HW 6, ME 440 Intermediate Vibration, Fall 2017

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December 30, 2019

0.1 Problem 1

Problem 1

Download the ANSYS input file "*1DOF_spring_mass-problem_18p1.txt*" from Canvas and step through the ANSYS tutorial "*Intro to ANSYS modal analysis*" that is also posted to Canvas. Using the parameters defined in the text file, analytically determine the natural frequency of the 1 degree of freedom system. Show your work for this calculation and then compare the analytical and finite element results. And then answer the following questions:

a) Does ANSYS provide the frequency (f) or the circular frequency (ω)?
b) Can we verify the amplitude of displacement shown on Slide 10 of the "Intro to ANSYS modal analysis" slides? Why or why not?

The input file to ANSYS is given to us in plain text file as the following

```
/filnam, 1DOF_spring_mass
/title, 1 Degree of freedom spring mass example
/prep7
!element type
et,1,mass21                !element type no.1 is mass21
et,2,combin14                !element type no.2 is combination 14 (this is a spring element)
! model parameters
mass = 10                 ! mass of mass element
k = 10                        ! spring stiffness
initial_l = 2               ! initial spring length (equilibrium length)
n_modes = 1                ! number of modes wanted
!real constants
r,1,mass                 ! real constant set 1 is for the point mass
r,2,k,,,,,initial_l         ! real constant set 2 is for the spring
!create nodes
n,1,0,0,0                 ! Node 1 is at x=0, y=0, z=0
n,2,initial_l,0,0         ! Node 2 is at x=initial_l, y=0, z=0
!create elements
type,2                        ! specify element type of subsequently defined elements
real,2                        ! specify real constant set of subsequently defined elements
e,1,2                       ! define element to start at node 1 and end at node 2
type,1                        ! specify element type of subsequently defined elements
real,1                        ! specify real constant set of subsequently defined elements
e,2                        ! define element to be created at node 2
!displacement boundary conditions
nsel,s,loc,x,0                         !select node at x = 0
d,all,ux,0                         !displacement of selected node in x-dir is 0
d,all,uy,0                         !displacement of selected node in y-dir is 0
d,all,uz,0                         !displacement of selected node in z-dir is 0
nsel,s,loc,x,initial_l                 !select node at x = initial_l
d,all,uy,0                         !displacement of selected node in y-dir is 0
d,all,uz,0                         !displacement of selected node in z-dir is 0
allsel
finish
/solu                                  !select static load solution
antype,modal
modopt,lanb,n_modes
solve
```

```
finish
/post1
```

0.1.1 Part (1)

For a mass-spring system the equation of motion is

$$\ddot{x} + \omega_n^2 x = 0$$

Where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10}{10}} = 1$ rad/sec. Since $\omega_n = 2\pi f_n$, hence $f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} = 0.1592$ Hz. Therefore the frequency given by ANSYS is in Hz and not the circular frequency rad/sec.

0.1.2 Part (2)

Unable to verify this result. At first I thought ANSYS uses gravity and the spring is vertically connected, therefore the static displacement would be

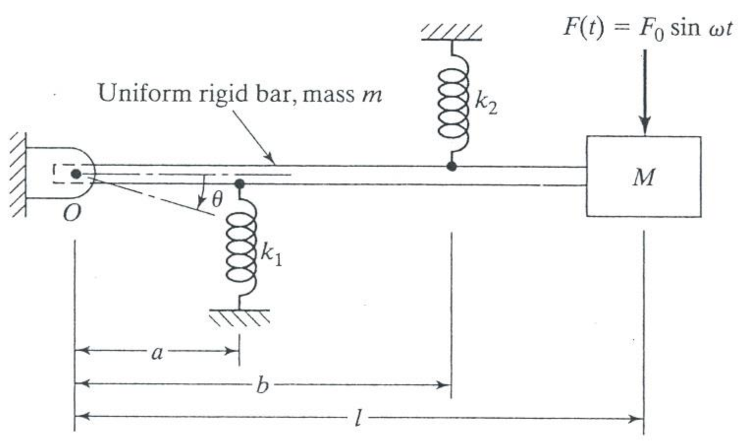
$$x_{st} = \frac{W}{k}$$

Where W is the weight attached to end of spring. But this gives $x_{st} = \frac{mg}{k} = \frac{10g}{10} = g$. And depending on units used (ANSYS do not use units and assumes that the input is using correct units), then value shown which is 0.316228 should be numerical value of g. But this would not be valid number using any units. Unable to find out how ANSYS came up with this value.

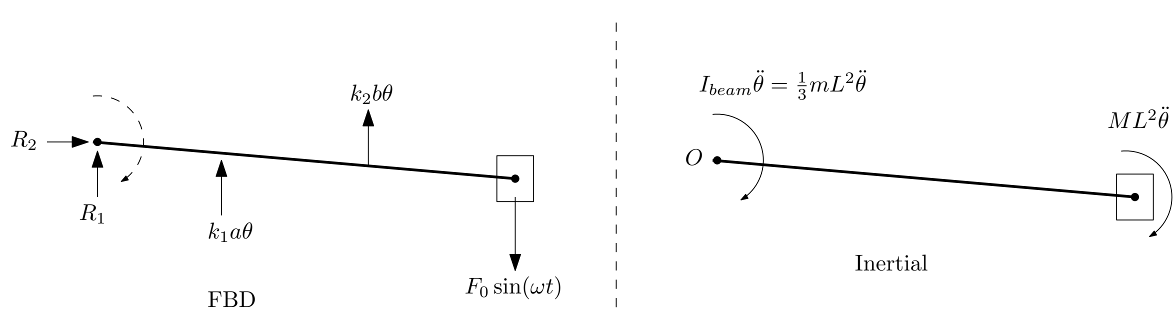
0.2 Problem 2

Problem 2

Derive the equation of motion and find the steady-state response $\{\theta(t)\}$ of the system shown below for rotational motion about the hinge O for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1$ m, $M = 50$ kg, $m = 10$ kg, $F_o = 500$ N and $\omega = 1000$ rpm. Give the steady-state response in the simplest form possible.



The free body diagram and the inertial diagram are given below. It is assumed that motion is measured from equilibrium position with the mass already in attached to springs. Hence the weight of the beam do not show up in the FBD.



Taking moments around hinge at point o and using anti-clockwise as positive gives (assuming small angle θ)

$$k_1 (a\theta) a + k_2 (b\theta) b - F_0 \sin(\omega t) L = -\left(\frac{1}{3} mL^2 + ML^2\right)\ddot{\theta}$$

$$\left(\frac{1}{3} mL^2 + ML^2\right)\ddot{\theta} + \theta\left(k_1 a^2 + k_2 b^2\right) = F_0 \sin(\omega t) L$$

In standard form, the above becomes

$$m_{eq}\ddot{\theta} + k_{eq}\theta = F_0 \sin\omega t$$

Where

$$\begin{aligned}\omega_n^2 &= \frac{k_{eq}}{m_{eq}}\\ &= \frac{k_1 a^2 + k_2 b^2}{L^2\left(\frac{1}{3}m + M\right)}\end{aligned}$$

This model is single degree of freedom system, undamped, with forced input. Hence we know its solution is given by

$$\theta(t) = \theta_h(t) + \theta_p(t)$$

Where $\theta_p(t)$ is particular solution and $\theta_h(t)$ is homogenous solution. We know that

$$\theta_h(t) = c_1 \cos\omega_n t + c_2 \sin\omega_n t$$

And assuming $\theta_p(t) = X \sin\omega t$. Now we need to check if $\omega \neq \omega_n$ so to decide on which solution to pick. Using the numerical values given

$$\begin{aligned}k_{eq} &= k_1 a^2 + k_2 b^2\\ &= (5000)(0.25)^2 + (5000)(0.5)^2\\ &= 1562.5 \text{ N/m}\end{aligned}$$

And

$$\begin{aligned}M_{eq} &= L^2\left(\frac{1}{3}m + M\right)\\ &= (1)^2\left(\left(\frac{1}{3}\right)(10) + 50\right)\\ &= 53.333 \text{ kg}\end{aligned}$$

Hence

$$\omega_n = \sqrt{\frac{k_{eq}}{M_{eq}}} = \sqrt{\frac{1562.5}{53.333}} = 5.413 \text{ rad/sec}$$

But the forcing frequency is given as

$$\omega = 1000\left(\frac{2\pi}{rev}\right)\left(\frac{\min}{60}\right) = 1000\left(\frac{2\pi}{60}\right) = 104.72 \text{ rad/sec}$$

Hence $\underline{\omega \neq \omega_n}$. We also see $\omega > \omega_n$ which means $r > 1$ where $r = \frac{\omega}{\omega_n}$, so we also expect that particular solution displacement maximum displacement to be negative. Now we use the standard solution, which is

$$\theta_p(t) = X \sin\omega t$$

Where

$$\begin{aligned} X &= \frac{F_0}{k_{eq} - m_{eq}\omega^2} \\ &= \frac{F_0}{m_{eq}} \frac{1}{\frac{k_{eq}}{m_{eq}} - \omega^2} \\ &= \frac{F_0}{m_{eq}} \frac{1}{\omega_n^2 - \omega^2} \\ &= \frac{F_0}{m_{eq}\omega_n^2} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \\ &= \frac{F_0}{k_{eq}} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \end{aligned}$$

Calling $\frac{\omega}{\omega_n} = r$, which is the standard notation and since $\frac{F_0}{k_{eq}} = x_{st}$ the static deflection, then the above becomes

$$X = \frac{x_{st}}{1 - r^2}$$

We notice again, since $r > 1$ in this problem, then X is negative. It is out of phase with the forcing function. The particular solution can now be written as

$$\begin{aligned} \theta_p(t) &= X \sin \omega t \\ &= \frac{x_{st}}{1 - r^2} \sin \omega t \end{aligned}$$

And the total solution is

$$\theta(t) = \overbrace{c_1 \cos \omega_n t + c_2 \sin \omega_n t}^{\text{homogeneous}} + \overbrace{\frac{x_{st}}{1 - r^2} \sin \omega t}^{\text{particular}} \tag{1}$$

Assuming initial conditions are $\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$, then (1) at $t = 0$ becomes

$$\theta_0 = c_1$$

Hence solution becomes

$$\theta(t) = \theta_0 \cos \omega_n t + c_2 \sin \omega_n t + \frac{x_{st}}{1 - r^2} \sin \omega t$$

Taking derivative

$$\theta'(t) = \omega_n \theta_0 \sin \omega_n t + \omega_n c_2 \cos \omega_n t + \omega \frac{x_{st}}{1 - r^2} \cos \omega t$$

At $t = 0$ the above becomes

$$\dot{\theta}_0 = \omega_n c_2 + \omega \frac{x_{st}}{1 - r^2}$$

Hence

$$\begin{aligned} c_2 &= \frac{\dot{\theta}_0}{\omega_n} - \frac{\omega}{\omega_n} \frac{x_{st}}{1 - r^2} \\ &= \frac{\dot{\theta}_0}{\omega_n} - \frac{r}{1 - r^2} x_{st} \end{aligned}$$

Therefore the solution now becomes (again, this is for $\omega \neq \omega_n$)

$$\theta(t) = \overbrace{\theta_0 \cos \omega_n t + \left(\frac{\dot{\theta}_0}{\omega_n} - \frac{r}{1 - r^2} x_{st} \right) \sin \omega_n t}^{\text{homogeneous}} + \overbrace{\left(\frac{x_{st}}{1 - r^2} \right) \sin \omega t}^{\text{particular}} \tag{2}$$

The problem now asks for steady state solution. It is not clear to me what is this meant to be, since there is no damping in the system, and hence the full solution remain for all time. Therefore, will show the full solution (using zero initial conditions) and will also show the particular solution.

This is a plot of the full solution, assuming that all initial conditions are zero. Therefore, this is a plot of this solution

$$\theta(t) = -\frac{F_0}{k_{eq}} \frac{r}{1 - r^2} \sin \omega_n t + \left(\frac{F_0}{k_{eq}} \frac{1}{1 - r^2} \right) \sin \omega t$$

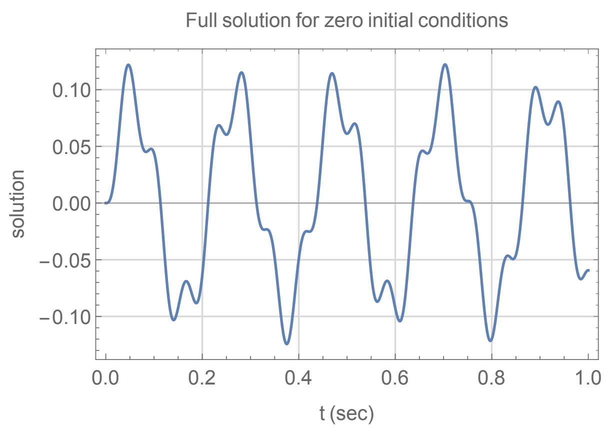
Obtained from (2) by setting $\theta_0 = 0, \dot{\theta}_0 = 0$

$$\theta(t) = -\frac{500}{1562.5}\left(\frac{3.5744}{1-(3.5744)^2}\right)\sin(5.413t) + \left(\frac{500}{1562.5}\left(\frac{1}{1-(3.5744)^2}\right)\right)\sin(104.72t)$$
$$= 0.09713\sin(5.413t) - 0.0272\sin(104.72t)$$

Here is a plot of the full solution for the first 1 second

```
In[15]:= x[t_] := 0.09713 Sin[29.297 t] - 0.0272 Sin[104.72 t];
  p = Plot[x[t], {t, 0, 1}, Frame → True,
    FrameLabel → {{"solution", None}, {"t (sec)", "Full solution for zero initial conditions"}},
    BaseStyle → 12, GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[16]=



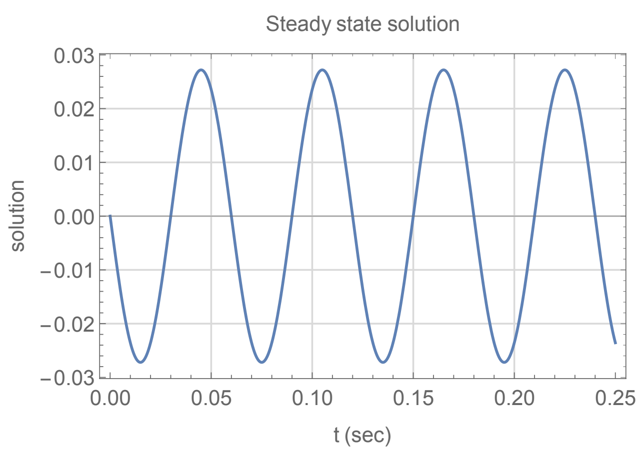
The particular solution (steady state?) is

$$\theta_p(t) = \frac{x_{st}}{1-r^2}\sin\omega t$$
$$= 0.0272\sin(104.72t)$$

Here is a plot of the particular solution for the first 0.25 second

```
In[61]:= x[t_] := -0.0272 Sin[104.72 t];
  p = Plot[x[t], {t, 0, .25}, Frame → True,
    FrameLabel → {{"solution", None}, {"t (sec)", "Steady state solution"}}, BaseStyle → 12,
    GridLines → Automatic, GridLinesStyle → LightGray]
```

Out[62]=



0.3 Problem 3

Problem 3

A spring-mass system with m = 10 kg and k = 5000 N/m is subjected to a harmonic force of amplitude 250 N and frequency ω. If the maximum amplitude of the mass is observed to be 100 mm, find the value of ω.

The equation of motion (assuming $\sin(\omega t)$ for the force) is[1]

$$m\ddot{x} + kx = F_0 \sin(\omega t)$$

Where $k = 5000$ N/m, $m = 10$ kg, $F_0 = 250$ N. We know the solution to the above is given by (but we here have to assume that $\omega \neq \omega_n$)

$$x(t) = \overbrace{x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1-r^2} x_{st}\right) \sin \omega_n t}^{\text{homogeneous}} + \overbrace{\left(\frac{x_{st}}{1-r^2}\right) \sin \omega t}^{\text{particular}}$$

Looking now at only the steady state solution (in this case, it is the particular solution) then we see that

$$x_{ss}(t) = \left(\frac{x_{st}}{1-r^2}\right) \sin \omega t$$

Hence maximum is

$$x_{\max}(t) = \frac{x_{st}}{1-r^2}$$

we are told that $x_{\max} = 0.1$ meter, and . But $r = \frac{\omega}{\omega_n}$ and $x_{st} = \frac{F_0}{k}$. Therefore the above becomes

$$x_{\max} = \frac{F_0}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

In the above equation everything is known except for ω. Solving for ω gives

$$\begin{aligned} 1 - \left(\frac{\omega}{\omega_n}\right)^2 &= \frac{F_0}{kx_{\max}} \\ \left(\frac{\omega}{\omega_n}\right)^2 &= 1 - \frac{F_0}{kx_{\max}} \\ \omega^2 &= \left(1 - \frac{F_0}{kx_{\max}}\right) \omega_n^2 \end{aligned}$$

But $\omega_n = \sqrt{\frac{k}{m}}$, hence

$$\omega = \sqrt{\frac{k}{m}} \sqrt{\left(1 - \frac{F_0}{kx_{\max}}\right)}$$

Substituting numerical values

$$\begin{aligned} \omega &= \sqrt{\frac{5000}{10}} \sqrt{\left(1 - \frac{250}{(5000)(0.1)}\right)} \\ &= 22.361\sqrt{0.5} \\ &= 15.812 \text{ rad/sec} \end{aligned}$$

[1]The general solution changes depending on if the forcing function is sin or cos. But the particular solution is the same.

ODE	solution
$m\ddot{x} + kx = F_0 \cos \omega t$	$x(t) = \left(x_0 - \frac{x_{st}}{1-r^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{x_{st}}{1-r^2} \cos \omega t$
$m\ddot{x} + kx = F_0 \sin \omega t$	$x(t) = x_0 \cos \omega_n t + \left(\frac{\dot{x}_0}{\omega_n} - \frac{r}{1-r^2} x_{st}\right) \sin \omega_n t + \frac{x_{st}}{1-r^2} \sin \omega t$