HW 9, ME 440 Intermediate Vibration, Fall 2017

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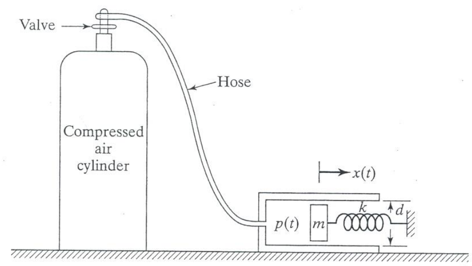
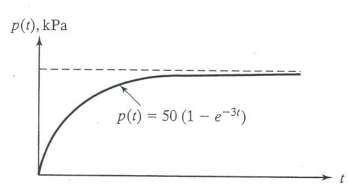
0.1 Problem 1

Problems 2/3 (due Friday, November 17th by 4pm)

A compressed air cylinder is connected to the spring-mass system shown in Figure (a) below. Due to a small leak in the valve, the pressure on the piston, *p(t)*, builds up as indicated in Figure (b) shown below. Assume $m = 10$ kg, $k = 1000$ N/m and $d = 0.1$ m and that all initial conditions are zero.



(a)



(b)

Solve for the complete response of the piston by using direct integration.

Since this is an undamped system, the equation of motion is

$$m\ddot{x} + kx = F(t)$$

Where $F(t) = Ap(t)$ and $p(t)$ is the pressure. Therefore

$$F(t) = \left(50\times 10^3\right) A\left(1 - e^{-3t}\right)$$

The term 50×10^3 was added above because the units were given in kPa and need to convert them to Pa. The equation of motion becomes

$$\begin{aligned} m\ddot{x} + kx &= \left(50\times 10^3\right) A\left(1 - e^{-3t}\right) \\ &= \left(50\times 10^3\right) A - \left(50\times 10^3\right) Ae^{-3t} \end{aligned}$$

To simplify notations, let $\beta = \left(50\times 10^3\right) A$. The above now becomes

$$m\ddot{x} + kx = \beta - \beta e^{-3t} \tag{1}$$

The solution to the above can be found by adding the two particular solutions of

$$m\ddot{x} + kx = \beta \tag{2}$$

And

$$m\ddot{x} + kx = -\beta e^{-3t} \tag{3}$$

To the homogeneous solution of $m\ddot{x} + kx = 0$. This can be done since the ODE is linear. The particular solution to (2) is found by assuming $x_p(t) = C_1$ where C_1 is some constant and substituting this into (1) and solving for C_1 gives $kC_1 = \beta$ or $C_1 = \frac{\beta}{k}$, hence

$$x_{p,1}(t) = \frac{\beta}{k} \tag{4A}$$

The particular solution to (2) is now found. From the lookup table, assuming $x_p(t) = C_1 e^{-3t}$ and substituting this into (2), and since $\dot{x}_p = -3C_1 e^{-3t}$ and $\ddot{x}_p = 9C_1 e^{-3t}$ gives

$$\begin{aligned} 9mC_1 e^{-3t} + kC_1 e^{-3t} &= -\beta e^{-3t} \\ 9mC_1 + kC_1 &= -\beta \\ C_1 &= \frac{-\beta}{9m + k} \end{aligned}$$

Therefore

$$x_{p,2}(t) = \frac{-\beta}{9m + k} e^{-3t} \tag{4B}$$

Now that the particular solutions are known (4A,4B), they are added to the homogeneous solution (which is known) and the complete solution for (1) is

$$\begin{aligned} x(t) &= \overbrace{A\cos\omega_n t + B\sin\omega_n t}^{x_h(t)} + \overbrace{x_{p,1}(t) + x_{p,2}(t)}^{x_p(t)} \\ &= A\cos\omega_n t + B\sin\omega_n t + \frac{\beta}{k} - \frac{\beta}{9m + k} e^{-3t} \end{aligned} \tag{5}$$

Initial conditions are now applied to determine A, B. Since $x(0) = 0$ the above becomes

$$\begin{aligned} 0 &= A + \frac{\beta}{k} - \frac{\beta}{9m + k} \\ A &= \frac{\beta}{9m + k} - \frac{\beta}{k} \end{aligned}$$

The solution (5) becomes

$$x(t) = \left(\frac{\beta}{9m+k} - \frac{\beta}{k}\right)\cos\omega_n t + B\sin\omega_n t + \frac{\beta}{k} - \frac{\beta}{9m+k}e^{-3t} \tag{6}$$

Taking derivative of the above

$$\dot{x}(t) = -\omega_n\left(\frac{\beta}{9m+k} - \frac{\beta}{k}\right)\sin\omega_n t + \omega_n B\cos\omega_n t + 3\frac{\beta}{9m+k}e^{-3t}$$

Since $\dot{x}(0) = 0$ then

$$\begin{aligned} 0 &= \omega_n B + 3\frac{\beta}{9m+k} \\ B &= \frac{-3\beta}{(9m+k)\,\omega_n} \end{aligned}$$

Substituting this in (6) gives the final solution

$$x(t) = \left(\frac{\beta}{9m+k} - \frac{\beta}{k}\right)\cos\omega_n t - \frac{3\beta}{(9m+k)\,\omega_n}\sin\omega_n t + \frac{\beta}{k} - \frac{\beta}{9m+k}e^{-3t} \tag{7}$$

Since

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$$

And

$$\begin{aligned} \beta &= \left(50\times 10^3\right) A \\ &= \left(50\times 10^3\right)\pi\left(\frac{0.1}{2}\right)^2 \\ &= 392.70 \end{aligned}$$

Then numerically, the solution (7) is

$$\begin{aligned} x(t) &= \left(\frac{392.70}{90+1000} - \frac{392.70}{1000}\right)\cos 10t - \frac{3\,(392.70)}{(90+1000)\,10}\sin 10t + \frac{392.70}{1000} - \frac{392.70}{90+1000}e^{-3t} \\ &= -0.032\cos 10t - 0.108\sin 10t + 0.393 - 0.360e^{-3t} \end{aligned}$$

Below is a plot of the above to illustrate the solution for some arbitrary time t.

```
In[86]:= d = 0.1;
m = 10;
k = 1000;
wn = Sqrt[k / m];
A0 = Pi (d / 2) ^2;
beta = 50000 * Pi * (d / 2) ^2

Out[91]= 392.699
```

```
In[94]:= x[t_] := (beta/(9 m + k) - beta/k) Cos[wn t] - (3 beta)/((9 m + k) wn) Sin[wn t] + (beta/k - beta/(9 m + k)) Exp[-3 t];
Plot[x[t], {t, 0, 10}, Frame → True,
 FrameLabel → {{"x(t)", None}, {"time (sec)", "Solution for probem 2, direct integration method"}},
 GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, BaseStyle → 12]
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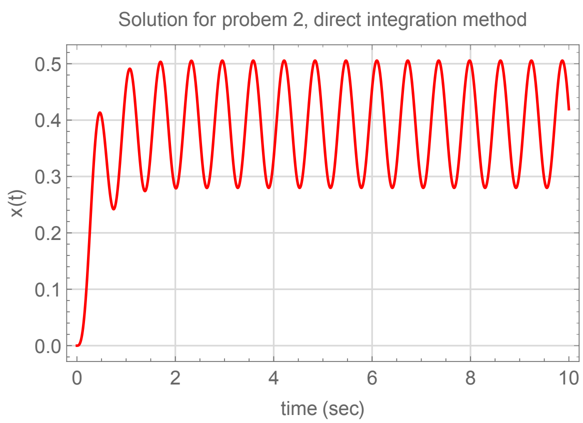
Out[95]=



0.2 Problem 2

Problem 3

Set up both integrals (both options) for solving for the response of the piston by using Duhamel's integral. You do NOT need to complete either of the integrations.

The force on the piston is

$$F(t) = Ap(t)$$

Where A is the area of the piston which is $A = \pi \left(\frac{d}{2}\right)^2$. Since this is undamped system, the equation of motion is

$$m\ddot{x} + kx = F(t)$$

To solve using Duhamel integration, the impulse response $g(t) = \frac{1}{m\omega_n} \sin(\omega_n t)$ is used. The integration is done using the two options.

0.2.1 Option 1

$$
\begin{aligned}
x_{conv}(t) &= \int_0^t F(\tau)\, g(t-\tau)\, d\tau \\
&= \frac{A}{m\omega_n} \int_0^t p(t) \sin(\omega_n (t-\tau))\, d\tau \\
&= \frac{A}{m\omega_n} \int_0^t 50(1000)\left(1-e^{-3\tau}\right) \sin(\omega_n (t-\tau))\, d\tau
\end{aligned}
$$

Where $50(1000)$ is used since the units are in kPa. The above becomes

$$
\begin{aligned}
x_{conv}(t) &= \left(5\times 10^4\right) \frac{A}{m\omega_n} \int_0^t \left(1-e^{-3\tau}\right) \sin(\omega_n (t-\tau))\, d\tau \\
&= \left(5\times 10^4\right) \frac{A}{m\omega_n} \left(\int_0^t \sin(\omega_n (t-\tau))\, d\tau - \int_0^t e^{-3\tau} \sin(\omega_n (t-\tau))\, d\tau\right) \qquad (1)
\end{aligned}
$$

The first integral in (1) becomes

$$
\begin{aligned}
\int_0^t \sin(\omega_n (t-\tau))\, d\tau &= -\left(\frac{\cos(\omega_n (t-\tau))}{-\omega_n}\right)_0^t \\
&= \frac{1}{\omega_n}\left(\cos(\omega_n (t-\tau))\right)_0^t \\
&= \frac{1}{\omega_n}\left(\cos(\omega_n (t-t)) - \cos(\omega_n t)\right) \\
&= \frac{1}{\omega_n}\left(1-\cos(\omega_n t)\right) \qquad (2)
\end{aligned}
$$

The second integral in (1) is found using the handout integration tables

$$
\int e^{ax} \sin(b+cx)\, dx = \frac{ae^{ax}\sin(b+cx)}{a^2+c^2} - \frac{ce^{ax}\cos(b+cx)}{a^2+c^2}
$$

In this case $a=-3$ and $b=\omega_n t$ and $c=-\omega_n$. The above becomes after substitution

$$
\begin{aligned}
\int_0^t e^{-3\tau} \sin(\omega_n (t-\tau))\, d\tau &= \left(\frac{-3e^{-3\tau}\sin(\omega_n (t-\tau))}{9+\omega_n^2} - \frac{-\omega_n e^{-3\tau}\cos(\omega_n (t-\tau))}{9+\omega_n^2}\right)_0^t \\
&= \frac{1}{9+\omega_n^2}\left(-3e^{-3\tau}\sin(\omega_n (t-\tau)) + \omega_n e^{-3\tau}\cos(\omega_n (t-\tau))\right)_0^t \\
&= \frac{1}{9+\omega_n^2}\left(\omega_n e^{-3t} - \left(-3\sin(\omega_n t) + \omega_n \cos(\omega_n t)\right)\right) \\
&= \frac{\omega_n e^{-3t} + 3\sin(\omega_n t) - \omega_n \cos(\omega_n t)}{9+\omega_n^2} \qquad (3)
\end{aligned}
$$

Substituting (2,3) into (1) gives the final result

$$
x_{conv}(t) = \left(5\times 10^4\right) \frac{A}{m\omega_n}\left(\frac{1}{\omega_n}\left(1-\cos(\omega_n t)\right) - \frac{\omega_n e^{-3t} + 3\sin(\omega_n t) - \omega_n \cos(\omega_n t)}{9+\omega_n^2}\right) \qquad (4)
$$

Because initial conditions are zero the solution is

$$\begin{aligned} x(t) &= x_h(t) + x_{cov(t)} \\ &= x_{cov(t)} \end{aligned}$$

Substituting all the numerical values, and since $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ then (4) becomes

$$\begin{aligned} x(t) &= \left(5\times 10^4\right)\frac{\pi\left(\frac{0.1}{2}\right)^2}{(10)(10)}\left(\frac{1}{10}(1-\cos(10t)) - \frac{10e^{-3t}+3\sin(10t)-10\cos(10t)}{109}\right) \\ &= 3.927\left(\frac{1}{10}(1-\cos(10t)) - \frac{10e^{-3t}+3\sin(10t)-10\cos(10t)}{109}\right) \\ &= 3.927\left(\frac{1}{10}(1-\cos(10t)) + \frac{10}{109}\cos(10t) - \frac{10}{109}e^{-3t} - \frac{3}{109}\sin(10t)\right) \\ &= 3.927\left(\frac{1}{10} - \frac{10}{109}e^{-3t} - \frac{3}{109}\sin(10t) - \frac{9}{1090}\cos(10t)\right) \end{aligned}$$

This is a plot of the above, which agrees with plot from the direct integration method. This verifies the above result

```
In[102]:= d = 0.1;
m = 10;
k = 1000;
wn = Sqrt[k / m];
A0 = Pi (d / 2) ^2;
```

$$\text{In[108]:= xconv[t\_] := } \frac{50 * 1000\, A0}{m\, wn}\left(\frac{1}{wn}(1 - \mathrm{Cos}[wn\, t]) - \frac{1}{9 + wn^2}(wn\, \mathrm{Exp}[-3\, t] + 3\, \mathrm{Sin}[wn\, t] - wn\, \mathrm{Cos}[wn\, t])\right)$$

```
In[110]:= Plot[xconv[t], {t, 0, 10}, Frame → True,
  FrameLabel → {{"x(t)", None}, {"time (sec)", "Solution for probem 3 option 1"}},
  GridLines → Automatic, GridLinesStyle → LightGray, PlotStyle → Red, BaseStyle → 12]
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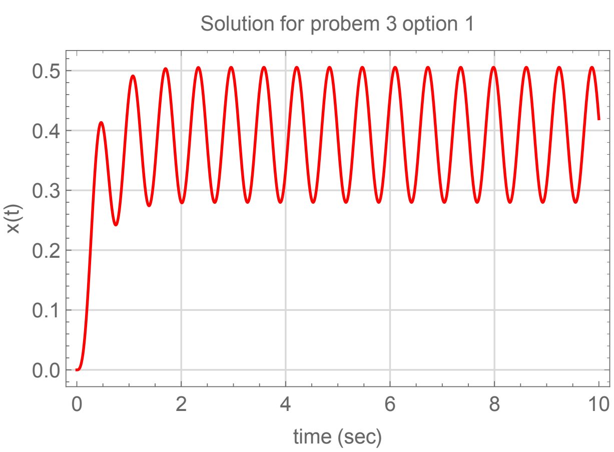
Out[110]=



0.2.2 Option 2

$$
\begin{aligned}
x_{conv}(t) &= \int_0^t F(t-\tau)\, g(\tau)\, d\tau \\
&= \frac{A}{m\omega_n} \int_0^t p(t-\tau) \sin(\omega_n \tau)\, d\tau \\
&= \frac{A}{m\omega_n} \int_0^t 50\,(1000) \left(1 - e^{-3(t-\tau)}\right) \sin(\omega_n \tau)\, d\tau
\end{aligned}
$$

Where $50\,(1000)$ is used, since the units are in kPa. The above becomes

$$
\begin{aligned}
x_{conv}(t) &= \left(5 \times 10^4\right) \frac{A}{m\omega_n} \int_0^t \left(1 - e^{-3(t-\tau)}\right) \sin(\omega_n \tau)\, d\tau \\
&= \left(5 \times 10^4\right) \frac{A}{m\omega_n} \left(\int_0^t \sin(\omega_n \tau)\, d\tau - \int_0^t e^{-3(t-\tau)} \sin(\omega_n \tau)\, d\tau \right) \qquad (1)
\end{aligned}
$$

The first integral in (1) is now evaluated

$$
\begin{aligned}
\int_0^t \sin(\omega_n \tau)\, d\tau &= -\frac{1}{\omega_n} \left(\cos(\omega_n \tau)\right)_0^t \\
&= \frac{-1}{\omega_n} \left(\cos(\omega_n t) - 1\right) \\
&= \frac{1}{\omega_n} \left(1 - \cos(\omega_n t)\right) \qquad (2)
\end{aligned}
$$

The second integral in (1) is

$$
\begin{aligned}
\int_0^t e^{-3(t-\tau)} \sin(\omega_n \tau)\, d\tau &= \int_0^t e^{-3t+3\tau} \sin(\omega_n \tau)\, d\tau \\
&= \int_0^t e^{-3t} e^{3\tau} \sin(\omega_n \tau)\, d\tau \\
&= e^{-3t} \int_0^t e^{3\tau} \sin(\omega_n \tau)\, d\tau \qquad (3)
\end{aligned}
$$

This integral is found using tables

$$
\int e^{ax} \sin(bx)\, dx = \frac{e^{ax} \left(a \sin(bx) - b \cos(bx)\right)}{a^2 + b^2}
$$

Where in this case $a = 3$ and $b = \omega_n$ Therefore (3) becomes

$$\begin{aligned}
e^{-3t}\int_0^t e^{3\tau}\sin(\omega_n\tau)\,d\tau &= e^{-3t}\left(\frac{e^{3\tau}(3\sin(\omega_n\tau)-\omega_n\cos(\omega_n\tau))}{9+\omega_n^2}\right)_0^t\\
&= \frac{e^{-3t}}{9+\omega_n^2}\left(e^{3\tau}(3\sin(\omega_n\tau)-\omega_n\cos(\omega_n\tau))\right)_0^t\\
&= \frac{e^{-3t}}{9+\omega_n^2}\left(e^{3t}(3\sin(\omega_n t)-\omega_n\cos(\omega_n t))-(-\omega_n)\right)\\
&= \frac{e^{-3t}}{9+\omega_n^2}\left(e^{3t}(3\sin(\omega_n t)-\omega_n\cos(\omega_n t))+\omega_n\right)\\
&= \frac{1}{9+\omega_n^2}\left(3\sin(\omega_n t)-\omega_n\cos(\omega_n t)+\omega_n e^{-3t}\right) \qquad (4)
\end{aligned}$$

Substituting (2,4) into (1) gives the final result

$$x_{conv}(t) = \left(5\times 10^4\right)\frac{A}{m\omega_n}\left(\frac{1}{\omega_n}(1-\cos(\omega_n t)) - \frac{3\sin(\omega_n t)-\omega_n\cos(\omega_n t)+\omega_n e^{-3t}}{9+\omega_n^2}\right) \qquad (5)$$

Because initial conditions are zero then

$$\begin{aligned}
x(t) &= x_h(t) + x_{cov(t)}\\
&= x_{cov(t)}
\end{aligned}$$

Comparing (5) above to equation (4) found using option (1) shows they are the same as expected.