HW 5, ME 440 Intermediate Vibration, Fall 2017

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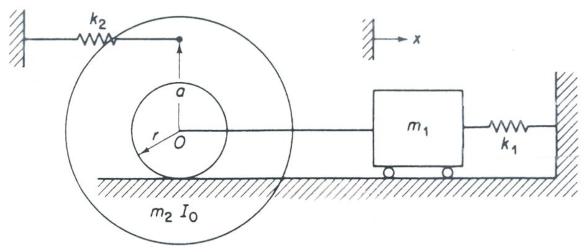
Problem 1

The stepped cylinder is connected to a spring of stiffness k_2 and an inextensible cable. The other end of the inextensible cable is attached to mass m_1. The stepped cylinder rolls without slip on the fixed surface. The mass m_1 rolls on 2 massless cylinders. Assume the system will be limited to small displacements. The total mass of the stepped cylinder is m_2 and it's mass moment of inertia about point O is I_0.



a) In preparation for using Newton's Second Law, sketch the free-body diagram(s) **and** inertial diagram for this system.
b) Using Newton's Laws exclusively, determine the differential equation of motion for small angular oscillations of the mass m_1 (in terms of the generalized coordinate x).

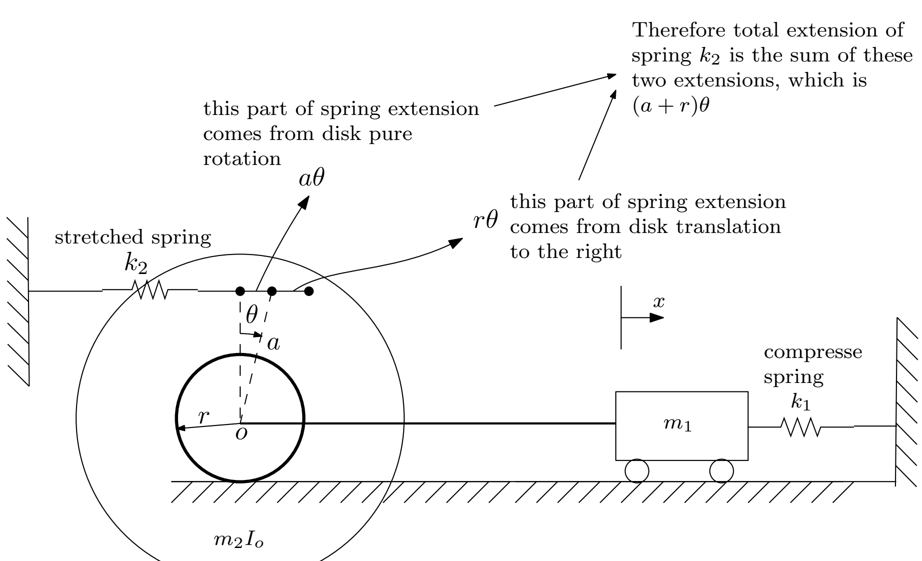
Problem 2

Repeat Problem 1 but use $T_{max} = U_{max}$ to find the natural frequency of the system.

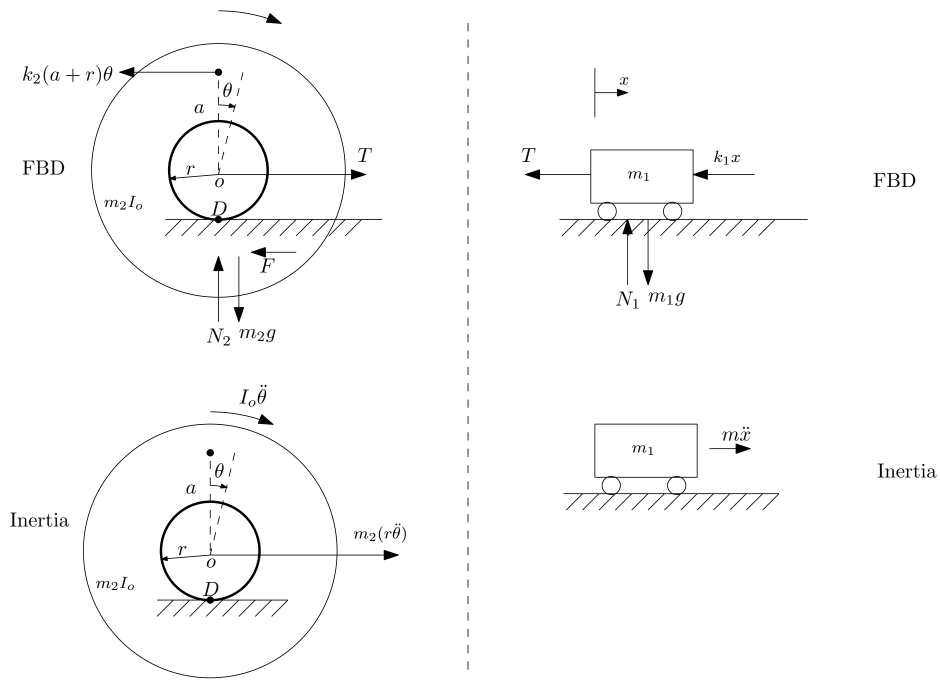
0.1 Problem 1

0.1.1 Part (A)

We start by assuming motion to the right, such that the small disk m_2 rotates clockwise as shown below. So the k_2 spring is stretched by amount $a\theta$ which come due to pure rotation, and it also stretch by $r\theta$ due to disk translation to the right at same time, therefore the spring k_1 will stretch by amount $(a + r)\theta$ and the k_1 spring will be compressed by amount x.



Based on the above, the following is the free body diagram for m_2 and m_1 and the corresponding kinematic diagrams. This assumes small angle θ and that springs remain straight.



0.1.2 Part (B)

Since cable is inextensible, then the constraint is that $x = r\theta$. Starting from the FBD for m_1

$$\begin{aligned}\sum F_x &= m_1\ddot{x}\\ -T - k_1 x &= m_1\ddot{x}\\ m_1\ddot{x} + k_1 x &= -T \end{aligned} \tag{1}$$

We do not need to resolve forces in vertical direction, since no motion is in that direction. To find T, which is the tension in cable, we go back to m_2 and find T.

We can do this part in two ways, either by taking moments around the instantaneous center of zero velocity which is point D at bottom of the small cylinder shown in the diagram, or we can take moments around the C.M. of the disk and then use another equation to solve for the friction F. We will show both methods, and that they give the same result.

<u>Method one, using instantaneous center of zero velocity</u>

Take moments around point D as shown in figure in order to not have to account for the friction force F and the N_2 force on m_2 and using positive as anti-clockwise gives

$$\begin{aligned}\sum M_D &= -I_D\ddot{\theta}\\ k_2\left(a+r\right)\theta\left(a+r\right) - Tr &= -\overbrace{\left(I_o + m_2 r^2\right)}^{\text{parallel axes}}\ddot{\theta}\\ T &= \frac{k_2\left(a+r\right)^2\theta + \left(I_o + m_2 r^2\right)\ddot{\theta}}{r}\end{aligned}$$

But due to constraint, then $\theta = \frac{x}{r}, \ddot{\theta} = \frac{\ddot{x}}{r}$. Hence the above can be written as

$$\begin{aligned}T &= \frac{k_2\frac{x}{r}\left(a+r\right)^2 + \left(I_o + m_2 r^2\right)\frac{\ddot{x}}{r}}{r}\\ &= \frac{x k_2\left(a+r\right)^2}{r^2} + \frac{\left(I_o + m_2 r^2\right)\ddot{x}}{r^2}\end{aligned} \tag{2}$$

Substituting (2) into (1) gives

$$m_1\ddot{x} + k_1 x = -\left(\frac{xk_2\,(a+r)^2}{r^2} + \frac{\left(I_o + m_2 r^2\right)\ddot{x}}{r^2}\right)$$

$$\begin{aligned}
m_1\ddot{x} + \frac{\left(I_o + m_2 r^2\right)\ddot{x}}{r^2} + k_1 x + \frac{xk_2\,(a+r)^2}{r^2} &= 0\\
\ddot{x}\left(m_1 + \frac{\left(I_o + m_2 r^2\right)}{r^2}\right) + x\left(k_1 + \frac{k_2\,(a+r)^2}{r^2}\right) &= 0\\
\ddot{x}\left(\frac{m_1 r^2 + \left(I_o + m_2 r^2\right)}{r^2}\right) + x\left(\frac{k_1 r^2 + k_2\,(a+r)^2}{r^2}\right) &= 0
\end{aligned}$$

Hence

$$\ddot{x}\left(m_1 r^2 + \left(I_o + m_2 r^2\right)\right) + x\left(k_1 r^2 + k_2\,(a+r)^2\right) = 0$$

In standard form

$$\ddot{x} + x\frac{k_1 r^2 + k_2\,(a+r)^2}{r^2\,(m_1 + m_2) + I_o} = 0 \tag{3}$$

Or

$$\ddot{x} + \omega_n^2 x = 0$$

Where

$$\omega_n^2 = \frac{r^2 k_1 + k_2\,(a+r)^2}{r^2\,(m_1 + m_2) + I_o}$$

Method two, moments around center of mass

Using this method. We start by taking moments around the center of mass of the disk m_2 and using positive as anti-clockwise gives

$$\begin{aligned}
\sum M_o &= -I_o\ddot{\theta}\\
(k_2\,(a+r)\,\theta)\,a - Fr &= -I_o\ddot{\theta}\\
F &= \frac{1}{r}\left(I_o\ddot{\theta} + (k_2\,(a+r)\,\theta)\,a\right) \qquad (4)
\end{aligned}$$

Now resolving forces in the x direction for m_2, gives (with positive to the right)

$$\begin{aligned}
\sum F_x &= m_2 r\ddot{\theta}\\
T - k_2\,(a+r)\,\theta - F &= m_2 r\ddot{\theta} \qquad (5)
\end{aligned}$$

Plugging (4) into (5) gives T

$$T - k_2\,(a+r)\,\theta - \frac{1}{r}\left(I_o\ddot{\theta} + (k_2\,(a+r)\,\theta)\,a\right) = m_2 r\ddot{\theta}$$

Solving for T gives

$$T = m_2 r\ddot{\theta} + \frac{1}{r}\left(I_o\ddot{\theta} + (k_2\,(a+r)\,\theta)\,a\right) + k_2\,(a+r)\,\theta$$

We now use the constraint that $x = r\theta$ to write everything in x. Hence $\theta = \frac{x}{r}, \ddot{\theta} = \frac{\ddot{x}}{r}$ and the above now becomes

$$\begin{aligned}
T &= m_2 r\frac{\ddot{x}}{r} + \frac{1}{r}\left(I_o\frac{\ddot{x}}{r} + \left(k_2\,(a+r)\,\frac{x}{r}\right)a\right) + k_2\,(a+r)\,\frac{x}{r}\\
&= m_2\ddot{x} + \frac{1}{r^2}\left(I_o\ddot{x} + (k_2\,(a+r)\,x)\,a\right) + k_2\,(a+r)\,\frac{x}{r}
\end{aligned}$$

Now that we found T, we go back to the equation of motion for m_1 in (1) and substitute the above into it, the result becomes

$$\begin{aligned}
m_1\ddot{x} + k_1 x &= -T\\
&= -\left(m_2\ddot{x} + \frac{1}{r^2}\left(I_o\ddot{x} + (k_2\,(a+r)\,x)\,a\right) + k_2\,(a+r)\,\frac{x}{r}\right)
\end{aligned}$$

Collecting terms

$$
\begin{aligned}
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+k_1x+\frac{1}{r^2}\left(\left(k_2\left(a+r\right)x\right)a\right)+k_2\left(a+r\right)\frac{x}{r}&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{1}{r^2}\left(k_2\left(a+r\right)a\right)+k_2\left(a+r\right)\frac{1}{r}\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left[\left(a+r\right)a+r\left(a+r\right)\right]\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left[a^2+ra+ar+r^2\right]\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left[a^2+2ar+r^2\right]\right)&=0\\
\ddot{x}\left(m_1+m_2+\frac{I_o}{r^2}\right)+x\left(k_1+\frac{k_2}{r^2}\left(a+r\right)^2\right)&=0
\end{aligned}
$$

Or

$$
\begin{aligned}
\ddot{x}\left(r^2\left(m_1+m_2\right)+I_o\right)+x\left(r^2k_1+k_2\left(a+r\right)^2\right)&=0\\
\ddot{x}+x\frac{r^2k_1+k_2\left(a+r\right)^2}{r^2\left(m_1+m_2\right)+I_o}&=0
\end{aligned}
$$

Which is the same equation of motion found in the first method.

0.2 Problem 2

In Rayleigh energy method, we ignore any friction, and assume motion is simple harmonic motion (which is valid, since there is no damping).

The Kinetic energy T of the system is (since disk rolls with no slip)

$$
T=\overbrace{\frac{1}{2}I_o\dot{\theta}^2+\frac{1}{2}m_2v_{cg}^2}^{\text{disk}}+\overbrace{\frac{1}{2}m_1\dot{x}^2}^{\text{cart}}
$$

But $v_{cg}=r\dot{\theta}$, hence the above becomes

$$
T=\frac{1}{2}I_o\dot{\theta}^2+\frac{1}{2}m_2\left(r\dot{\theta}\right)^2+\frac{1}{2}m_1\dot{x}^2
$$

But due to constraint, then $\theta=\frac{x}{r}$, then $\dot{\theta}=\frac{\dot{x}}{r}$ and the above becomes

$$
\begin{aligned}
T&=\frac{1}{2}I_o\left(\frac{\dot{x}}{r}\right)^2+\frac{1}{2}m_2\left(r\frac{\dot{x}}{r}\right)^2+\frac{1}{2}m_1\dot{x}^2\\
&=\frac{1}{2}I_o\frac{\dot{x}^2}{r^2}+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}m_1\dot{x}^2\\
&=\frac{1}{2}\dot{x}^2\left(\frac{I_o}{r^2}+m_2+m_1\right) \qquad (1)
\end{aligned}
$$

The potential energy is

$$
\begin{aligned}
U&=\frac{1}{2}k_2\left(\left(a+r\right)\theta\right)^2+\frac{1}{2}k_1x^2\\
&=\frac{1}{2}k_2\left(\left(a+r\right)\frac{x}{r}\right)^2+\frac{1}{2}k_1x^2\\
&=\frac{1}{2}k_2\left(a+r\right)^2\frac{x^2}{r^2}+\frac{1}{2}k_1x^2 \qquad (2)
\end{aligned}
$$

To find $T_{\max}$ and $U_{\max}$, we now assume m_1 undergoes simple harmonic motion given by $x\left(t\right)=X_{\max}\sin\left(\omega_nt\right)$. Hence $\dot{x}=X_{\max}\omega_n\cos\omega_nt$. Therefore

$$
\begin{aligned}
\dot{x}_{\max}&=X_{\max}\omega_n\\
x_{\max}&=X_{\max}
\end{aligned}
$$

Therefore using these into (1) and (2) gives

$$T_{\max} = \frac{1}{2}\left(\dot{x}_{\max}\right)^2\left(\frac{I_o}{r^2} + m_2 + m_1\right)$$

$$U_{\max} = \frac{1}{2}k_2\left(a+r\right)^2\frac{x_{\max}^2}{r^2} + \frac{1}{2}k_1 x_{\max}^2$$

Or

$$T_{\max} = \frac{1}{2}\left(X_{\max}\omega_n\right)^2\left(\frac{I_o}{r^2} + m_2 + m_1\right)$$

$$U_{\max} = \frac{1}{2}X_{\max}^2\left(\frac{k_2\left(a+r\right)^2}{r^2} + k_1\right)$$

Hence

$$T_{\max} = U_{\max}$$

$$\frac{1}{2}\left(X_{\max}\omega_n\right)^2\left(\frac{I_o}{r^2} + m_2 + m_1\right) = \frac{1}{2}X_{\max}^2\left(\frac{k_2\left(a+r\right)^2}{r^2} + k_1\right)$$

$$\omega_n^2\left(\frac{I_o}{r^2} + m_2 + m_1\right) = \frac{k_2\left(a+r\right)^2 + r^2k_1}{r^2}$$

Solving for ω_n^2

$$\omega_n^2 = \frac{k_2\left(a+r\right)^2 + r^2k_1}{I_o + r^2\left(m_2+m_1\right)}$$

Therefore the equation of motion for m_2 is

$$\ddot{x} + \omega_n^2 x = 0$$

$$\ddot{x} + \frac{k_2\left(a+r\right)^2 + r^2k_1}{I_o + r^2\left(m_2+m_1\right)}x = 0$$

Comparing this to the solution found in first problem, we see they are the same. The Rayleigh energy method was much simpler in this case. But we have to ignore any friction, and assume motion is harmonic, which is reasonable, since this is single degree of freedom system.