HW2 ME 739 Introduction to robotics

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Department of Mechanical Engineering
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Contents

List of Tables

0.1 Problem 1

Problem 1.

Frame {A} and frame {B} are initially coincident. Frame {B} is rotated through the following sequence of rotations:

1. $+\alpha$ degree rotation about $\hat{z}_A$
2. $-\beta$ degree rotation about $\hat{x}_B$
3. $-\alpha$ degree rotation about $\hat{z}_A$
4. $+\gamma$ degree rotation about $\hat{y}_B$
5. $+\beta$ degree rotation about $\hat{x}_A$

where $\alpha = +45$ degrees; $\beta = -30$ degrees; $\gamma = +90$ degrees

Evaluate the rotation transformation (matrix) that describes the orientation of frame {B} relative to frame {A} following this sequence of rotations.

When the rotation R_i is around a fixed frame, it is pre-multiplied by the current sequence of rotations. If the rotation R_i is around the current frame, it is post-multiplied by the current sequence of rotations.

1. $R = R_{z,\alpha}$
2. Since rotation is around current frame, it is post multiplied giving $R = R_{z,\alpha} R_{x,-\beta}$
3. Since rotation is around fixed frame, it is pre-multiplied, giving $R = R_{z,-\alpha} R_{z,\alpha} R_{x,-\beta}$
4. Since rotation now is about current frame, it is post multiplied giving $R = R_{z,-\alpha} R_{z,\alpha} R_{x,-\beta} R_{y,\gamma}$
5. Since rotation now is about fixed frame, it is pre-multiplied giving $R = R_{x,\beta} R_{z,-\alpha} R_{z,\alpha} R_{x,-\beta} R_{y,\gamma}$

Now that the rotation sequence is completed, the sequence of rotations are evaluated. Before that, some simplification is made as follows

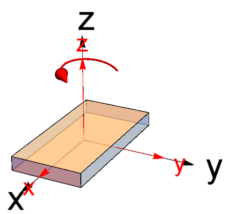
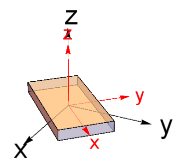
$$
\begin{aligned}
R &= R_{x,\beta} \overbrace{R_{z,-\alpha} R_{z,\alpha}}^{I} R_{x,-\beta} R_{y,\gamma} \\
&= \overbrace{R_{x,\beta} R_{x,-\beta}}^{I} R_{y,\gamma} \\
&= R_{y,\gamma}
\end{aligned}
$$

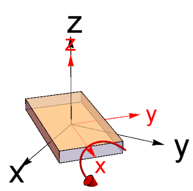
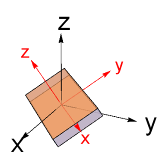
Therefore

$$
R = R_{y,\gamma} = \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} = \begin{pmatrix} \cos 90^0 & 0 & \sin 90^0 \\ 0 & 1 & 0 \\ -\sin 90^0 & 0 & \cos 90^0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}
$$

Only angle γ was needed in finding the final result. The above result shows that the final orientation is as if one rotation of $+90$ degree was made around the fixed y axes. The following diagram shows the result after each rotation, which confirms the above result.[1]

[1] Source code that generated these plot is in the appendix if needed.

+45 around fixed z	after rotation	Final rotation matrix
		$\begin{pmatrix} 0.707107 & -0.707107 & 0. \\ 0.707107 & 0.707107 & 0. \\ 0. & 0. & 1. \end{pmatrix}$

30° around current x	after rotation	Final rotation matrix
		$\begin{pmatrix} 0.707107 & -0.612372 & 0.353553 \\ 0.707107 & 0.612372 & -0.353553 \\ 0. & 0.5 & 0.866025 \end{pmatrix}$

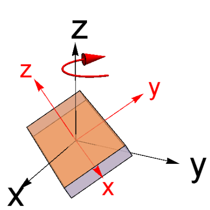
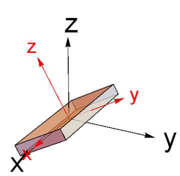
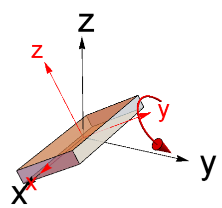
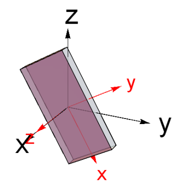
-45° around fixed z	after rotation	Final rotation matrix
		$\begin{pmatrix} 1. & 0. & 0. \\ 0. & 0.866025 & -0.5 \\ 0. & 0.5 & 0.866025 \end{pmatrix}$

Figure 1: Graphical representation of problem 1 rotations

+90 around current y	after rotation	Final rotation matrix
		$\begin{pmatrix} 0. & 0. & 1. \\ 0.5 & 0.866025 & 0. \\ -0.866025 & 0.5 & 0. \end{pmatrix}$

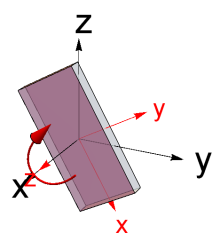
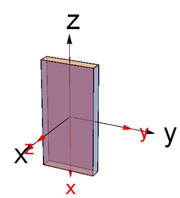
-30° around fixed x	after rotation	Final rotation matrix
		$\begin{pmatrix} 0 & 0 & 1. \\ 0 & 1. & 0 \\ -1. & 0 & 0 \end{pmatrix}$

Figure 2: Graphical representation of problem 1 rotations

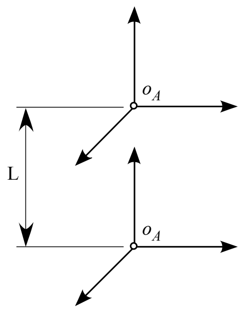
0.2 Problem 2

Problem 2.

The D-H parameters (*d*, *a*, *α*, and *θ*) and the homogeneous transformation which results (see below – also in Kinematics lecture notes) cannot be used to represent a *general* rigid-body transformation.

Homogeneous transformation matrix using DH convention:

$$T = \begin{bmatrix} c_\theta & -s_\theta c_\alpha & s_\theta s_\alpha & a c_\theta \\ s_\theta & c_\theta c_\alpha & -c_\theta s_\alpha & a s_\theta \\ 0 & s_\alpha & c_\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



► Explain why this is the case. You can use physical and/or mathematical arguments.

► For the rigid-body transformation shown to the right, label the unit vectors of the two reference frames such that the D-H parameters cannot describe their relative transformation.

0.2.1 Part a

The homogeneous transformation matrix based on the use of the DH convention contains only 4 parameters d, a, α, θ. Since a general rigid body transformation requires 6 parameters (3 angles for orientations, and 3 for translation), this implies the DH homogeneous transformation matrix can not be used to represent any arbitrary rigid body transformation. However, the DH convention can be used to represent any rigid body transformation that meets two conditions, as specified on page 78 of the text book. These are

1. The x_i axis of the i^{th} frame is $\perp$ to the z_{i-1} axis of the $i-1$ frame.
2. The x_i axis of the i^{th} frame intersects the z_{i-1} axis of the $i-1$ frame

0.2.2 Part b

At least one of the above two DH constraints need to be violated in order to come up with a configuration that cannot be described using the DH transformation matrix. This is done by making x_1 axis not perpendicular to z_0 axis. The following diagram illustrates this

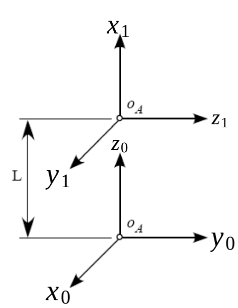
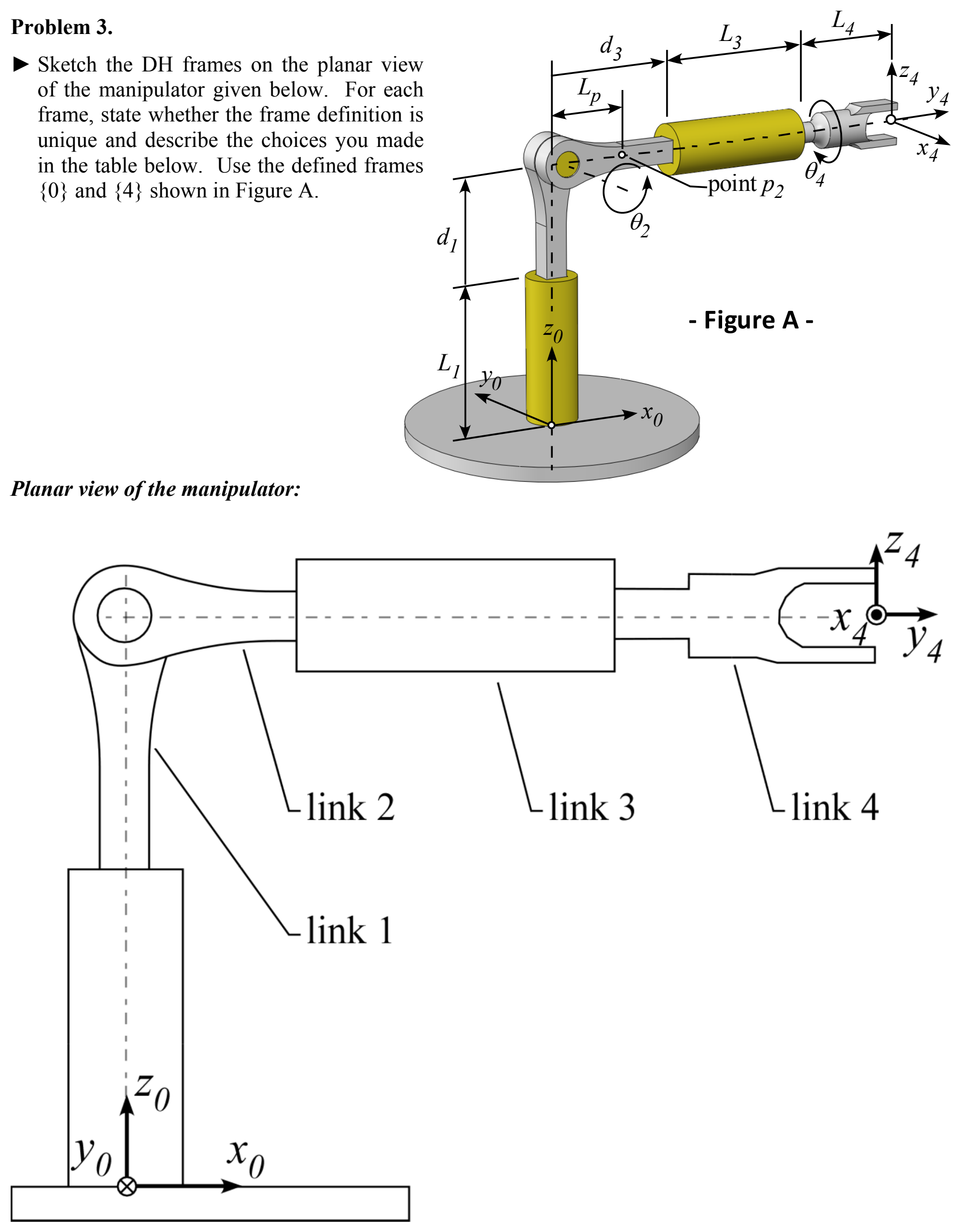


Figure 3: Problem 2 setup of axis

0.3 Problem 3


Problem 3.

► Sketch the DH frames on the planar view of the manipulator given below. For each frame, state whether the frame definition is unique and describe the choices you made in the table below. Use the defined frames {0} and {4} shown in Figure A.

Planar view of the manipulator:

Figure 4: Problem 3 description

The first step is to assign the z_i axis for each link. There are 5 links numbered $0, 1, 2, 3, 4$ and four joints numbered $1, 2, 3, 4$. Link 0 is fixed and does not move. Frame i is attached to link i but its z_i axis is oriented along the line of motion of joint $i+1$. Once all the z axis are established, then the rest of the frames are configured by insuring that that each x axis is perpendicular to the last frame z axis.

So the rule to follow is that x_i axis must be perpendicular to z_{i-1} axis.

The frames are drawn below. The following choices were made. z_0 intersects z_1 (case 3 in the textbook), hence x_1 is arbitrary. The origin o_1 is the point of intersection of z_1 and z_0 as shown in the problem statement.

Next, z_1 and z_2 also intersect (case 3). The choice of x_2 is arbitrary. The origin o_2 could be located also where z_1 intersect z_2. But any point along z_2 will also work. Here o_2 was placed at point p_2.

Next, z_2 is parallel to z_3. This is case (2) in the textbook. There are infinitely many common normals. Here origin o_3 can be anywhere along z_3. It is placed at start of the end effector as shown.

The following diagram shows the frames locations.

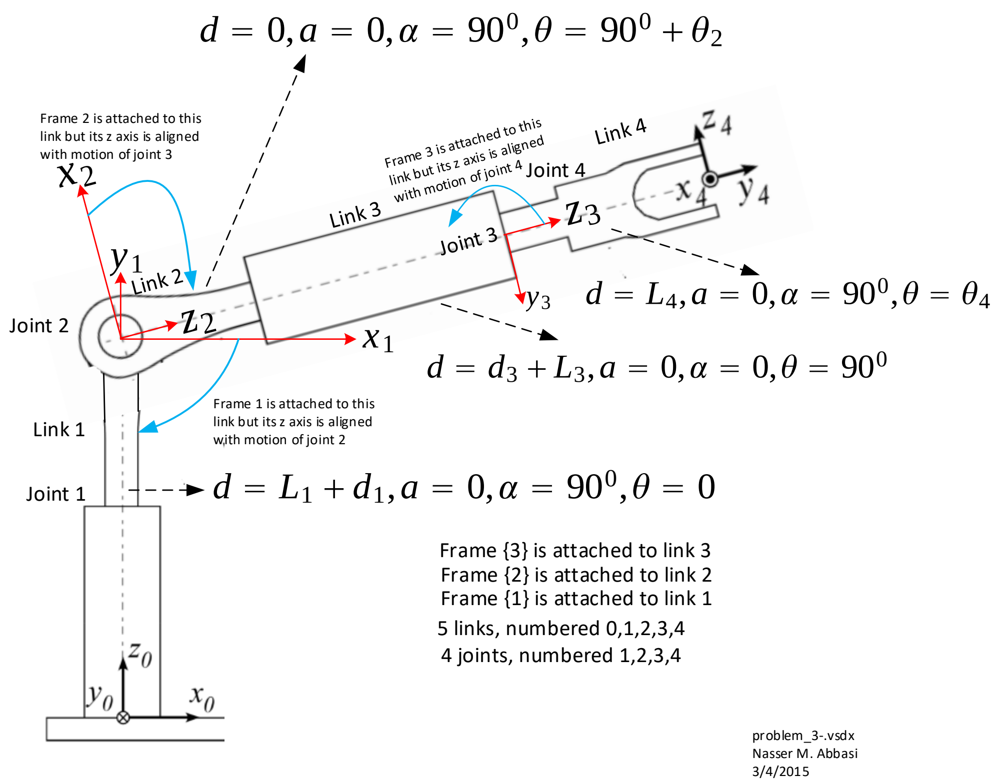


Figure 5: Frames assignments for problem 3

0.4 Problem 4

Evaluate the DH parameters for the manipulator and frame definitions developed in the previous problem

i	θ_i	d_i	a_i	α_i
1	0	$L_1 + d_1$	0	90^0
2	$90^0 + \theta_2$	0	0	90^0
3	90^0	$L_3 + d_3$	0	0
4	θ_4	L_4	0	90^0

The homogeneous transformation matrices T_i^{i-1} are now evaluated, and then T_4^0 is found in order to verify that the location of the end effector. To verify, when $\theta_2 = 0, \theta_4 = 0$, then the end effector x, y, z position relative to the base frame should be located at

$$
\begin{aligned}
x &= d_3 + L_3 + L_4 \\
y &= 0 \\
z &= L_1 + d_1
\end{aligned}
$$

These are The homogeneous transformation matrices T_i^{i-1}

$$
T_1^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_1 + L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
T_2^1 = \begin{pmatrix} -\sin\theta_2 & 0 & \cos\theta_2 & 0 \\ \cos\theta_2 & 0 & \sin\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
T_3^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_3 + L_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$$
T_4^3 = \begin{pmatrix} \cos\theta_4 & 0 & \sin\theta_4 & 0 \\ \sin\theta_4 & 0 & -\cos\theta_4 & 0 \\ 0 & 1 & 0 & L_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

$T_4^0 = T_1^0 T_2^1 T_3^2 T_4^3$ was found and evaluated for $\theta_2 = 0$. The result is

$$
\boxed{\begin{pmatrix} 0 & 1 & 0 & d_3 + L_3 + L_4 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_1 + L_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}}
$$

Which shows the correct position vector of the end effector frame.

0.5 Problem 5

Problem 5

► Write a Matlab function which takes as its input the DH parameters and returns the associated 4x4 homogeneous transformation matrix, *T*. A possible function prototype is given below. To learn how to write a `function` in Matlab, type `help function` in the Matlab workspace.

```
function T = DH2T(d, a, alpha, theta);
```

► Write a Matlab script to verify that your function is working properly. The script should evaluate the *T* matrix for the set of DH parameters listed in the adjacent table. When your script is run, the *T* matrices should be printed to the workspace. This script, along with your function, are to be handed in to the course dropbox on the Learn@UW course page. As a possible starting point, your script might look like:

Link	d_i	a_i	α_i	θ_i
Case 1:	0	10	0	0
Case 2:	10	0	0	0
Case 3:	10	0	π	0
Case 4:	0	0	π	π

```
clear all; close all; clc
% Case 1:
d = 0;  a = 10; alpha = 0;  theta = 0;
T1 = DH2T(a,alpha,d,theta)
% Case 2:
d = 10; a = 0;  alpha = 0;  theta = 0;
T2 = DH2T(a,alpha,d,theta)
% Case 3:
d = 10; a = 0;  alpha = pi; theta = 0;
T3 = DH2T(a,alpha,d,theta)
% Case 4:
d = 0;  a = 0;  alpha = pi; theta = pi;
T4 = DH2T(a,alpha,d,theta)
```

Figure 6: Problem 5 description

0.5.1 part a

The following is the Matlab function **DH2T**

```
function T = DH2T(a,alpha,d,theta)
%generated the transformation matrix for a DH table row
%ME 739, Univ. Of Wisconsin, Madison. Spring 2015
T=[cos(theta)  -sin(theta)*cos(alpha)  sin(theta)*sin(alpha)   a*cos(theta);
   sin(theta)  cos(theta)*cos(alpha)   -cos(theta)*sin(alpha)  a*sin(theta);
   0           sin(alpha)              cos(alpha)              d;
   0           0                       0                       1
   ];
end
```

0.5.2 part b

The following is the script used and below it is the output generated.

```
clear all; close all; clc
%case 1
d=0; a=10; alpha=0; theta=0;
T1=DH2T(a,alpha,d,theta)

%case 2
d=10; a=10; alpha=0; theta=0;
T2=DH2T(a,alpha,d,theta)

%case 3
d=10; a=10; alpha=pi; theta=0;
T3=DH2T(a,alpha,d,theta)

%case 4
d=0; a=10; alpha=pi; theta=pi;
T4=DH2T(a,alpha,d,theta)
```

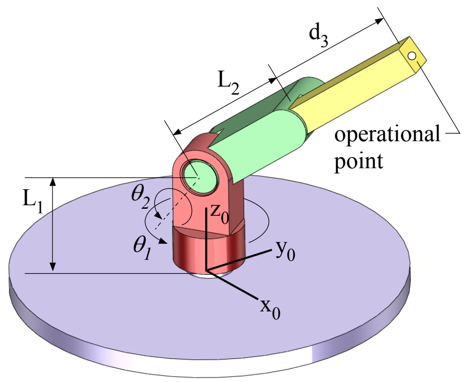
```
T1 =
   1     0     0    10
   0     1     0     0
   0     0     1     0
   0     0     0     1

T2 =
   1     0     0    10
   0     1     0     0
   0     0     1    10
   0     0     0     1

T3 =
   1             0             0            10
   0            -1   -1.2246e-16             0
   0    1.2246e-16            -1            10
   0             0             0             1

T4 =
   -1    1.2246e-16    1.4998e-32           -10
   1.2246e-16             1    1.2246e-16    1.2246e-15
   0    1.2246e-16            -1             0
   0             0             0             1
```

0.6 Problem 6



Problem 6.

For the manipulator shown, the forward kinematics are given as:

$$x_e = \cos\theta_1 \cos\theta_2 (L_2 + d_3)$$
$$y_e = \sin\theta_1 \cos\theta_2 (L_2 + d_3)$$
$$z_e = L_1 + \sin\theta_2 (L_2 + d_3)$$

► Develop the linear velocity Jacobian, J_v, using direct differentiation of the forward kinematics. Assume that the task and joint variable vectors are defined as

$$q = [\theta_1 \quad \theta_2 \quad d_3]^T$$
$$x = [x_e \quad y_e \quad z_e]^T$$

► For the manipulator configuration defined by the joint vector, $q = [\pi \quad \frac{1}{2}\pi \quad L_2]^T$, evaluate the linear velocity of the end-effector given the joint velocity vector $\dot{q} = [1 \quad 0 \quad 1]^T$.

► For what values of the joint variables is the linear velocity Jacobian, J_v, singular? Use physical and/or mathematical arguments to support your answer.

Figure 7: Problem 6 description

0.6.1 Part a

$$J_v = \begin{pmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial d_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial d_3} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \frac{\partial z}{\partial d_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial \theta_1} \cos\theta_1 \cos\theta_2 (L_2 + d_3) & \frac{\partial}{\partial \theta_2} \cos\theta_1 \cos\theta_2 (L_2 + d_3) & \frac{\partial}{\partial d_3} \cos\theta_1 \cos\theta_2 (L_2 + d_3) \\ \frac{\partial}{\partial \theta_1} \sin\theta_1 \cos\theta_2 (L_2 + d_3) & \frac{\partial}{\partial \theta_2} \sin\theta_1 \cos\theta_2 (L_2 + d_3) & \frac{\partial}{\partial d_3} \sin\theta_1 \cos\theta_2 (L_2 + d_3) \\ \frac{\partial}{\partial \theta_1} (L_1 + \sin\theta_2 (L_2 + d_3)) & \frac{\partial}{\partial \theta_2} (L_1 + \sin\theta_2 (L_2 + d_3)) & \frac{\partial}{\partial d_3} (L_1 + \sin\theta_2 (L_2 + d_3)) \end{pmatrix}$$

$$= \begin{pmatrix} -\sin\theta_1 \cos\theta_2 (L_2 + d_3) & -\cos\theta_1 \sin\theta_2 (L_2 + d_3) & \cos\theta_1 \cos\theta_2 \\ \cos\theta_1 \cos\theta_2 (L_2 + d_3) & -\sin\theta_1 \sin\theta_2 (L_2 + d_3) & \sin\theta_1 \cos\theta_2 \\ 0 & \cos\theta_2 (L_2 + d_3) & \sin\theta_2 \end{pmatrix}$$

0.6.2 Part b

$$
\begin{aligned}
v &= J_v\dot{q} \\
&= \begin{pmatrix} -\sin\theta_1\cos\theta_2\,(L_2+d_3) & -\cos\theta_1\sin\theta_2\,(L_2+d_3) & \cos\theta_1\cos\theta_2 \\ \cos\theta_1\cos\theta_2\,(L_2+d_3) & -\sin\theta_1\sin\theta_2\,(L_2+d_3) & \sin\theta_1\cos\theta_2 \\ 0 & \cos\theta_2\,(L_2+d_3) & \sin\theta_2 \end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} \\
&= \begin{pmatrix} -\sin\pi\cos\frac{\pi}{2}\,(L_2+L_2) & -\cos\pi\sin\frac{\pi}{2}\,(L_2+L_2) & \cos\pi\cos\frac{\pi}{2} \\ \cos\pi\cos\frac{\pi}{2}\,(L_2+L_2) & -\sin\pi\sin\frac{\pi}{2}\,(L_2+L_2) & \sin\pi\cos\frac{\pi}{2} \\ 0 & \cos\frac{\pi}{2}\,(L_2+L_2) & \sin\frac{\pi}{2} \end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} \\
&= \begin{pmatrix} 0 & 2L_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix} \\
&= \begin{pmatrix}0\\0\\1\end{pmatrix}
\end{aligned}
$$

0.6.3 Part(c)

Mathematically, J_v is singular when its determinant is zero, or when J_v has a zero eigenvalue. The determinant of the Jacobian [2] was found to be

$$
\boxed{|J_v| = (d_3 + L_2)^2 \cos\theta_2}
$$

The above is zero when $\cos\theta_2 = 0$. This occurs when $\theta_2 = \pm\frac{\pi}{2}$ (and any odd integer multiple of this value).

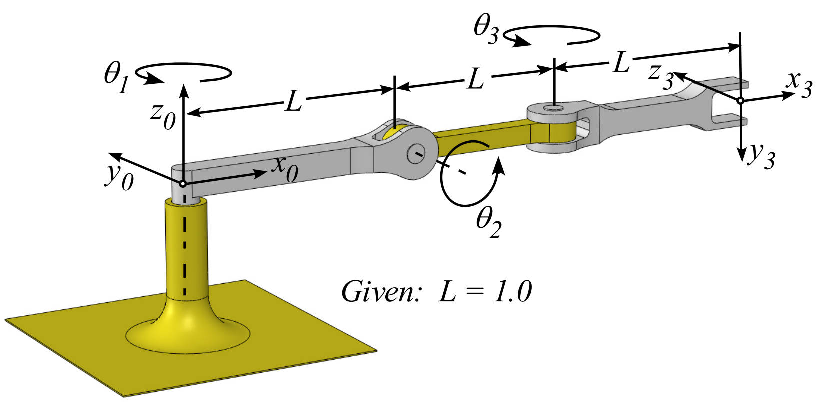
This implies that the singularity occurs when the arm is fully extended vertically in the upward position, or if physically possible, when the arm is fully extended vertically but in the downwards position.

[2]Using syms and simplification

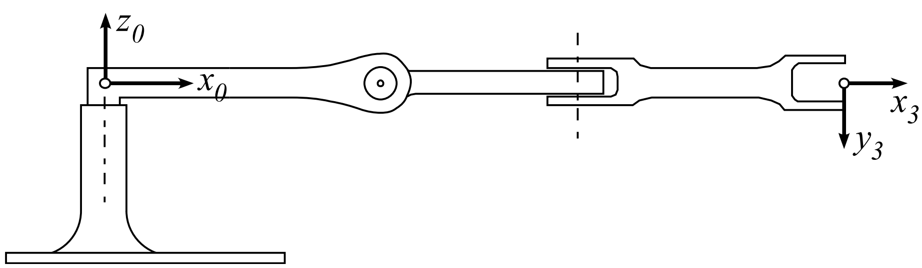
0.7 Problem 7

Problem 7.

Consider the three axis RRR manipulator shown in the figure below



► Derive the forward kinematics, T_3^0, of this manipulator as a function of the joint displacements and the geometric parameters shown in the figure. Sketch your intermediate frame definitions on the plane view of the manipulator shown below. Keep in mind that your frame definitions should be consistent with the conventions assumed when constructing the explicit form of the basic Jacobian.



► Evaluate the full basic Jacobian, J_0, for this manipulator. In this case, the basic Jacobian relates the joint space velocities, $\dot{q}$, and task space velocities $\dot{X}$.

$$\dot{X} = J_o\dot{q} \quad \text{where} \quad \dot{X} = [\dot{x} \quad \dot{y} \quad \dot{z} \quad \omega_x \quad \omega_y \quad \omega_z]^T \quad \text{and} \quad \dot{q} = [\dot{\theta}_1 \quad \dot{\theta}_2 \quad \dot{\theta}_3]^T$$

- For the manipulator configurations listed below, evaluate the required joint torques to react the applied end-effector forces and torques: $F = [f_x \quad 0 \quad 0 \quad 0 \quad 0 \quad \tau_z]^T$. Discuss you results.

Configuration	θ_1	θ_2	θ_3
1	0	30°	0
2	0	0	30°
3	0	90°	90°

- Evaluate the linear velocity Jacobian, J_v^1, expressed in frame {1}
- Using J_v^1 find the singularities of the manipulator (with respect to the end-effector's linear velocity)
- For each type of singularity that you identified explain the physical interpretation of the singularity - by sketching the arm in a singular configuration and describing the resulting limitation on its movement.
- For the manipulator above, a new task position representation has been defined as

  $u = 2x + 3y$

  $v = x + y - z$

  $w = z$

  Evaluate the *linear* velocity <u>analytical</u> Jacobian, J_a, for this new representation when the manipulator configuration is given as $\vec{q} = [\theta_1 \quad \theta_2 \quad \theta_3] = \left[0 \quad \frac{\pi}{2} \quad \frac{\pi}{2}\right]$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = J_a \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

Figure 8: Problem 7 description

0.7.1 Derive the forward kinematics

The frames are first assigned to each link by insuring that that the z axis of each frame follows the DH convention, which means z_i axis attached to link i is aligned in the direction of motion of joint $i+1$.

For a revolute joint, this will be its axis of spin, and for prismatic joint, this will be its direction of sliding. Once this is done the forward kinematics matrices are found using either the DH table method or using homogeneous transformation by direct inspection. In this case, the homogeneous transformation by direct inspection method was used. The frames are assigned as follows

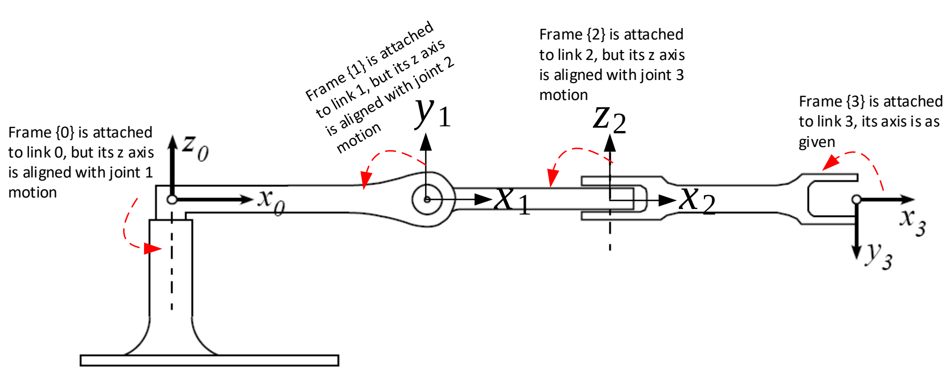


Figure 9: Frames assignments for problem 7

0.7.2 Determine forward kinematics matrices and evaluate full basic Jacobian J_0

Define the matrices T_1^0, T_2^1, T_3^2 using homogeneous transformation. Since $x' = J_0 q'$ then to find J_0 we need to first find forward kinematics. By inspection, we find each transformation matrix to be

$$T_1^0 = \begin{pmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & L\cos(\theta_1) \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & L\sin(\theta_1) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_2^1 = \begin{pmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) & L\cos(\theta_2) \\ \sin(\theta_2) & 0 & \cos(\theta_2) & L\sin(\theta_2) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^2 = \begin{pmatrix} \cos(\theta_3) & 0 & -\sin(\theta_3) & L\cos(\theta_3) \\ \sin(\theta_3) & 0 & \cos(\theta_3) & L\sin(\theta_3) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore

$$T_2^0 = T_1^0 T_2^1 = \begin{pmatrix} \cos(\theta_1)\cos(\theta_2) & -\sin(\theta_1) & -\cos(\theta_1)\sin(\theta_2) & L\cos(\theta_1) + L\cos(\theta_2)\cos(\theta_1) \\ \cos(\theta_2)\sin(\theta_1) & \cos(\theta_1) & -\sin(\theta_1)\sin(\theta_2) & L\sin(\theta_1) + L\cos(\theta_2)\sin(\theta_1) \\ \sin(\theta_2) & 0 & \cos(\theta_2) & L\sin(\theta_2) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^0 = T_2^0 T_3^2$$

$$= \begin{pmatrix} \cos(\theta_1)\cos(\theta_2)\cos(\theta_3) - \sin(\theta_1)\sin(\theta_3) & \cos(\theta_1)\sin(\theta_2) & -\cos(\theta_3)\sin(\theta_1) - \cos(\theta_1)\cos(\theta_2)\sin(\theta_3) & L(\cos(\theta_1)(\cos(\theta_2)(\cos(\theta_3)+1)+1) - \sin \\ \cos(\theta_2)\cos(\theta_3)\sin(\theta_1) + \cos(\theta_1)\sin(\theta_3) & \sin(\theta_1)\sin(\theta_2) & \cos(\theta_1)\cos(\theta_3) - \cos(\theta_2)\sin(\theta_1)\sin(\theta_3) & L((\cos(\theta_2)(\cos(\theta_3)+1)+1)\sin(\theta_1) + \cos \\ \cos(\theta_3)\sin(\theta_2) & -\cos(\theta_2) & -\sin(\theta_2)\sin(\theta_3) & L(\cos(\theta_3)+1)\sin(\theta_2) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In short notation the above can be written as

$$T_2^0 = \begin{pmatrix} C_1C_2 & -S_1 & -C_1S_2 & LC_1(C_2+1) \\ C_2S_1 & C_1 & -S_1S_2 & L(C_2+1)S_1 \\ S_2 & 0 & C_2 & LS_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T_3^0 = \begin{pmatrix} C_1C_2C_3 - S_1S_3 & C_1S_2 & -C_3S_1 - C_1C_2S_3 & L(C_1(C_2(C_3+1)+1) - S_1S_3) \\ C_2C_3S_1 + C_1S_3 & S_1S_2 & C_1C_3 - C_2S_1S_3 & L((C_2(C_3+1)+1)S_1 + C_1S_3) \\ C_3S_2 & -C_2 & -S_2S_3 & L(C_3+1)S_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we find the base Jacobian J_0 where $x' = J_0 q'$

$$\boxed{J_0 = \begin{pmatrix} J_{v_1} & J_{v_2} & J_{v_3} \\ J_{\omega_1} & J_{\omega_2} & J_{\omega_3} \end{pmatrix}}$$

where

$$J_{v_i}^0 = \epsilon_i z_{i-1}^0 + \hat{\epsilon}\left(z_{i-1}^0 \times (o_n^0 - o_{i-1}^0)\right)$$
$$J_{\omega_i}^0 = \hat{\epsilon} z_{i-1}^0$$

z_0^0 is given by $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and

$$J_{v_1} = z_0^0 \times (o_3^0 - o_0^0)$$
$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} L\left(\cos(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)-\sin(\theta_1)\sin(\theta_3)\right) \\ L\left(\sin(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)+\sin(\theta_3)\cos(\theta_1)\right) \\ L\sin(\theta_2)\left(\cos(\theta_3)+1\right) \end{pmatrix}$$
$$= \begin{pmatrix} -L\left(\sin(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)+\sin(\theta_3)\cos(\theta_1)\right) \\ L\left(\cos(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)-\sin(\theta_1)\sin(\theta_3)\right) \\ 0 \end{pmatrix}$$

$$J_{v_2} = z_1^0 \times (o_3^0 - o_1^0)$$
$$= \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix} \times \left(\begin{pmatrix} L\left(\cos(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)-\sin(\theta_1)\sin(\theta_3)\right) \\ L\left(\sin(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)+\sin(\theta_3)\cos(\theta_1)\right) \\ L\sin(\theta_2)\left(\cos(\theta_3)+1\right) \end{pmatrix} - \begin{pmatrix} L\cos(\theta_1) \\ L\sin(\theta_1) \\ 0 \end{pmatrix}\right)$$
$$= \begin{pmatrix} -L\sin(\theta_2)\cos(\theta_1)\left(\cos(\theta_3)+1\right) \\ -L\sin(\theta_1)\sin(\theta_2)\left(\cos(\theta_3)+1\right) \\ L\cos(\theta_2)\left(\cos(\theta_3)+1\right) \end{pmatrix}$$

$$J_{v_3} = z_2^0 \times (o_3^0 - o_2^0)$$
$$= \begin{pmatrix} \sin(\theta_2)\left(-\cos(\theta_1)\right) \\ -\sin(\theta_1)\sin(\theta_2) \\ \cos(\theta_2) \end{pmatrix} \times \left(\begin{pmatrix} L\left(\cos(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)-\sin(\theta_1)\sin(\theta_3)\right) \\ L\left(\sin(\theta_1)\left(\cos(\theta_2)\left(\cos(\theta_3)+1\right)+1\right)+\sin(\theta_3)\cos(\theta_1)\right) \\ L\sin(\theta_2)\left(\cos(\theta_3)+1\right) \end{pmatrix} - \begin{pmatrix} L\cos(\theta_1)\left(\cos(\theta_2)+1\right) \\ L\sin(\theta_1)\left(\cos(\theta_2)+1\right) \\ L\sin(\theta_2) \end{pmatrix}\right)$$
$$= \begin{pmatrix} -L\left(\sin(\theta_1)\cos(\theta_3)+\sin(\theta_3)\cos(\theta_1)\cos(\theta_2)\right) \\ L\left(\cos(\theta_1)\cos(\theta_3)-\sin(\theta_1)\sin(\theta_3)\cos(\theta_2)\right) \\ -L\sin(\theta_2)\sin(\theta_3) \end{pmatrix}$$

$$J_{w_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$J_{w_2} = \begin{pmatrix} \sin(\theta_1) \\ -\cos(\theta_1) \\ 0 \end{pmatrix}$$

$$J_{w_3} = \begin{pmatrix} \sin(\theta_2)\left(-\cos(\theta_1)\right) \\ -\sin(\theta_1)\sin(\theta_2) \\ \cos(\theta_2) \end{pmatrix}$$

Hence the Jacobian becomes

$$J_0 = \begin{pmatrix} -L\left(\left(\cos\left(\theta_2\right)\left(\cos\left(\theta_3\right)+1\right)+1\right)\sin\left(\theta_1\right)+\cos\left(\theta_1\right)\sin\left(\theta_3\right)\right) & -L\cos\left(\theta_1\right)\left(\cos\left(\theta_3\right)+1\right)\sin\left(\theta_2\right) & -L\left(\cos\left(\theta_3\right)\sin\left(\theta_1\right)+\cos\left(\theta_1\right)\cos\left(\theta_2\right)\mathrm{s}\right. \\ L\left(\cos\left(\theta_1\right)\left(\cos\left(\theta_2\right)\left(\cos\left(\theta_3\right)+1\right)+1\right)-\sin\left(\theta_1\right)\sin\left(\theta_3\right)\right) & -L\left(\cos\left(\theta_3\right)+1\right)\sin\left(\theta_1\right)\sin\left(\theta_2\right) & L\left(\cos\left(\theta_1\right)\cos\left(\theta_3\right)-\cos\left(\theta_2\right)\sin\left(\theta_1\right)\mathrm{si}\right. \\ 0 & L\cos\left(\theta_2\right)\left(\cos\left(\theta_3\right)+1\right) & -L\sin\left(\theta_2\right)\sin\left(\theta_3\right) \\ 0 & \sin\left(\theta_1\right) & -\cos\left(\theta_1\right)\sin\left(\theta_2\right) \\ 0 & -\cos\left(\theta_1\right) & -\sin\left(\theta_1\right)\sin\left(\theta_2\right) \\ 1 & 0 & \cos\left(\theta_2\right) \end{pmatrix}$$

Or in short notation

$$\boxed{J_0 = \begin{pmatrix} -L\left(\left(C_2\left(C_3+1\right)+1\right)S_1+C_1S_3\right) & -LC_1\left(C_3+1\right)S_2 & -L\left(C_3S_1+C_1C_2S_3\right) \\ L\left(C_1\left(C_2\left(C_3+1\right)+1\right)-S_1S_3\right) & -L\left(C_3+1\right)S_1S_2 & L\left(C_1C_3-C_2S_1S_3\right) \\ 0 & LC_2\left(C_3+1\right) & -LS_2S_3 \\ 0 & S_1 & -C_1S_2 \\ 0 & -C_1 & -S_1S_2 \\ 1 & 0 & C_2 \end{pmatrix}}$$

When $L=1$ the above becomes

$$\boxed{J_0 = \begin{pmatrix} -\left(C_2\left(C_3+1\right)+1\right)S_1-C_1S_3 & -C_1\left(C_3+1\right)S_2 & -C_3S_1-C_1C_2S_3 \\ C_1\left(C_2\left(C_3+1\right)+1\right)-S_1S_3 & -\left(C_3+1\right)S_1S_2 & C_1C_3-C_2S_1S_3 \\ 0 & C_2\left(C_3+1\right) & -S_2S_3 \\ 0 & S_1 & -C_1S_2 \\ 0 & -C_1 & -S_1S_2 \\ 1 & 0 & C_2 \end{pmatrix}}$$

0.7.3 Evaluate required joint torques for the configurations

Using the duality property where

$$\begin{aligned} x' &= J_0 q' \\ \tau &= J^T f \end{aligned}$$

We can determine τ for each Jacobian at each configuration. The following table gives the result of this computation

#	J_0	$J^T f$	τ …
$\begin{pmatrix}0\\30\\0\end{pmatrix}$	$\begin{pmatrix}0 & -L & 0\\ \left(1+\sqrt{3}\right)L & 0 & L\\ 0 & \sqrt{3}L & 0\\ 0 & 0 & -\frac{1}{2}\\ 0 & -1 & 0\\ 1 & 0 & \frac{\sqrt{3}}{2}\end{pmatrix}$	$\begin{pmatrix}0 & \left(1+\sqrt{3}\right)L & 0 & 0 & 0 & 1\\ -L & 0 & \sqrt{3}L & 0 & -1 & 0\\ 0 & L & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2}\end{pmatrix}\begin{pmatrix}f_x\\0\\0\\0\\0\\\tau_z\end{pmatrix}$	τ … $-L$ … $\sqrt{3}$ …
$\begin{pmatrix}0\\0\\30\end{pmatrix}$	$\begin{pmatrix}-\frac{L}{2} & 0 & -\frac{L}{2}\\ \left(2+\frac{\sqrt{3}}{2}\right)L & 0 & \frac{\sqrt{3}L}{2}\\ 0 & \left(1+\frac{\sqrt{3}}{2}\right)L & 0\\ 0 & 0 & 0\\ 0 & -1 & 0\\ 1 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}-\frac{L}{2} & \left(2+\frac{\sqrt{3}}{2}\right)L & 0 & 0 & 0 & 1\\ 0 & 0 & \left(1+\frac{\sqrt{3}}{2}\right)L & 0 & -1 & 0\\ -\frac{L}{2} & \frac{\sqrt{3}L}{2} & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}f_x\\0\\0\\0\\0\\\tau_z\end{pmatrix}$	$\tau_z -$ … 0 … $\tau_z -$ …
$\begin{pmatrix}0\\90\\90\end{pmatrix}$	$\begin{pmatrix}-L & -L & 0\\ L & 0 & 0\\ 0 & 0 & -L\\ 0 & 0 & -1\\ 0 & -1 & 0\\ 1 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}-L & L & 0 & 0 & 0 & 1\\ -L & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & -L & -1 & 0 & 0\end{pmatrix}\begin{pmatrix}f_x\\0\\0\\0\\0\\\tau_z\end{pmatrix}$	$\tau_z -$ … $-L$ … 0 …

0.7.4 Discussion of result

Since

$$f = \begin{pmatrix}f_x\\0\\0\\0\\0\\\tau_z\end{pmatrix}$$

then only entries in the J^T matrix which are not zero at location $(i,1)$ and $(i,6)$ where i is the row number of J^T will contribute to the joint torques. So for configuration 1 above, we see that J^T has non zero value in one of these locations in each row. But for configuration 2, the second row of J^T has zero in both the first column and the last column. This means no matter what joint torque is applied to joint 2, it will have no effect on end effector forces produced.

Similarly, for the third configuration, we see that the last row of J^T also has zero entry in the first and last column. This means no matter what joint torque is applied to joint 3, it will have no effect on end effector forces produced when in this configuration.

0.7.5 Evaluate linear velocity Jacobian J_v^1 expressed in frame 1

To find J_0^1 we need to transform J_0 to frame 1 using $J_0^i = \begin{pmatrix} R_0^i & 0 \\ 0 & R_0^i \end{pmatrix} J_0$ where in this case $i = 1$.

But $R_0^1 = (R_1^0)^{-1} = (R_1^0)^T$. We found T_1^0 from above, so we can extract R_1^0 part from it

$$T_1^0 = \begin{pmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & L\cos(\theta_1) \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & L\sin(\theta_1) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_1^0 = \begin{pmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) \\ \sin(\theta_1) & 0 & -\cos(\theta_1) \\ 0 & 1 & 0 \end{pmatrix}$$

Hence

$$\begin{pmatrix} R_0^i & 0 \\ 0 & R_0^i \end{pmatrix} = \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \sin(\theta_1) & -\cos(\theta_1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta_1) & \sin(\theta_1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \sin(\theta_1) & -\cos(\theta_1) & 0 \end{pmatrix}$$

mutliplying the above with J_0 found earlier gives

$$\begin{aligned} J_0^1 &= \begin{pmatrix} R_0^i & 0 \\ 0 & R_0^i \end{pmatrix} J_0 \\ &= \begin{pmatrix} -L\sin(\theta_3) & -L(\cos(\theta_3)+1)\sin(\theta_2) & -L\cos(\theta_2)\sin(\theta_3) \\ 0 & L\cos(\theta_2)(\cos(\theta_3)+1) & -L\sin(\theta_2)\sin(\theta_3) \\ -L(\cos(\theta_2)(\cos(\theta_3)+1)+1) & 0 & -L\cos(\theta_3) \\ 0 & 0 & -\sin(\theta_2) \\ 1 & 0 & \cos(\theta_2) \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

Hence $J_{0_v}^1$ is the top three rows given by

$$\boxed{J_{0_v}^1 = \begin{pmatrix} -L\sin(\theta_3) & -L(\cos(\theta_3)+1)\sin(\theta_2) & -L\cos(\theta_2)\sin(\theta_3) \\ 0 & L\cos(\theta_2)(\cos(\theta_3)+1) & -L\sin(\theta_2)\sin(\theta_3) \\ -L(\cos(\theta_2)(\cos(\theta_3)+1)+1) & 0 & -L\cos(\theta_3) \end{pmatrix}}$$

When $L = 1$ the above becomes

$$\boxed{J_{0_v}^1 = \begin{pmatrix} -\sin(\theta_3) & -\cos(\theta_3)\sin(\theta_2) - \sin(\theta_2) & -\cos(\theta_2)\sin(\theta_3) \\ 0 & \cos(\theta_3)\cos(\theta_2) + \cos(\theta_2) & -\sin(\theta_2)\sin(\theta_3) \\ -\cos(\theta_3)\cos(\theta_2) - \cos(\theta_2) - 1 & 0 & -\cos(\theta_3) \end{pmatrix}}$$

0.7.6 Find the singularities in J_v^1 and sketch the arm

The singularity of J_v^1 can be found mathematically by finding the conditions under which the determinant of J_v^1 is zero, or the conditions under which one of the eigevalues become zero. Or we can use geometry and consider the cases where the arm is in singular direction. Mathematically, the determinant of J_v^1 is

$$|J_v^1| = -L^3 \sin(\theta_3)(\cos(\theta_2)+1)(\cos(\theta_3)+1)$$

When $L = 1$ the above becomes

$$|J_v^1| = -(1+\cos\theta_2)(1+\cos\theta_3)\sin\theta_3$$

To make $|J_v^1|$, the above implies the corresponding joint angles have to be the following

$$\begin{aligned}\theta_2 &= \pm\pi \\ \theta_3 &= \{\pm\pi, 0\}\end{aligned}$$

The joint angle θ_1 can be any value since it does not contribute to making the determinant zero since the determinant does not depend on θ_1. The above shows there are a total of 5 configurations that will result in singularity.

The following diagram illustrates the above singularites found and also sketches the the singular direction.

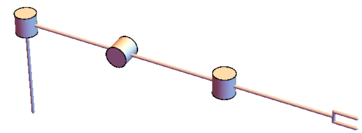
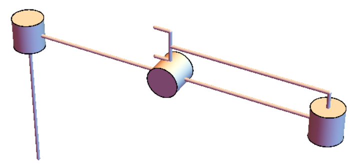
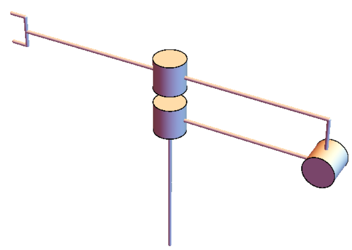
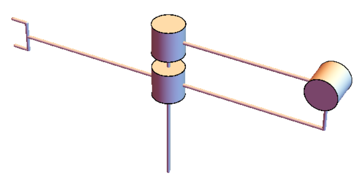
θ_1	θ_2	θ_3	J_v^1	configuration showing singular direction
any	any	0	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2L & 0 \\ -3L & 0 & -L \end{pmatrix}$	
any	any	$\pm\pi$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -L & 0 & L \end{pmatrix}$	
any	π	any	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2L & 0 \\ L & 0 & -L \end{pmatrix}$	
any	$-\pi$	any	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2L & 0 \\ L & 0 & -L \end{pmatrix}$	

Figure 10: Singular directions for problem 7

0.7.7 Find the linear analytic Jacobian J_a for new representation

The analytical linear velocity Jacobian J_a is given by

$$J_a = E_p(x_p)J_0$$

Where $E_p(x_p)$ is the representation matrix found from

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = E_p(x_p)\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Since we are given the expressions for u, v, w, then we differentiate them w.r.t time to determine $E_p(x_p)$ and obtain

$$\begin{aligned}\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} &= \begin{pmatrix} 2x' + 4y' \\ x' + y' - z' \\ z' \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}\end{aligned}$$

Hence

$$E_p(x_p) = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore $J_a = E_p(x_p)J_0$ becomes, at the required angles

$$\boxed{J_a = \begin{pmatrix} L & -2L & 0 \\ 0 & -L & L \\ 0 & 0 & -L \end{pmatrix}}$$

And at $L = 1$

$$J_a = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

0.8 Problem 8

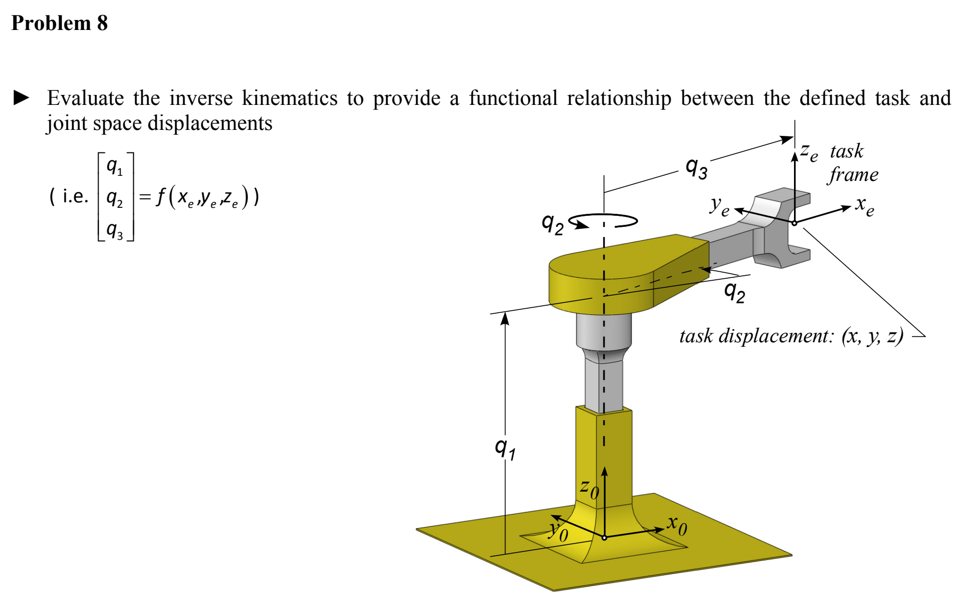
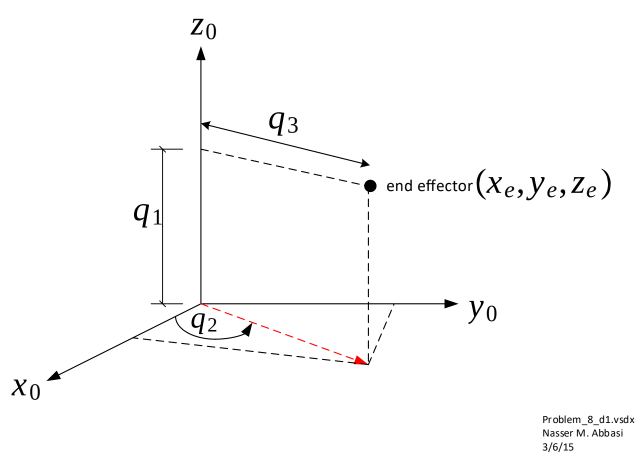


Figure 11: Problem 8 description

Using geometry as illustrated below



Hence based on the above diagram q_1 is found to be

$$q_1 = z_e$$

Taking the projection of the end effector vector on the xy plane gives

$$x_e = q_3 \cos(q_2)$$
$$y_e = q_3 \sin(q_2)$$

Dividing the second equation above by the first one gives $\frac{y_e}{x_e} = \tan(q_2)$. Hence $q_2 = \text{atantwo}(x_e, y_e)$. If $x = 0$ then there is no solution (singular direction). A second solution is $q_2 = \text{atantwo}(x_e, y_e) + \pi$

and finally $q_3 = \sqrt{x_e^2 + y_e^2}$. Therefore the two solutions are

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}_{\{1\}} = \begin{pmatrix} z_e \\ \operatorname{atantwo}(x_e, y_e) \\ \sqrt{x_e^2 + y_e^2} \end{pmatrix} \qquad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}_{\{2\}} = \begin{pmatrix} z_e \\ \operatorname{atantwo}(x_e, y_e) + \pi \\ \sqrt{x_e^2 + y_e^2} \end{pmatrix}$$

0.9 Appendix

Source code for problem 1

```
axes[x_, y_, z_, f_, a_] :=
  Graphics3D[Join[{Arrowheads[a]},
    Arrow[{{0, 0, 0}, #}] & /@ {{x, 0, 0}, {0, y, 0}, {0, 0, z}},
     {Text[Style["x", FontSize -> Scaled[f]], {1.2*x, 0.1*y, 0.1*z}],
     Text[Style["y", FontSize -> Scaled[f]], {0.1 x, 1.2*y, 0.1*z}],
     Text[Style["z", FontSize -> Scaled[f]], {0.1*x, 0.1*y, 1.2*z}]}]];

rotZ[a_] := {{Cos[a], -Sin[a], 0}, {Sin[a], Cos[a], 0}, {0, 0, 1}};
rotX[a_] := {{1, 0, 0}, {0, Cos[a], -Sin[a]}, {0, Sin[a], Cos[a]}};
rotY[a_] := {{Cos[a], 0, Sin[a]}, {0, 1, 0}, {-Sin[a], 0, Cos[a]}};
obj = Cuboid[{1, -.5, -.1}, {-1, .5, .1}];

show[mat_, arrow_, fixed_] := Show[Graphics3D[
    {
     GeometricTransformation[{
       {Opacity[0.7], obj},
       {Red, Arrow[{{0, 0, 0}, {1.2, 0, 0}}]},
       {Red, Arrow[{{0, 0, 0}, {0, 1.2, 0}}]},
       {Red, Arrow[{{0, 0, 0}, {0, 0, 1.2}}]},
       Text[Style["x", Red, FontSize -> Scaled[0.07]], {1.4, 0, 0}],
       Text[Style["y", Red, FontSize -> Scaled[0.07]], {0, 1.4, 0}],
       Text[Style["z", Red, FontSize -> Scaled[0.07]], {0, 0, 1.4}],
       If[fixed, Sequence @@ {}, arrow]
       }, mat
      ],
     If[fixed, arrow, Sequence @@ {}]
     },
    Boxed -> False,
    Axes -> None,
    ViewPoint -> {3, 1.5, 1.5},
    SphericalRegion -> True,
    ImageSize -> 250, ImageMargins -> 0, ImagePadding -> 0, PlotRangePadding -> None],
   axes[1.6, 1.6, 1.5, 0.1, 0.04]];

show[mat_] := Show[Graphics3D[
   {
    GeometricTransformation[{
      {Opacity[0.7], obj},
      {Red, Arrow[{{0, 0, 0}, {1.2, 0, 0}}]},
      {Red, Arrow[{{0, 0, 0}, {0, 1.2, 0}}]},
      {Red, Arrow[{{0, 0, 0}, {0, 0, 1.2}}]},
```

```
      Text[Style["x", Red, FontSize -> Scaled[0.07]], {1.4, 0, 0}],
      Text[Style["y", Red, FontSize -> Scaled[0.07]], {0, 1.4, 0}],
      Text[Style["z", Red, FontSize -> Scaled[0.07]], {0, 0, 1.4}]
      }, mat
     ]
    },
   Boxed -> False,
   Axes -> None,
   ViewPoint -> {3, 1.5, 1.5},
   SphericalRegion -> True,
   ImageSize -> 200, ImageMargins -> 0, ImagePadding -> 0, PlotRangePadding -> None],
  axes[1.6, 1.6, 1.5, 0.1, 0.04]];

p1 = Grid[{
   {Style["+45 around fixed z", Bold, 14], Style["after rotation ", Bold, 14], Style["Final rotation m
   {
    show[rotZ[0], makeCurvedArrow[.5, "z", 1], True],
    show[rotZ[45 Degree]],
    MatrixForm[N@rotZ[45 Degree]]
    }
   }, Spacings -> {1, 1}, Frame -> All, FrameStyle -> LightGray
  ]

p2 = Grid[{
   {Style["30 around current x", Bold, 14], Style["after rotation ", Bold, 14],
    Style["Final rotation matrix", Bold, 14]},
   {
    show[rotZ[45 Degree], makeCurvedArrow[.5, "x", 1], False],
    show[rotZ[45 Degree].rotX[30 Degree]],
    MatrixForm[N@rotZ[45 Degree].rotX[30 Degree]]
    }}, Spacings -> {1, 1}, Frame -> All, FrameStyle -> LightGray
  ]

p3 = Grid[{
   {Style["-45 around fixed z", Bold, 14], Style["after rotation ", Bold, 14],
    Style["Final rotation matrix", Bold, 14]},
   {
    show[rotZ[45 Degree].rotX[30 Degree], makeCurvedArrow[.5, "z", -1], True],
    show[rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree]],
    MatrixForm[N@rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree]]
    }}, Spacings -> {1, 1}, Frame -> All, FrameStyle -> LightGray
  ]

p4 = Grid[{
   {Style["+90 around current y", Bold, 14], Style["after rotation ", Bold, 14], Style["Final rotation
   {
    show[rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree], makeCurvedArrow[.5, "y", 1], False],
    show[rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree].rotY[90 Degree]],
    MatrixForm[N@rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree].rotY[90 Degree]]
    }}, Spacings -> {1, 1}, Frame -> All, FrameStyle -> LightGray
  ]

p5 = Grid[{
   {Style["-30 around fixed x", Bold, 14], Style["after rotation ", Bold, 14],
```

```
  Style["Final rotation matrix", Bold, 14]},
 {
  show[rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree].rotY[90 Degree], makeCurvedArrow[.5, "x", -1
  show[rotX[-30 Degree].rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree].rotY[90 Degree]],
  MatrixForm[Chop[N@rotX[-30 Degree].rotZ[-45 Degree].rotZ[45 Degree].rotX[30 Degree].rotY[90 Degree
  }
 }, Spacings -> {1, 1}, Frame -> All, FrameStyle -> LightGray]
```