Homework #11
EMA 545, Spring 2013

Problem 1.)
Consider the N-DOF system modeled by the system of equations

$$[M]\{\ddot{q}\}+[C]\{\dot{q}\}+[K]\{q\}=\{Q\}$$

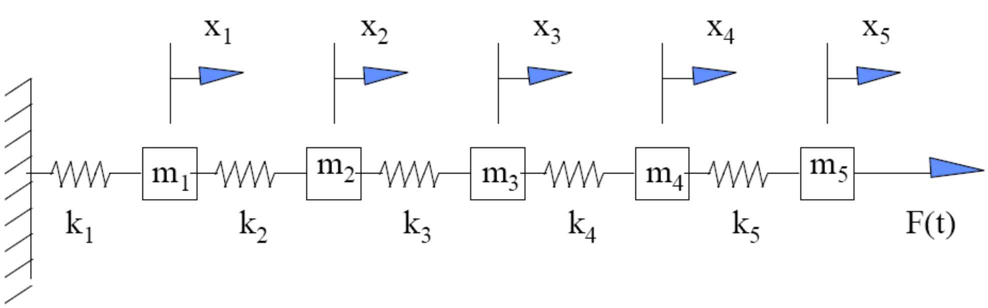
with [C] matrix given by:

$$[C]=[M][\Phi]\begin{bmatrix} 2\varsigma_1\omega_1 & 0 & 0 & \cdots & 0 \\ 0 & 2\varsigma_2\omega_2 & & & \vdots \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & & 0 & 2\varsigma_N\omega_N \end{bmatrix}[\Phi]^T[M].$$

If we transform to normal coordinates using $\{q\}=[\Phi]\{\eta\}$, show that the N-coupled equations transform into N uncoupled differential equations of the form

$$\ddot{\eta}_j+2\zeta_j\omega_j\dot{\eta}+\omega_j^2\eta_j=\{\Phi_j\}^T\{Q\} \qquad j=1,N$$

Problem 2.)
The 5-DOF system shown below can be thought of as a lumped-element approximation of a fixed-free elastic bar. (This is similar to Example Problem 4.4, which treats a fixed-fixed bar, and similar to example problem 7.2, where the exact solution is derived.)



In parallel with each of the springs k_i which are drawn, there are viscous dampers c_i which are not shown. The equations of motion for this system are easily found to be:

$$[\mathrm{m}]\underline{\ddot{\mathrm{x}}}+[\mathrm{c}]\underline{\dot{\mathrm{x}}}+[\mathrm{k}]\underline{\mathrm{x}}=\underline{\mathrm{F}}$$

where [m] is a diagonal matrix having entries m_i, and [k] is the banded matrix:

$$\begin{bmatrix} (k1+k2) & -k2 & 0 & 0 & 0 \\ -k2 & (k2+k3) & -k3 & 0 & 0 \\ 0 & -k3 & (k3+k4) & -k4 & 0 \\ 0 & 0 & -k4 & (k4+k5) & -k5 \\ 0 & 0 & 0 & -k5 & k5 \end{bmatrix}$$

The damping matrix has the same form as [k], but with viscous damping coefficients c_i in place of stiffness coefficients k_i. The forcing vector $\underline{F}$ is a 5 by 1 vector of zeros, except for the last entry: $\underline{F}=[0 \quad 0 \quad 0 \quad 0 \quad F(t)]^T$.

The numerical value for k_2 through k_5 is 1 N/m, the numerical value of k_1 = 2 N/m, the numerical value for m_1 through m_5 is 1 kg, and the numerical values for the viscous damping coefficients $c_i = 0.1*k_i$, i=1,5 (in units of N-s/m).

(a) Find the natural frequencies and mass-normalized modes of the system.
(b) Find the magnitude and phase of the steady-state response $x_5(t)$ assuming the forcing to be harmonic, with amplitude 1 N and with a frequency from 0 to $1.2*\omega_5$. Plot the magnitude and phase of the response, clearly indicating the location of the natural frequencies.
(c) Repeat the analysis in (b), but use the strategy described in Problem 1 to create a [C] matrix that gives 2% modal damping to each mode. Overlay the frequency response of this system with that which you found in (b).
(d) Compare your answer for part (c) to that obtained using a structural damping model and a loss factor of γ=0.04.

(You will need the following to compare this problem with problem 3 below.) As discussed in Example Problem 4.4, the relationship between the lumped spring stiffnesses and the parameters EA and L are as follows: k_i = N*EA/L, i=2,N where N is the number of masses. The spring adjacent to a fixed point, because it is only 1/2 the length of the other springs has a stiffness twice as high, k_1 = 2N*EA/L. The lumped masses are equal to the total bar mass divided by N: $m_i = \rho AL/N$, where ρ is the mass density of the bar.

Problem 3.) (40 points)

Use a three-term Ritz series to predict the first 3 natural frequencies and natural modes of a fixed-free bar of length L, elastic modulus E, and constant cross-sectional area A.

(5 pts) a.) Use the potential and kinetic energy expressions in the book (eq. 6.1.1 and 6.1.2) to derive the expressions for the mass and stiffness matrices in eq. 6.1.11 and 6.1.13.

(5 pts) b.) Use the following Ritz basis functions to find the 3x3 mass and stiffness matrices:

$$\psi_1(x) = x/L, \quad \psi_2(x) = (x/L)^2, \text{ and } \psi_3(x) = (x/L)^3$$

Hint: Use the pattern described in Example Problem 6.1. In that problem, a uniform bar that is fixed at x=0 is studied; however, in that problem, there is an extra spring and dashpot at the right end of the bar, x=L. Note also that Example Problem 6.1 uses a different set of basis functions.

(10 pts) c.) Repeat the analysis using the following basis functions. I suggest using a computer package to estimate the numerical terms in the mass and stiffness matrices.

$$\psi_n = \sin\left(\alpha_n \frac{x}{L}\right), \qquad \alpha_n = \left(\frac{2n-1}{2}\right)\pi, \qquad n = 1, 2, 3$$

What do you notice about the M and K matrices using this set of basis functions?

(10 pts) d.) Compare the natural frequencies obtained parts (b) and (c) of this problem with those obtained in Problem 2. (Use the relationships given in the problem statement above to find EA/L and ρAL values that agree with those used in problem 2.)

(10 pts) e.) Generate a plot of the mode shapes of the systems based on the models in (b) and (c), and also overlay the mode shapes obtained in problem 2. How do the three sets of results compare?