ with $y(0)=1, y'(0) = y''(0) = 0$ are needed to evaluate $\int_0^1 y(x)\,dx$ correct to three decimal places?

<u>Answer</u>

$$y''' - x^3 y = 0$$

Since x is an ordinary point, we use

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^n \\
y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n \\
y'' &= \sum_{n=0}^{\infty} n\,(n+1)\, a_{n+1} x^{n-1} = \sum_{n=1}^{\infty} n\,(n+1)\, a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+1)\,(n+2)\, a_{n+2} x^n \\
y''' &= \sum_{n=0}^{\infty} n\,(n+1)\,(n+2)\, a_{n+2} x^{n-1} = \sum_{n=1}^{\infty} n\,(n+1)\,(n+2)\, a_{n+2} x^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1)\,(n+2)\,(n+3)\, a_{n+3} x^n
\end{aligned}$$

Hence the ODE becomes

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1)\,(n+2)\,(n+3)\, a_{n+3} x^n - x^3 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=0}^{\infty} (n+1)\,(n+2)\,(n+3)\, a_{n+3} x^n - \sum_{n=0}^{\infty} a_n x^{n+3} &= 0 \\
\sum_{n=0}^{\infty} (n+1)\,(n+2)\,(n+3)\, a_{n+3} x^n - \sum_{n=3}^{\infty} a_{n-3} x^n &= 0
\end{aligned}$$

For $n = 0$

$$\begin{aligned}
(1)\,(2)\,(3)\, a_3 &= 0 \\
a_3 &= 0
\end{aligned}$$

For $n = 1$

$$\begin{aligned}
(2)\,(3)\,(4)\, a_4 &= 0 \\
a_4 &= 0
\end{aligned}$$

For $n = 2$

$$a_5 = 0$$

For $n \geq 3$, recursive equation is used

$$\begin{aligned}
(n+1)\,(n+2)\,(n+3)\, a_{n+3} - a_{n-3} &= 0 \\
a_{n+3} &= \frac{a_{n-3}}{(n+1)\,(n+2)\,(n+3)}
\end{aligned}$$

Or, for $n = 3$

$$a_6 = \frac{a_0}{(4)\,(5)\,(6)}$$

For $n = 4$

$$a_7 = \frac{a_1}{(4+1)\,(4+2)\,(4+3)} = \frac{a_1}{(5)\,(6)\,(7)}$$

For $n = 5$

$$a_8 = \frac{a_2}{(5+1)\,(5+2)\,(5+3)} = \frac{a_2}{(6)\,(7)\,(8)}$$

For $n = 6$

$$a_9 = \frac{a_3}{(6+1)\,(6+2)\,(6+3)} = 0$$

For $n = 7$

$$a_{10} = \frac{a_4}{(n+1)\,(n+2)\,(n+3)} = 0$$

For $n = 8$

$$a_{11} = \frac{a_5}{(8+1)\,(8+2)\,(8+3)} = 0$$

For $n = 9$

$$a_{12} = \frac{a_6}{(9+1)\,(9+2)\,(9+3)} = \frac{a_6}{(10)\,(11)\,(12)} = \frac{a_0}{(4)\,(5)\,(6)\,(10)\,(11)\,(12)}$$

For $n = 10$

$$a_{13} = \frac{a_7}{(10+1)\,(10+2)\,(10+3)} = \frac{a_7}{(11)\,(12)\,(13)} = \frac{a_1}{(5)\,(6)\,(7)\,(11)\,(12)\,(13)}$$

For $n = 11$

$$a_{14} = \frac{a_8}{(11+1)(11+2)(11+3)} = \frac{a_8}{(12)(13)(14)} = \frac{a_2}{(6)(7)(8)(12)(13)(14)}$$

And so on. Hence the series is

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + 0x^3 + 0x^4 + 0x^5 + \frac{a_0}{(4)(5)(6)}x^6 + \frac{a_1}{(5)(6)(7)}x^7 + \frac{a_2}{(6)(7)(8)}x^8 \\ &+ 0 + 0 + 0 + \frac{a_0}{(4)(5)(6)(10)(11)(12)}x^{12} + \frac{a_1}{(5)(6)(7)(11)(12)(13)}x^{13} \\ &+ \frac{a_2}{(6)(7)(8)(12)(13)(14)}x^{14} + 0 + 0 + 0 + \cdots \end{aligned}$$

Or

$$y(x) = a_0 + a_1 x + a_2 x^2 + \frac{a_0}{(4)(5)(6)}x^6 + \frac{a_1}{(5)(6)(7)}x^7 + \frac{a_2}{(6)(7)(8)}x^8 + \frac{a_0}{(4)(5)(6)(10)(11)(12)}x^{12} + \frac{a_1}{(5)(6)(7)(11)(12)(13)}x^{13} + \frac{a_2}{(6)(7)(8)(12)(13)(14)}x^{14} + \cdots$$

Or

$$\begin{aligned} y(x) &= a_0\left(1 + \frac{x^6}{(4)(5)(6)} + \frac{x^{12}}{(4)(5)(6)(10)(11)(12)} + \cdots\right) \\ &+ a_1\left(x + \frac{x^7}{(5)(6)(7)} + \frac{x^{13}}{(5)(6)(7)(11)(12)(13)} + \cdots\right) \\ &+ a_2\left(x^2 + \frac{x^8}{(6)(7)(8)} + \frac{x^{14}}{(6)(7)(8)(12)(13)(14)} + \cdots\right) \end{aligned}$$

$$\begin{aligned} y(x) &= a_0\left(1 + \frac{1}{120}x^6 + \frac{1}{158400}x^{12} + \cdots\right) \\ &+ a_1\left(x + \frac{1}{210}x^7 + \frac{1}{360360}x^{13} + \cdots\right) \\ &+ a_2\left(x^2 + \frac{1}{336}x^8 + \frac{1}{733\,824}x^{14} + \cdots\right) \end{aligned}$$

We now apply initial conditions $y(0) = 1, y'(0) = y''(0) = 0$. When $y(0) = 1$

$$1 = a_0$$

Hence solution becomes

$$\begin{aligned} y(x) &= \left(1 + \frac{1}{120}x^6 + \frac{1}{158400}x^{12} + \cdots\right) \\ &+ a_1\left(x + \frac{1}{210}x^7 + \frac{1}{360360}x^{13} + \cdots\right) \\ &+ a_2\left(x^2 + \frac{1}{336}x^8 + \frac{1}{733\,824}x^{14} + \cdots\right) \end{aligned}$$

Taking derivative

$$\begin{aligned} y'(x) &= \left(\frac{6}{120}x^5 + \frac{12}{158400}x^{11} + \cdots\right) \\ &+ a_1\left(1 + \frac{7}{210}x^6 + \frac{13}{360360}x^{12} + \cdots\right) \\ &+ a_2\left(2x + \frac{8}{336}x^7 + \frac{12}{733\,824}x^{13} + \cdots\right) \end{aligned}$$

Applying $y'(0) = 0$ gives

$$0 = a_1$$

And similarly, Applying $y''(0) = 0$ gives $a_2 = 0$. Hence the solution is

$$y(x) = a_0\left(1 + \frac{x^6}{(4)(5)(6)} + \frac{x^{12}}{(4)(5)(6)(10)(11)(12)} + \cdots\right)$$
$$= 1 + \frac{x^6}{(4)(5)(6)} + \frac{x^{12}}{(4)(5)(6)(10)(11)(12)} + \frac{x^{18}}{(4)(5)(6)(10)(11)(12)(16)(17)(18)} + \cdots$$
$$= 1 + \frac{x^6}{120} + \frac{x^{12}}{158\,400} + \frac{x^{18}}{775\,526\,400} + \cdots$$

We are now ready to answer the question. We will do the integration by increasing the number of terms by one each time. When the absolute difference between each increment becomes less than 0.001 we stop. When using one term For

$$\int_0^1 y(x)\,dx = \int_0^1 dx$$
$$= 1$$

When using two terms

$$\int_0^1 1 + \frac{x^6}{120} dx = \left(x + \frac{x^7}{120(7)}\right)_0^1$$
$$= 1 + \frac{1}{840}$$
$$= \frac{841}{840}$$
$$= 1.001190476$$

Difference between one term and two terms is 0.001190476. When using three terms

$$\int_0^1 1 + \frac{x^6}{120} + \frac{x^{12}}{158\,400} dx = \left(x + \frac{x^7}{840} + \frac{x^{13}}{2059\,200}\right)_0^1$$
$$= 1 + \frac{1}{840} + \frac{1}{2059\,200}$$
$$= \frac{14\,431\,567}{14\,414\,400}$$
$$= 1.001190962$$

Comparing the above result, with the result using two terms, we see that only two terms are needed since the change in accuracy did not affect the first three decimal points. Hence we need only this solution with two terms only

$$y(x) = 1 + \frac{1}{120}x^6$$

0.6 Problem 3.24

0.6.1 part e

Problem Find series expansion of all the solutions to the following differential equation about $x = 0$. Try to sum in closed form any infinite series that appear.

$$2xy'' - y' + x^2 y = 0$$

Solution

$$y'' - \frac{1}{2x}y' + \frac{x}{2}y = 0$$

The only singularity is in $p(x)$ is $x = 0$. We will now check if it is removable. (i.e. regular)

$$\lim_{x\to 0} xp(x) = \lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2}$$

Therefore $x=0$ is regular singular point. Hence we try Frobenius series

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r} \\ y'(x) &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} \\ y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \end{aligned}$$

Substituting the above in $2x^2y'' - xy' + x^3y = 0$ results in

$$\begin{aligned} 2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} - x \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} + x^3 \sum_{n=0}^{\infty} a_n x^{n+r} &= 0 \\ \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+3} &= 0 \\ \sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n x^{n+r} - \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=3}^{\infty} a_{n-3} x^{n+r} &= 0 \qquad (1) \end{aligned}$$

The first step is to obtain the indicial equation. As the nature of the roots will tell us how to proceed. The indicial equation is obtained from setting $n=0$ in (1) with the assumption that $a_0 \neq 0$. Setting $n=0$ in (1) gives

$$\begin{aligned} 2(n+r)(n+r-1) a_n - (n+r) a_n &= 0 \\ 2(r)(r-1) a_0 - r a_0 &= 0 \end{aligned}$$

Since $a_0 \neq 0$ then we obtain the indicial equation (quadratic in r)

$$\begin{aligned} 2(r)(r-1) - r &= 0 \\ r(2(r-1)-1) &= 0 \\ r(2r-3) &= 0 \end{aligned}$$

Hence roots are

$$\begin{aligned} r_1 &= 0 \\ r_2 &= \frac{3}{2} \end{aligned}$$

Since $r_1 - r_2$ is not an integer, then we know we can now construct two linearly independent solutions

$$\begin{aligned} y_1(x) &= x^{r_1} \sum_{n=0} a_n x^n \\ y_2(x) &= x^{r_2} \sum_{n=0} b_n x^n \end{aligned}$$

Or

$$\begin{aligned} y_1(x) &= \sum_{n=0} a_n x^n \\ y_2(x) &= \sum_{n=0} b_n x^{n+\frac{3}{2}} \end{aligned}$$

Notice that the coefficients are not the same. Since we now know r_1, r_2, we will use the above series solution to obtain $y_1(x)$ and $y_2(x)$.

For $y_1(x)$ where $r_1 = 0$

$$\begin{aligned} y_1(x) &= \sum_{n=0} a_n x^n \\ y'(x) &= \sum_{n=0} n a_n x^{n-1} = \sum_{n=1} n a_n x^{n-1} = \sum_{n=0} (n+1) a_{n+1} x^n \\ y''(x) &= \sum_{n=0} n(n+1) a_{n+1} x^{n-1} = \sum_{n=1} n(n+1) a_{n+1} x^{n-1} = \sum_{n=0} (n+1)(n+2) a_{n+2} x^n \end{aligned}$$

Substituting the above in $2x^2y'' - xy' + x^3y = 0$ results in

$$
\begin{aligned}
2x^2\sum_{n=0}(n+1)(n+2)a_{n+2}x^n - x\sum_{n=0}(n+1)a_{n+1}x^n + x^3\sum_{n=0}a_nx^n &= 0\\
\sum_{n=0}2(n+1)(n+2)a_{n+2}x^{n+2} - \sum_{n=0}(n+1)a_{n+1}x^{n+1} + \sum_{n=0}a_nx^{n+3} &= 0\\
\sum_{n=2}2(n-1)(n)a_nx^n - \sum_{n=1}na_nx^n + \sum_{n=3}a_{n-3}x^n &= 0
\end{aligned}
$$

For $n = 1$ (index starts at $n = 1$).

$$
\begin{aligned}
-na_nx^n &= 0\\
-a_1 &= 0\\
a_1 &= 0
\end{aligned}
$$

For $n = 2$

$$
\begin{aligned}
2(n-1)(n)a_nx^n - na_nx^n &= 0\\
2(2-1)(2)a_2 - 2a_2 &= 0\\
2a_2 &= 0\\
a_2 &= 0
\end{aligned}
$$

For $n \geq 3$ we have recursive formula

$$
\begin{aligned}
2(n-1)(n)a_n - na_n + a_{n-3} &= 0\\
a_n &= \frac{-a_{n-3}}{n(2n-3)}
\end{aligned}
$$

Hence, for $n = 3$

$$
a_3 = \frac{-a_0}{(3)(6-3)} = \frac{-a_0}{9}
$$

For $n = 4$

$$
a_4 = \frac{-a_1}{n(2n-3)} = 0
$$

For $n = 5$

$$
a_5 = \frac{-a_2}{n(2n-3)} = 0
$$

For $n = 6$

$$
a_6 = \frac{-a_3}{n(2n-3)} = \frac{-a_3}{(6)(12-3)} = \frac{-a_3}{54} = \frac{a_0}{(54)(9)} = \frac{1}{486}a_0
$$

And for $n = 7, 8$ we also obtain $a_7 = 0, a_8 = 0$, but for a_9

$$
a_9 = \frac{-a_6}{9(2(9)-3)} = \frac{-a_6}{135} = \frac{-a_0}{(135)(486)} = -\frac{1}{65\,610}a_0
$$

And so on. Hence from $\sum_{n=0}a_nx^n$ we obtain

$$
\begin{aligned}
y_1(x) &= a_0 - \frac{-a_0}{9}x^3 + \frac{1}{486}a_0x^6 - \frac{1}{65\,610}a_0x^9 + \cdots\\
&= a_0\left(1 - \frac{1}{9}x^3 + \frac{1}{486}x^6 - \frac{1}{65\,610}x^9 + \cdots\right)
\end{aligned}
$$

Now that we found $y_1(x)$.

<u>For $y_2(x)$</u> with $r_2 = \frac{3}{2}$.

$$
\begin{aligned}
y(x) &= \sum_{n=0}b_nx^{n+\frac{3}{2}}\\
y'(x) &= \sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{1}{2}}\\
y''(x) &= \sum_{n=0}\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n-\frac{1}{2}}
\end{aligned}
$$

Substituting this into $2x^2y'' - xy' + x^3y = 0$ gives

$$2x^2\sum_{n=0}\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n-\frac{1}{2}} - x\sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{1}{2}} + x^3\sum_{n=0}b_nx^{n+\frac{3}{2}} = 0$$

$$\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n-\frac{1}{2}+2} - \sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{1}{2}+1} + \sum_{n=0}b_nx^{n+\frac{3}{2}+3} = 0$$

$$\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} - \sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} + \sum_{n=0}b_nx^{n+\frac{9}{2}} = 0$$

$$\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} - \sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} + \sum_{n=3}b_{n-3}x^{n+\frac{9}{2}-3} = 0$$

$$\sum_{n=0}2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} - \sum_{n=0}\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} + \sum_{n=3}b_{n-3}x^{n+\frac{3}{2}} = 0$$

Now that all the x terms have the same exponents, we can continue.

For $n = 0$

$$2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_0x^{\frac{3}{2}} - \left(n+\frac{3}{2}\right)b_0x^{\frac{3}{2}} = 0$$

$$2\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)b_0x^{\frac{3}{2}} - \left(\frac{3}{2}\right)b_0x^{\frac{3}{2}} = 0$$

$$\frac{3}{2}b_0x^{\frac{3}{2}} - \left(\frac{3}{2}\right)b_0x^{\frac{3}{2}} = 0$$

$$0b_0 = 0$$

Hence b_0 is arbitrary.

For $n = 1$

$$2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} - \left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} = 0$$

$$2\left(1+\frac{1}{2}\right)\left(1+\frac{3}{2}\right)b_1 - \left(1+\frac{3}{2}\right)b_1 = 0$$

$$5b_1 = 0$$

$$b_1 = 0$$

For $n = 2$

$$2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} - \left(n+\frac{3}{2}\right)b_nx^{n+\frac{3}{2}} = 0$$

$$2\left(2+\frac{1}{2}\right)\left(2+\frac{3}{2}\right)b_2 - \left(2+\frac{3}{2}\right)b_2 = 0$$

$$14b_2 = 0$$

$$b_2 = 0$$

For $n \geq 3$ we use the recursive formula

$$2\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)b_n - \left(n+\frac{3}{2}\right)b_n + b_{n-3} = 0$$

$$b_n = \frac{-b_{n-3}}{n(2n+3)}$$

Hence for $n = 3$

$$b_3 = \frac{-b_0}{3(9)} = \frac{-b_0}{27}$$

For $n = 4, n = 5$ we will get $b_4 = 0$ and $b_5 = 0$ since $b_1 = 0$ and $b_2 = 0$.

For $n = 6$

$$b_6 = \frac{-b_3}{6(12+3)} = \frac{-b_3}{90} = \frac{b_0}{27(90)} = \frac{b_0}{2430}$$

For $n = 7, n = 8$ we will get $b_7 = 0$ and $b_8 = 0$ since $b_4 = 0$ and $b_5 = 0$

For $n=9$

$$a_9=\frac{-b_6}{n(2n+3)}=\frac{-b_6}{9(18+3)}=\frac{-b_6}{189}=\frac{-b_0}{2430(189)}=\frac{-b_0}{459270}$$

And so on. Hence, from $y_2(x)=\sum_{n=0} b_n x^{n+\frac{3}{2}}$ the series is

$$\begin{aligned} y_2(x) &= b_0 x^{\frac{3}{2}}-\frac{b_0}{27}x^{3+\frac{3}{2}}+\frac{b_0}{2430}x^{6+\frac{3}{2}}-\frac{b_0}{459\,270}x^{9+\frac{3}{2}}+\cdots \\ &= b_0 x^{\frac{3}{2}}\left(1-\frac{x^3}{27}+\frac{x^6}{2430}-\frac{x^9}{459270}+\cdots\right) \end{aligned}$$

The final solution is

$$y(x)=y_1(x)+y_2(x)$$

Or

$$\boxed{y(x)=a_0\left(1-\frac{1}{9}x^3+\frac{1}{486}x^6-\frac{1}{65\,610}x^9+\cdots\right)+b_0x^{\frac{3}{2}}\left(1-\frac{x^3}{27}+\frac{x^6}{2430}-\frac{x^9}{459270}+\cdots\right)} \tag{2}$$

Now comes the hard part. Finding closed form solution.

The Taylor series of $\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)$ is (using CAS)

$$\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)\approx 1-\frac{1}{9}x^3+\frac{1}{486}x^6-\frac{1}{65610}x^9+\cdots \tag{3}$$

And the Taylor series for $\sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)$ is (Using CAS)

$$\begin{aligned} \sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right) &\approx \frac{1}{3}x^{\frac{3}{2}}\sqrt{2}-\frac{1}{81}x^{\frac{9}{2}}\sqrt{2}+\frac{1}{7290}x^{\frac{15}{2}}\sqrt{2}-\cdots \\ &= x^{\frac{3}{2}}\left(\frac{1}{3}\sqrt{2}-\frac{1}{81}x^3\sqrt{2}+\frac{1}{7290}x^6\sqrt{2}-\cdots\right) \\ &= \frac{1}{3}\sqrt{2}x^{\frac{3}{2}}\left(1-\frac{1}{27}x^3+\frac{1}{2430}x^6-\cdots\right) \end{aligned} \tag{4}$$

Comparing (3,4) with (2) we see that (2) can now be written as

$$y(x)=a_0\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)+\frac{b_0}{\frac{1}{3}\sqrt{2}}\sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)$$

Or letting $c=\frac{b_0}{\frac{1}{3}\sqrt{2}}$

$$\boxed{y(x)=a_0\cos\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)+c_0\sin\left(\frac{1}{3}x^{\frac{3}{2}}\sqrt{2}\right)}$$

This is the closed form solution. The constants a_0, c_0 can be found from initial conditions.

0.6.2 part f

Problem Find series expansion of all the solutions to the following differential equation about $x=0$. Try to sum in closed form any infinite series that appear.

$$(\sin x)y''-2(\cos x)y'-(\sin x)y=0$$

Solution

In standard form

$$y''-2\left(\frac{\cos x}{\sin x}\right)y'-y=0$$

Hence the singularities are in $p(x)$ only and they occur when $\sin x=0$ or $x=0,\pm\pi,\pm 2\pi,\cdots$

but we just need to consider $x=0$. Let us check if the singularity is removable.

$$\lim_{x\to 0} xp(x) = 2\lim_{x\to 0} x\frac{\cos x}{\sin x} = 2\lim_{x\to 0} x\frac{1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots} = 2\lim_{x\to 0}\frac{1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots}{1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots} = 2$$

Hence the singularity is regular. So we can use Frobenius series

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^{n+r}\\
y'(x) &= \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}\\
y''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}
\end{aligned}$$

Substituting the above in $\sin(x)y'' - 2\cos(x)y' - \sin(x)y = 0$ results in

$$\sin(x)\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - 2\cos(x)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sin(x)\sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

Now using Taylor series for $\sin x, \cos x$ expanded around 0, the above becomes

$$\begin{aligned}
&\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}\\
&-2\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{(2m)!}\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}\\
&-\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}a_n x^{n+r} = 0 \qquad (1)
\end{aligned}$$

We need now to evaluate products of power series. Using what is called Cauchy product rule, where

$$f(x)g(x) = \left(\sum_{m=0}^{\infty} b_m x^m\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_m a_n x^{m+n} \qquad (2)$$

Applying (2) to first term in (1) gives

$$\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)(n+r-1)}{(2m+1)!}a_n x^{2m+n+r-1} \qquad (3)$$

Applying (2) to second term in (1) gives

$$\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{(2m)!}\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)}{(2m)!}a_n x^{2m+n+r-1} \qquad (4)$$

Applying (2) to the last term in (1) gives

$$\sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!}\sum_{n=0}^{\infty}a_n x^{n+r} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m a_n}{(2m+1)!}x^{2m+n+r+1} \qquad (5)$$

Substituting (3,4,5) back into (1) gives

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)(n+r-1)}{(2m+1)!}a_n x^{2m+n+r-1}\\
&-2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)}{(2m)!}a_n x^{2m+n+r-1}\\
&-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m a_n}{(2m+1)!}x^{2m+n+r+1} = 0
\end{aligned}$$

We now need to make all x exponents the same. This gives

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)(n+r-1)}{(2m+1)!}a_n x^{2m+n+r-1}\\
&-2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m(n+r)}{(2m)!}a_n x^{2m+n+r-1}\\
&-\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^m a_{n-2}}{(2m+1)!}x^{2m+n+r-1} = 0 \qquad (6)
\end{aligned}$$

The first step is to obtain the indicial equation. As the nature of the roots will tell us how to proceed. The indicial equation is obtained from setting $n = m = 0$ in (6) with the assumption that $a_0 \neq 0$. This results in

$$\frac{(-1)^m (n+r)(n+r-1)}{(2m+1)!} a_n x^{2m+n+r-1} - 2\frac{(-1)^m (n+r)}{(2m)!} a_n x^{2m+n+r-1} = 0$$
$$(r)(r-1)a_0 - 2ra_0 = 0$$
$$a_0\left(r^2 - r - 2r\right) = 0$$

Since $a_0 \neq 0$ then we obtain the indicial equation (quadratic in r)

$$r^2 - 3r = 0$$
$$r(r-3) = 0$$

Hence roots are

$$r_1 = 3$$
$$r_2 = 0$$

(it is always easier to make $r_1 > r_2$). Since $r_1 - r_2 = 3$ is now an integer, then this is case II part (b) in textbook, page 72. In this case, the two linearly independent solutions are

$$y_1(x) = x^3 \sum_{n=0} a_n x^n = \sum_{n=0} a_n x^{n+3}$$
$$y_2(x) = ky_1(x)\ln(x) + x^{r_2}\sum_{n=0} b_n x^n = ky_1(x)\ln(x) + \sum_{n=0} b_n x^n$$

Where k is some constant. Now we will find y_1. From (6), where now we set $r = 3$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m (n+3)(n+2)}{(2m+1)!} a_n x^{2m+n++2}$$
$$-2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^m (n+3)}{(2m)!} a_n x^{2m+n+2}$$
$$-\sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{(-1)^m a_{n-2}}{(2m+1)!} x^{2m+n+2} = 0$$

For $m = 0, n = 1$

$$\frac{(-1)^m (n+3)(n+2)}{(2m+1)!} a_n - 2\left(\frac{(-1)^m (n+3)}{(2m)!} a_n\right) = 0$$
$$(4)(3)a_1 - 2(4a_1) = 0$$
$$a_1 = 0$$

For $m = 0, n \geq 2$ we obtain recursive equation

$$(n+3)(n+2)a_n - 2(n+3)a_n - a_{n-2} = 0$$
$$a_n = \frac{-a_{n-2}}{(n+3)(n+2) - 2(n+3)} = \frac{-a_{n-2}}{n(n+3)}$$

Hence, for $m = 0, n = 2$

$$a_2 = \frac{-a_0}{10}$$

For $m = 0, n = 3$

$$a_3 = \frac{-a_1}{n(n+3)} = 0$$

For $m = 0, n = 4$

$$a_4 = \frac{-a_2}{4(7)} = \frac{a_0}{280}$$

For $m = 0, n = 5$

$$a_5 = 0$$

For $m = 0, n = 6$

$$a_6 = \frac{-a_4}{6(6+3)} = \frac{-a_0}{6(6+3)(280)} = \frac{-a_0}{15120}$$

For $m = 0, n = 7$, $a_7 = 0$ and for $m = 0, n = 8$

$$a_8 = \frac{-a_6}{8(8+3)} = \frac{1}{1330560} a_0$$

And so on. Since $y_1(x) = \sum_{n=0} a_n x^{n+3}$, then the first solution is now found. It is

$$
\begin{aligned}
y_1(x) &= a_0 x^3 + a_1 x^4 + a_2 x^5 + a_3 x^6 + a_4 x^7 + a_5 x^8 + a_6 x^9 + a_7 x^{10} + a_8 x^{11} + \cdots \\
&= a_0 x^3 + 0 - \frac{a_0}{10} x^5 + 0 + \frac{a_0}{280} x^7 + 0 - \frac{a_0}{15120} x^9 + 0 + \frac{1}{1330560} a_0 x^{11} + \cdots \\
&= a_0 x^3 \left(1 - \frac{x^2}{10} + \frac{x^5}{280} - \frac{x^6}{15120} + \frac{x^8}{1330560} - \cdots\right) \qquad (7)
\end{aligned}
$$

The second solution can now be found from

$$
y_2 = k y_1(x) \ln(x) + \sum_{n=0} b_n x^n
$$

I could not find a way to convert the complete solution to closed form solution, or even find closed form for $y_1(x)$. The computer claims that the closed form final solution is

$$
\begin{aligned}
y(x) &= y_1(x) + y_2(x) \\
&= a_0 \cos(x) + b_0 \left(-\sqrt{\cos^2 x - 1} + \cos x \ln\left(\cos(x) + \sqrt{\cos^2 x - 1}\right)\right)
\end{aligned}
$$

Which appears to imply that (7) is $\cos(x)$ series. But it is not. Converting series solution to closed form solution is hard. Is this something we are supposed to know how to do? Other by inspection, is there a formal process to do it?