HW2 ECE 719 Optimal systems

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Electrical engineering department
University of Wisconsin, Madison

Instructor: Professor B Ross Barmish

By

Nasser M. Abbasi

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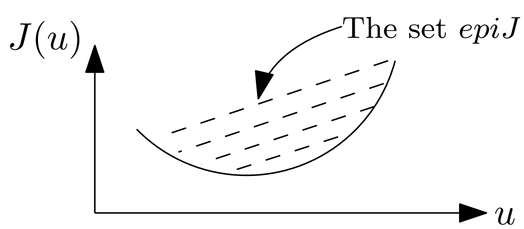
0.1 Problem 1

PROBLEM DESCRIPTION

> Barmish
>
> **ECE 719 – Homework Epigraph**
>
> Give a function $J:\mathbf{R}^n\to\mathbf{R}$, we recall that its *epigraph* is the a set in $\mathbf{R}^{n+1}$ given by
> $$epi\ J=\{(u,\alpha)\in\mathbf{R}^{n+1}:\alpha\geq J(u)\}.$$
> Now prove that J is a convex function if and only if *epi* J is a convex set.

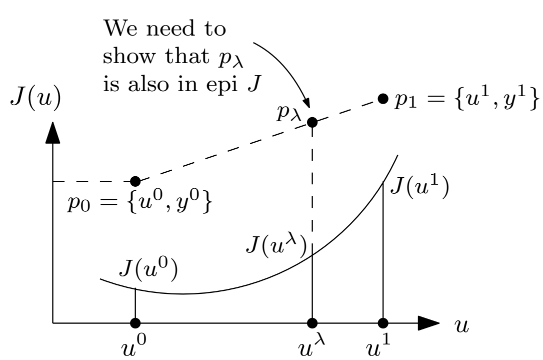
SOLUTION The following diagram illustrates epi J for $n=1$. In words, it is the set of all points above the curve of the function $J(u)$



This is an iff proof, hence we need to show the following

1. Given J is convex function, then show that epi J is a convex set.
2. Given that epi J is a convex set, then show that J is a convex function.

Proof of first direction We pick any two arbitrary points in epi J, such as $p_0=(u^0,y^0)$ and $p_1=(u^1,y^1)$. To show epi J is a convex set, we need now to show that any point on the line between p_0,p_1 is also in epi J. The point between them is given by $p_\lambda=\left(u^\lambda,y^\lambda\right)$ where $\lambda\in[0,1]$. The following diagram helps illustrates this for $n=1$.



The point p_λ is given by

$$\begin{aligned}\left(u^\lambda, y^\lambda\right) &= (1-\lambda)\, p_0 + \lambda p_1 \\ &= (1-\lambda)\left(u^0, y^0\right) + \lambda\left(u^1, y^1\right) \\ &= \left((1-\lambda)\, u^0 + \lambda u^1, (1-\lambda)\, y^0 + \lambda y^1\right)\end{aligned}$$

Therefore $y^\lambda = (1-\lambda)\, y^0 + \lambda y^1$. Since p_0, p_1 are in epi J, then by the definition of epi J, we know that $y^0 \geq J\left(u^0\right)$ and $y^1 \geq J\left(u^1\right)$. Therefore we conclude that

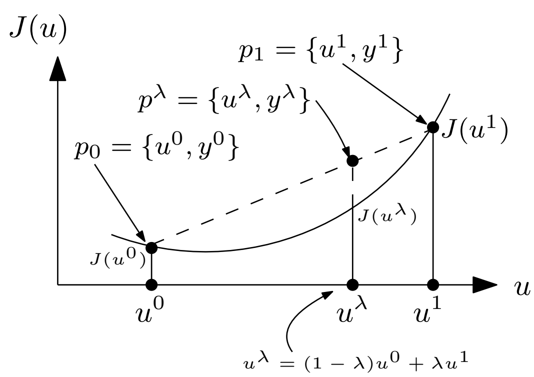
$$y^\lambda \geq (1-\lambda)\, J\left(u^0\right) + \lambda J\left(u^1\right) \tag{1}$$

But since we assumed J is a convex function, then we also know that $(1-\lambda)\, J\left(u^0\right) + \lambda J\left(u^1\right) \geq J\left(u^\lambda\right)$ where $u^\lambda = (1-\lambda)\, u^0 + \lambda u^1$. Therefore (1) becomes

$$y^\lambda \geq J\left(u^\lambda\right)$$

This implies the arbitrary point p_λ is in epi J.

We now need to proof the other direction. Given that epi J is a convex set, then show that J is a convex function. Since epi J is a convex set, we pick two arbitrary points in epi J, such as p_0, p_1. We can choose $p_0 = \left(u^0, J\left(u^0\right)\right)$ and $p_1 = \left(u^1, J\left(u^1\right)\right)$. These are still in epi J, but on the lower bound, on the edge with $J(u)$ curve.



Since p_0, p_1 are two points in a convex set, then any point p^λ on a line between them is also in epi J (by definition of a convex set). And since $p^\lambda = (1-\lambda)\, p_0 + \lambda p_1$ this implies

$$\begin{aligned}p^\lambda &= \left(u^\lambda, y^\lambda\right) \\ &= \left((1-\lambda)\, p_0 + \lambda p_1\right) \\ &= \left((1-\lambda)\left(u^0, J\left(u^0\right)\right) + \lambda\left(u^1, J\left(u^1\right)\right)\right) \\ &= \left((1-\lambda)\, u^0 + \lambda u^1, \overbrace{(1-\lambda)\, J\left(u^0\right) + J\left(u^1\right)}^{y^\lambda}\right)\end{aligned} \tag{1}$$

Since p^λ is in epi J then by definition of epi J

$$y^\lambda \geq J\left(u^\lambda\right) \tag{2}$$

But from (1) we see that $y^{\lambda} = (1-\lambda) J\left(u^0\right) + J\left(u^1\right)$, therefore (2) is the same as writing

$$(1-\lambda) J\left(u^0\right) + J\left(u^1\right) \geq J\left(u^{\lambda}\right) \tag{3}$$

But $u^{\lambda} = (1-\lambda) u^0 + \lambda u^1$, hence the above becomes

$$(1-\lambda) J\left(u^0\right) + J\left(u^1\right) \geq J\left((1-\lambda) u^0 + \lambda u^1\right)$$

But the above is the definition of a convex function. Therefore $J(u)$ is a convex function. QED.

0.2 Problem 2

PROBLEM DESCRIPTION

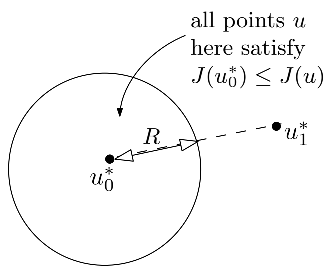
Barmish

ECE 719 – Homework Unique Minimum

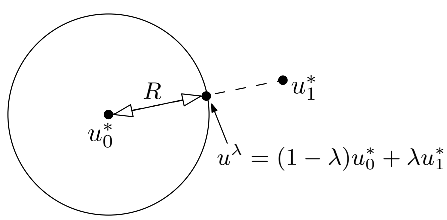
Suppose $J : \mathbf{R}^n \rightarrow \mathbf{R}$ is strictly convex. Then prove the following: If a minimizing element $u^* \in \mathbf{R}^n$ exists, it must be unique.

SOLUTION Let u_0^* and u_1^* be any two different minimizing elements in $\Re^n$ such that $J\left(u_0^*\right) < J\left(u_1^*\right)$. We will show that this leads to contradiction. Since u_0^* is a minimizer, then there exists some $R > 0$, such that all points u that satisfy $\|u^* - u\| \leq R$ also satisfy

$$J\left(u_0^*\right) \leq J(u)$$



We will consider all points along the line joining u_0^*, u_1^*, and pick one point u^λ that satisfies $\left\|u^* - u^\lambda\right\| \leq R$, where $\lambda \in [0,1]$ is selected to make the convex mixture $u^\lambda = (1-\lambda)\, u_0^* + \lambda u_1^*$ satisfied. Therefore any $\lambda \leq \frac{R}{\|u_0^* - u_1^*\|}$ will put u^λ inside the sphere of radius R.



Hence now we can say that

$$J\left(u_0^*\right) \leq J\left(u^\lambda\right) \tag{1}$$

But given that $J(u)$ is a strict convex function, then

$$J(u^\lambda) < (1-\lambda)\, J\left(u_0^*\right) + \lambda J\left(u_1^*\right) \tag{2}$$

Since we assumed that $J\left(u_0^*\right) < J\left(u_1^*\right)$, then if we replace $J\left(u_1^*\right)$ by $J\left(u_0^*\right)$ in the RHS of (2), it will change from < to ≤ resulting in

$$J(u^{\lambda}) \leq (1-\lambda) J\left(u_0^*\right) + \lambda J\left(u_0^*\right)$$
$$J(u^{\lambda}) \leq J\left(u_0^*\right) \tag{3}$$

We see that equations (3) and (1) are a contradiction. Therefore our assumption is wrong and there can not be more than one minimizing element and u_0^* must be the same as u_1^*.

0.3 Problem 3

PROBLEM DESCRIPTION

Barmish

ECE 719 – Homework Global Minimum

Preamble: Suppose $J : \mathbf{R}^n \rightarrow \mathbf{R}$. A point $u^* \in \mathbf{R}^n$ is said to be a *local minimum of* J if there exists some suitably small $\delta > 0$ leading to satisfaction of the following condition:

$$J(u^*) \leq J(u)$$

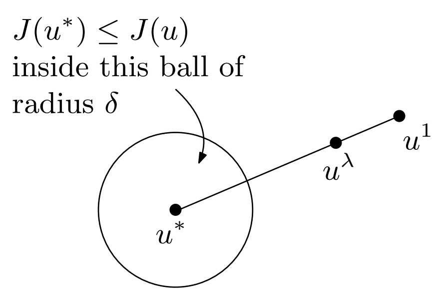
for all u such that $||u - u^*|| < \delta$. Said another way, u^* is a minimizing element over a suitably small open neighborhood. For the case when $J(u^*) \leq J(u)$ for all u, we call u^* a *global minimum* of J.

The Homework Problem: Suppose $J : \mathbf{R}^n \rightarrow \mathbf{R}$ is convex. Prove that every local minimum of J is a global minimum.

SOLUTION We are given that $J(u^*) \leq J(u)$ for all u such that $||u^* - u|| < \delta$. Let us pick any arbitrary point u^1, outside ball of radius δ. Then any point on the line between u^* and u^1 is given by

$$u^\lambda = (1 - \lambda)\, u^* + \lambda u^1$$

In picture, so far we have this setup

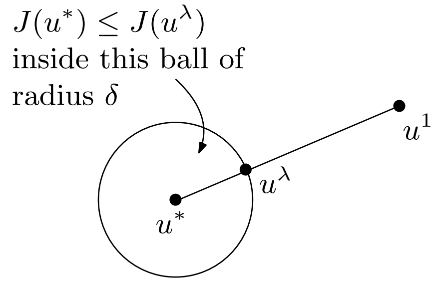


We now need to show that $J(u^*) \leq J\left(u^1\right)$ even though u^1 is outside the ball. Since J is a convex function, then

$$J\left(u^\lambda\right) \leq (1 - \lambda)\, J(u^*) + \lambda J\left(u^1\right) \tag{1}$$

We can now select λ to push u^λ to be inside the ball. We are free to change λ as we want while keeping u^1 fixed, outside the ball. If we do this we then we have

$$J(u^*) \leq J\left(u^\lambda\right)$$



Hence (1) becomes

$$J(u^*) \leq (1-\lambda) J(u^*) + \lambda J\left(u^1\right) \tag{2}$$

Where we replaced $J\left(u^\lambda\right)$ by $J(u^*)$ in (1) and since $J(u^*) \leq J\left(u^\lambda\right)$ the $\leq$ relation remained valid. Simplifying (2) gives

$$J(u^*) \leq J(u^*) - \lambda J(u^*) + \lambda J\left(u^1\right)$$
$$\lambda J(u^*) \leq \lambda J\left(u^1\right)$$

For non-zero λ this means $J(u^*) \leq J\left(u^1\right)$. This completes the proof, since u^1 was arbitrary point anywhere. Hence u^* is global minimum. QED

0.4 Problem 4

PROBLEM DESCRIPTION

Barmish

ECE 719 – Homework Multiple Combinations

For a convex function $J:\mathbf{R}^n \to \mathbf{R}$, prove the following condition is satisfied: Given any points $u^1, u^2, ..., u^N \in \mathbf{R}^n$ and any scalars $\lambda_1, \lambda_2, ..., \lambda_N \geq 0$ such that

$$\sum_{i=1}^{N} \lambda_i = 1,$$

it follows that

$$J\left(\sum_{i=1}^{N} \lambda_i u^i\right) \leq \sum_{i=1}^{N} \lambda_i J(u^i).$$

SOLUTION

We need to show that $J\left(\sum_{i=1}^{N}\lambda_i u^i\right) \leq \sum_{i=1}^{N}\lambda_i J\left(u^i\right)$ where $\sum_{i=1}^{N}\lambda_i = 1$. Proof by induction. For $N = 1$ and since $\lambda_1 = 1$, then we have

$$J\left(u^1\right) = J\left(u^1\right)$$

The case for $N = 2$ comes for free, from the definition of J being a convex function

$$J\left((1-\lambda)\,u^1 + \lambda u^2\right) \leq (1-\lambda)\,J\left(u^1\right) + \lambda J\left(u^2\right) \tag{A}$$

By making $(1-\lambda) \equiv \lambda_1, \lambda \equiv \lambda_2$, the above can be written as

$$J\left(\lambda_1 u^1 + \lambda_2 u^2\right) \leq \lambda_1 J\left(u^1\right) + \lambda_2 J\left(u^2\right)$$

We now assume it is true for $N = k-1$. In other words, the inductive hypothesis below is given as true

$$J\left(\sum_{i=1}^{k-1}\lambda_i u^i\right) \leq \sum_{i=1}^{k-1}\lambda_i J\left(u^i\right) \tag{*}$$

Now we have to show it will also be true for $N = k$, which is

$$\begin{aligned}
\sum_{i=1}^{k}\lambda_i J\left(u^i\right) &= \lambda_1 J\left(u^1\right) + \lambda_1 J\left(u^1\right) + \cdots + \lambda_k J\left(u^k\right)\\
&= (1-\lambda_k)\left(\frac{\lambda_1}{(1-\lambda_k)}J\left(u^1\right) + \frac{\lambda_1}{(1-\lambda_k)}J\left(u^1\right) + \cdots + \frac{\lambda_{k-1}}{(1-\lambda_k)}J\left(u^{k-1}\right) + \frac{\lambda_k}{(1-\lambda_k)}J\left(u^k\right)\right)\\
&= (1-\lambda_k)\left(\frac{\lambda_1}{(1-\lambda_k)}J\left(u^1\right) + \frac{\lambda_1}{(1-\lambda_k)}J\left(u^1\right) + \cdots + \frac{\lambda_{k-1}}{(1-\lambda_k)}J\left(u^{k-1}\right)\right) + \lambda_k J\left(u^k\right)\\
&= (1-\lambda_k)\left(\sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}J\left(u^i\right)\right) + \lambda_k J\left(u^k\right) \qquad (1)
\end{aligned}$$

Now we take advantage of the inductive hypothesis Eq. (*) on $k-1$, which says that

$J\left(\sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}u^i\right) \leq \sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}J\left(u^i\right)$. Using this in (1) changes it to $\geq$ relation

$$\sum_{i=1}^{k}\lambda_i J\left(u^i\right) \geq (1-\lambda_k)\, J\left(\sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}u^i\right) + \lambda_k J\left(u^k\right) \tag{2}$$

We now take advantage of the case of $N=2$ in (A) by viewing RHS of (2) as $(1-\lambda_k)\, J\left(u^1\right) + \lambda_k J\left(u^2\right)$, where we let $u^1 \equiv \sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}u^i, u^2 \equiv u^k$. Hence we conclude that

$$(1-\lambda_k)\, J\left(\sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}u^i\right) + \lambda_k J\left(u^k\right) \geq J\left((1-\lambda_k)\sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}u^i + \lambda_k u^k\right) \tag{3}$$

Using (3) in (2) gives (the $\geq$ relation remains valid, even more now, since we replaced something in RHS of (2), by something smaller)

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i J\left(u^i\right) &\geq J\left((1-\lambda_k)\sum_{i=1}^{k-1}\frac{\lambda_i}{(1-\lambda_k)}u^i + \lambda_k u^k\right)\\ &= J\left(\left(\sum_{i=1}^{k-1}\lambda_i u^i\right) + \lambda_k u^k\right)\end{aligned}$$

Hence

$$\sum_{i=1}^{k}\lambda_i J\left(u^i\right) \geq J\left(\sum_{i=1}^{k}\lambda_i u^i\right)$$

QED.

0.5 Problem 5

PROBLEM DESCRIPTION

Barmish

ECE 719 – Homework Hessian

For $u \in \mathbf{R}^n$, define

$$J(u) = -(u_1 u_2 u_3 \cdots u_n)^{1/n}.$$

Prove that $J(u)$ is convex on the positive orthant; i.e., the set defined by $u_i > 0$ for $i = 1, 2, ..., n$.

SOLUTION

Assuming $J(u)$ is twice continuously differentiable (C^2) in $u_1, u_2, \cdots, u_n$, then if we can show that the Hessian $\nabla^2 J(u)$ is positive semi-definite on $u_i > 0$, then this implies $J(u)$ is convex. The first step is to determined $\nabla^2 J(u)$.

$$\begin{aligned}
\frac{\partial J}{\partial u_i} &= -\frac{1}{n}(u_1 u_2 \cdots u_n)^{\frac{1}{n}-1} \prod_{k=1,k\neq i}^{n} u_k = \frac{1}{n}\frac{J(u)}{(u_1 u_2 \cdots u_n)} \prod_{k=1,k\neq i}^{n} u_k = \frac{1}{n}\frac{J(u)}{\prod_{k=1}^{n} u_k} \prod_{k=1,k\neq i}^{n} u_k \\
&= \frac{1}{n}\frac{J(u)}{u_i}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial^2 J}{\partial u_i^2} &= \frac{1}{n}\frac{\left(\frac{1}{n}\frac{J(u)}{u_i}\right)}{u_i} - \frac{1}{n}\frac{J(u)}{u_i^2} \\
&= \frac{1}{n^2}\frac{J(u)}{u_i^2} - \frac{1}{n}\frac{J(u)}{u_i^2} \\
&= \frac{1}{n}\frac{J(u)}{u_i^2}\left(\frac{1}{n} - 1\right)
\end{aligned}$$

And the cross derivatives are

$$\begin{aligned}
\frac{\partial^2 J}{\partial u_i \partial u_j} &= \frac{\partial}{\partial u_j}\left(\frac{1}{n}\frac{J(u)}{u_i}\right) \\
&= \frac{1}{n}\frac{\frac{1}{n}\frac{J(u)}{u_j}}{u_i} \\
&= \frac{1}{n^2}\frac{J(u)}{u_i u_j}
\end{aligned}$$

Therefore

$$\nabla^2 J(u) = \begin{pmatrix} \frac{1}{n^2}\frac{J(u)}{u_1^2}(1-n) & \frac{1}{n^2}\frac{J(u)}{u_1 u_2} & \cdots & \frac{1}{n^2}\frac{J(u)}{u_1 u_n} \\ \frac{1}{n^2}\frac{J(u)}{u_2 u_1} & \frac{1}{n^2}\frac{J(u)}{u_2^2}(1-n) & \cdots & \frac{1}{n^2}\frac{J(u)}{u_2 u_n} \\ \vdots & \cdots & \ddots & \vdots \\ \frac{1}{n^2}\frac{J(u)}{u_n u_1} & \frac{1}{n^2}\frac{J(u)}{u_n u_2} & \cdots & \frac{1}{n^2}\frac{J(u)}{u_n^2}(1-n) \end{pmatrix}$$

Now we need to show that $\nabla^2 J(u)$ is positive semi-definite. For $n = 1$, the above reduces to

$$\nabla^2 J(u) = \frac{J(u)}{u_1^2}(1-1) = 0$$

Hence non-negative. This is the same as saying the second derivative is zero. For $n = 2$

$$\nabla^2 J(u) = \begin{pmatrix} \frac{1}{4}J(u)\frac{1-2}{u_1^2} & \frac{1}{u_1 u_2}\frac{1}{4}J(u) \\ \frac{1}{u_2 u_1}\frac{1}{4}J(u) & \frac{1}{4}J(u)\frac{1-2}{u_2^2} \end{pmatrix} = \begin{pmatrix} \frac{-1}{u_1^2}\frac{1}{4}J(u) & \frac{1}{u_1 u_2}\frac{1}{4}J(u) \\ \frac{1}{u_2 u_1}\frac{1}{4}J(u) & \frac{-1}{u_2^2}\frac{1}{4}J(u) \end{pmatrix}$$

The first leading minor is $\frac{-1}{4u_1^2}J(u)$, which is positive, since $J(u) < 0$ and $u_i > 0$ (given). The second leading minor is

$$\Delta_2 = \begin{vmatrix} \frac{-1}{u_1^2}\frac{1}{4}J(u) & \frac{1}{u_1 u_2}\frac{1}{4}J(u) \\ \frac{1}{u_2 u_1}\frac{1}{4}J(u) & \frac{-1}{u_2^2}\frac{1}{4}J(u) \end{vmatrix} = 0$$

Hence all the leasing minors are non-negative. Which means $\nabla^2 J(u)$ is semi-definite. We will look at $n = 3$

$$\nabla^2 J(u) = \begin{pmatrix} \frac{-2}{u_1^2}\frac{1}{9}J(u) & \frac{1}{u_1 u_2}\frac{1}{9}J(u) & \frac{1}{u_1 u_3}\frac{1}{9}J(u) \\ \frac{1}{u_2 u_1}\frac{1}{9}J(u) & \frac{-2}{u_2^2}\frac{1}{9}J(u) & \frac{1}{u_2 u_3}\frac{1}{9}J(u) \\ \frac{1}{u_3 u_1}\frac{1}{9}J(u) & \frac{1}{u_3 u_2}\frac{1}{9}J(u) & \frac{-2}{u_3^2}\frac{1}{9}J(u) \end{pmatrix}$$

The first leading minor is $\frac{-2}{9u_1^2}J(u)$, which is positive again, since $J(u) < 0$ for $u_i > 0$ (given). And the second leading minor is $\frac{1}{27}J^2\frac{u^2}{u_1^2 u_2^2}$

which is positive, since all terms are positive. The third leading minor is

$$\Delta_3 = \begin{vmatrix} \frac{-2}{u_1^2}\frac{1}{9}J(u) & \frac{1}{u_1 u_2}\frac{1}{9}J(u) & \frac{1}{u_1 u_3}\frac{1}{9}J(u) \\ \frac{1}{u_2 u_1}\frac{1}{9}J(u) & \frac{-2}{u_2^2}\frac{1}{9}J(u) & \frac{1}{u_2 u_3}\frac{1}{9}J(u) \\ \frac{1}{u_3 u_1}\frac{1}{9}J(u) & \frac{1}{u_3 u_2}\frac{1}{9}J(u) & \frac{-2}{u_3^2}\frac{1}{9}J(u) \end{vmatrix} = 0$$

Hence non-of the leading minors are negative. Therefore $\nabla^2 J(u)$ is semi-definite. The same pattern repeats for higher values of n. All leading minors are positive, except the last leading minor will be zero.

0.5.1 Appendix

Another way to show that $\nabla^2 J(u)$ is positive semi-definite is to show that $x^T\left(\nabla^2 J(u)\right)x \geq 0$ for any vector x. (since $\nabla^2 J(u)$ is symmetric).

$$x^T\left(\nabla^2 J(u)\right)x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \frac{1}{n}\frac{J(u)}{u_1^2}\left(\frac{1}{n}-1\right) & \frac{1}{n^2}\frac{J(u)}{u_1 u_2} & \cdots & \frac{1}{n^2}\frac{J(u)}{u_1 u_n} \\ \frac{1}{n^2}\frac{J(u)}{u_2 u_1} & \frac{1}{n}\frac{J(u)}{u_2^2}\left(\frac{1}{n}-1\right) & \cdots & \frac{1}{n^2}\frac{J(u)}{u_2 u_n} \\ \vdots & \cdots & \ddots & \vdots \\ \frac{1}{n^2}\frac{J(u)}{u_n u_1} & \frac{1}{n^2}\frac{J(u)}{u_n u_2} & \cdots & \frac{1}{n}\frac{J(u)}{u_n^2}\left(\frac{1}{n}-1\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Now the idea is to set $n = 1, 2, 3, \cdots$ and show that the resulting values ≥ 0 always. For $n = 1$, we obtain 0 as before. For $n = 2$, let

$\Delta = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} \frac{1}{n}\frac{J(u)}{u_1^2}\left(\frac{1}{n}-1\right) & \frac{1}{n^2}\frac{J(u)}{u_1 u_2} \\ \frac{1}{n^2}\frac{J(u)}{u_2 u_1} & \frac{1}{n}\frac{J(u)}{u_2^2}\left(\frac{1}{n}-1\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Expanding gives

$$\begin{aligned} \Delta &= \begin{pmatrix} x_1 \frac{1}{n}\frac{J(u)}{u_1^2}\left(\frac{1}{n}-1\right) + x_2 \frac{1}{n^2}\frac{J(u)}{u_2 u_1} & x_1 \frac{1}{n^2}\frac{J(u)}{u_1 u_2} + x_2 \frac{1}{n}\frac{J(u)}{u_2^2}\left(\frac{1}{n}-1\right) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= x_1^2 \frac{1}{n}\frac{J(u)}{u_1^2}\left(\frac{1}{n}-1\right) + x_1 x_2 \frac{1}{n^2}\frac{J(u)}{u_2 u_1} + x_2 x_1 \frac{1}{n^2}\frac{J(u)}{u_1 u_2} + x_2^2 \frac{1}{n}\frac{J(u)}{u_2^2}\left(\frac{1}{n}-1\right) \\ &= x_1^2 \frac{1}{2}\frac{J(u)}{u_1^2}\left(\frac{1}{2}-1\right) + x_1 x_2 \frac{1}{4}\frac{J(u)}{u_2 u_1} + x_2 x_1 \frac{1}{4}\frac{J(u)}{u_1 u_2} + x_2^2 \frac{1}{2}\frac{J(u)}{u_2^2}\left(\frac{1}{2}-1\right) \end{aligned}$$

The RHS above becomes, and by replacing $J(u) = -\sqrt{u_1 u_2}$ for $n = 2$

$$\begin{aligned} -\frac{1}{4}x_1^2\frac{J(u)}{u_1^2} + x_1 x_2 \frac{1}{2}\frac{J(u)}{u_2 u_1} - \frac{1}{4}x_2^2\frac{J(u)}{u_2^2} &= \frac{1}{4}x_1^2\frac{\sqrt{u_1 u_2}}{u_1^2} - x_1 x_2 \frac{1}{2}\frac{\sqrt{u_1 u_2}}{u_2 u_1} + \frac{1}{4}x_2^2\frac{\sqrt{u_1 u_2}}{u_2^2} \\ &= \left(\frac{1}{\sqrt{4}}\frac{(u_1 u_2)^{\frac{1}{4}}}{u_1}x_1 - \frac{1}{\sqrt{4}}\frac{(u_1 u_2)^{\frac{1}{4}}}{u_2}x_2\right)^2 \end{aligned}$$

Where we completed the square in the last step above. Hence $x^T\left(\nabla^2 J(u)\right)x \geq 0$. The same process can be continued for n higher. Hence $\nabla^2 J(u)$ is positive semi-definite.