HW 10, ME 440 Intermediate Vibration, Fall 2017

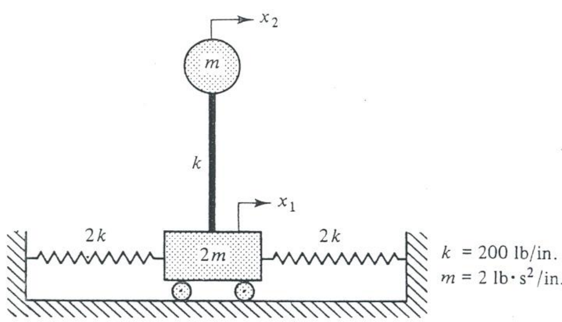
Nasser M. Abbasi

December 30, 2019

Problem 1. Use Newton's Law to determine the equation of motion. Solve for the natural frequencies and mode shapes without using a computer (solve by hand). Use your hand written solution to write out the 2x2 modal matrix (normalized) and the 2x2 Ω matrix.

Problem 2. Solve for the natural frequencies and mode shapes using Matlab. (Include a screen shot of your Matlab output.)

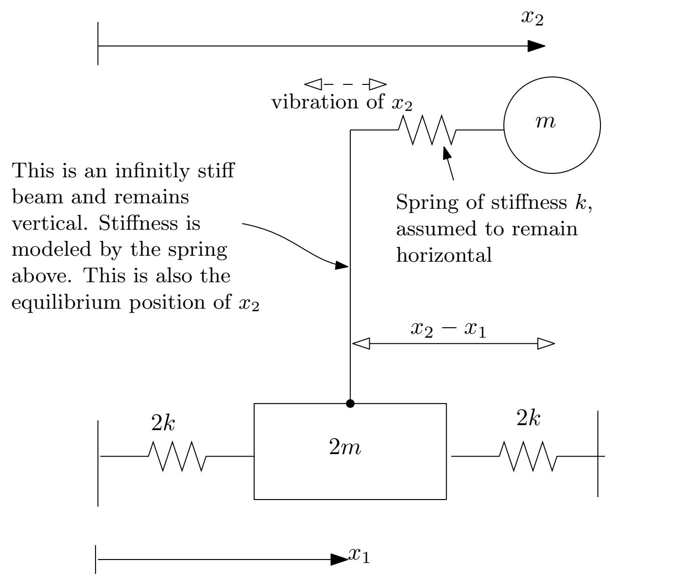
The sphere of mass m is attached to the end of a cantilevered beam that is fixed to a carriage of mass $2m$ as shown in the figure below. The generalized coordinates of the system are the absolute displacements x_1 and x_2 of the carriage and sphere, respectively. Determine (a) the mass and stiffness matrices of the system, and (b) the system's natural circular frequencies and modal matrix $[u]$ if $k = 200$ lb / in. and $m = 2$ lb·s^2 / in.



Partial answer: $\omega_2 = 16.68$ rad/s

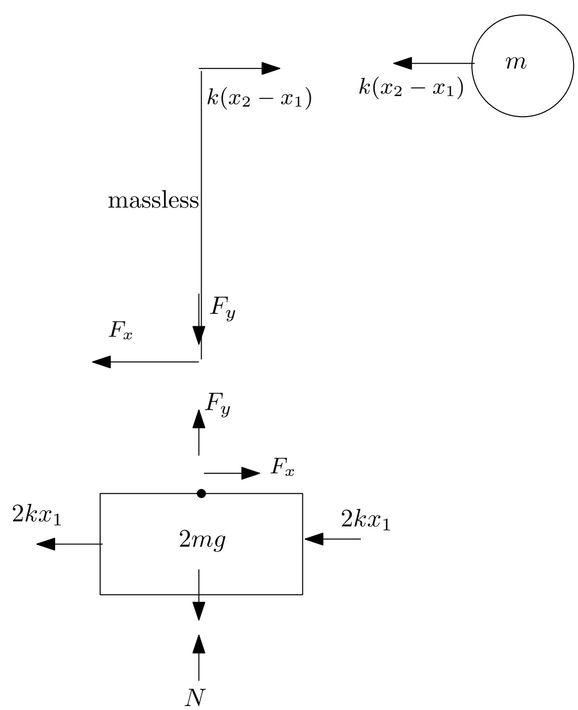
0.1 Problem 1

To make it easier to obtain the equation of motions, the top mass m is modeled as attached to spring of stiffness k which is in turn attached to an infinitely stiff vertical massless beam. This way the vibration of the mass m at the top can be more easily modeled.



Simplified model of the original system

Based on the above diagram, we now obtain the free body diagram as follows. In this, we assume that $x_2 > x_1$ and both as positive. Hence spring k attached to m is in tension.



The top mass m vibrates in horizontal direction only. Hence this assumes the spring will remain horizontal and we must assume that $x_2 - x_1$ remain small for this model to be realistic.

From this free body diagram we see now that the reaction force F_x is equal to $k(x_2 - x_1)$. (By resolving forces in the x direction for the massless beam).

Therefore

$$F_x = k(x_2 - x_1)$$

And the equation of motion for x_2 is

$$\begin{aligned} m\ddot{x}_2 &= -k(x_2 - x_1) \\ m\ddot{x}_2 + kx_2 - kx_1 &= 0 \end{aligned} \tag{1}$$

The equation of motion for the cart is

$$\begin{aligned} 2m\ddot{x}_1 &= -4kx_1 + F_x \\ 2m\ddot{x}_1 &= -4kx_1 + k(x_2 - x_1) \\ 2m\ddot{x}_1 + 5kx_1 - kx_2 &= 0 \end{aligned} \tag{2}$$

Writing (1) and (2) in matrix form

$$\boxed{\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 5k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

Or

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1000 & -200 \\ -200 & 200 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The first step is to find the eigenvalues (which are the square of the natural frequency) for the system.

Let

$$\begin{aligned} A &= M^{-1}K \\ &= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1000 & -200 \\ -200 & 200 \end{bmatrix} \end{aligned}$$

But

$$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \frac{1}{\det(M)} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= \frac{1}{8} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

Hence

$$A = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1000 & -200 \\ -200 & 200 \end{bmatrix}$$
$$= \begin{bmatrix} 250 & -50 \\ -100 & 100 \end{bmatrix}$$

Now we will find the eigenvalues of A (these will be the ω_n^2 values). To find the eigenvalues of A, we solve

$$\det([A] - \lambda [I]) = 0$$
$$\det\left(\begin{bmatrix} 250 & -50 \\ -100 & 100 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) =$$
$$\begin{vmatrix} 250 - \lambda & -50 \\ -100 & 100 - \lambda \end{vmatrix} =$$
$$(250 - \lambda)(100 - \lambda) - 5000 = 0$$
$$\lambda^2 - 350\lambda + 20\,000 = 0$$

Hence

$$\lambda = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{350}{2} \pm \frac{\sqrt{350^2 - 4(20\,000)}}{2}$$
$$= 175 \pm 103.08$$
$$= \{71.92, 278.08\}$$

Therefore, the eigenvalues are

$$\lambda = \omega_n^2 = \{71.92, 278.08\} \tag{3}$$

The natural frequencies of the system are the sqrt of the eigenvalues. Therefore

$$\omega_n = \left\{\sqrt{71.92}, \sqrt{278.08}\right\}$$
$$= \{8.4806, 16.676\}$$

Hence

$$\omega_{n(1)} = 8.4806 \text{ rad/sec}$$
$$\omega_{n(2)} = 16.676 \text{ rad/sec}$$

The next step is to find the eigenvectors. These are also called the shape vectors, or the u vectors. Each eigenvalue will generate one eigenvector. We need to solve

$$[A]\{u\} = \lambda\{u\}$$

For each eigenvalue, we find the corresponding eigenvector.

For $\lambda = 71.92$, we obtain the equation

$$\begin{bmatrix} 250 & -50 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} = 71.92 \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix}$$

From first equation

$$250u_{11} - 50u_{21} = 71.92u_{11}$$

We always let $u_{11} = 1$. Therefore

$$\begin{aligned} 250 - 50u_{21} &= 71.92 \\ u_{21} &= \frac{250 - 71.92}{50} \\ &= 3.5616 \end{aligned}$$

Therefore, the first eigenvector is

$$\vec{u}_1 = \begin{Bmatrix} 1 \\ 3.5616 \end{Bmatrix}$$

For $\lambda = 278.08$, we obtain the equation

$$\begin{bmatrix} 250 & -50 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} = 278.08 \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix}$$

From first equation

$$250u_{12} - 50u_{22} = 278.08u_{12}$$

We always let $u_{12} = 1$. Hence

$$\begin{aligned} 250 - 50u_{22} &= 278.08 \\ u_{22} &= \frac{250 - 278.08}{50} \\ &= -0.561\,6 \end{aligned}$$

Therefore, the second eigenvector is

$$\vec{u}_2 = \begin{Bmatrix} 1 \\ -0.561\,6 \end{Bmatrix}$$

Therefore the <u>modal matrix</u> $[u]$ is

$$\boxed{u = \begin{bmatrix} 1 & 1 \\ 3.5616 & -0.5616 \end{bmatrix}}$$

And Ω matrix is

$$\begin{aligned} \Omega &= \begin{bmatrix} \omega_{n(1)}^2 & 0 \\ 0 & \omega_{n(2)}^2 \end{bmatrix} \\ &= \begin{bmatrix} 71.92 & 0 \\ 0 & 278.08 \end{bmatrix} \end{aligned}$$

And the system of equations written in principle coordinates q is

$$\left\{\ddot{q}\right\} + [\Omega]\left\{q\right\} = \{0\}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} + \begin{bmatrix} 71.92 & 0 \\ 0 & 278.08 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which is now decoupled. The solution in normal coordinates is

$$\begin{aligned} \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} &= A_1 \begin{Bmatrix} u_{11} \\ u_{21} \end{Bmatrix} \cos\left(\omega_{n(1)}t - \phi_1\right) + A_2 \begin{Bmatrix} u_{12} \\ u_{22} \end{Bmatrix} \cos\left(\omega_{n(2)}t - \phi_2\right) \\ &= A_1 \begin{Bmatrix} 1 \\ 3.5616 \end{Bmatrix} \cos\left(8.481t - \phi_1\right) + A_2 \begin{Bmatrix} 1 \\ -0.561\,6 \end{Bmatrix} \cos\left(16.676t - \phi_2\right) \end{aligned}$$

0.1.1 Appendix

This is derivation of the same equations of motions using energy method. (In this example, this method is much simpler to use to find equation of motions). The kinetic energy of the system is

$$T = \frac{1}{2} m\dot{x}_2^2 + \frac{1}{2}(2m)\dot{x}_1^2$$

And the potential energy comes only from the springs, since we assumed the top mass m remain horizontal as it vibrates back and forth

$$U = \frac{1}{2} 4kx_1^2 + \frac{1}{2} k(x_2 - x_1)^2$$

Therefore the Lagrangian is

$$\begin{aligned} \Gamma &= T - U \\ &= \frac{1}{2} m\dot{x}_2^2 + m\dot{x}_1^2 - \frac{1}{2}(4k)x_1^2 - \frac{1}{2}k(x_2 - x_1)^2 \end{aligned}$$

<u>EQM for x_1</u>

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\Gamma}{\dot{x}_1}\right)-\frac{\partial\Gamma}{x_1}&=0\\
\frac{d}{dt}\left(2m\dot{x}_1\right)-\left(-4kx_1+k\left(x_2-x_1\right)\right)&=0\\
2m\ddot{x}_1-\left(-4kx_1+kx_2-kx_1\right)&=0\\
2m\ddot{x}_1-\left(-5kx_1+kx_2\right)&=0\\
2m\ddot{x}_1+5kx_1-kx_2&=0 \qquad (1)
\end{aligned}$$

<u>EQM for x_2</u>

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\Gamma}{\dot{x}_2}\right)-\frac{\partial\Gamma}{x_2}&=0\\
\frac{d}{dt}\left(m\dot{x}_2\right)-\left(-k\left(x_2-x_1\right)\right)&=0\\
m\ddot{x}_2-\left(-kx_2+kx_1\right)&=0\\
m\ddot{x}_2+kx_2-kx_1&=0 \qquad (2)
\end{aligned}$$

In Matrix form (1,2) becomes

$$\begin{bmatrix}2m & 0\\ 0 & m\end{bmatrix}\begin{Bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{Bmatrix}+\begin{bmatrix}5k & -k\\ -k & k\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Which is the same exact result obtained earlier.

0.2 Problem 2

The Matlab code is the following

```
%Solve HW 10, problem 2 using Matlab
%Nasser M. Abbasi, ME 440, Fall 2017
%see HW 10 for more details.

m = 2;
k = 200;

mass_mat = [2*m 0;
            0   m]

stiffness_mat = [5*k -k;
                 -k   k]

A_mat = inv(mass_mat) * stiffness_mat

[eig_vectors, eig_values] = eig(A_mat);

natural_frequencies = sqrt(diag( eig_values))

eig_vectors(:,1) = eig_vectors(:,1)/eig_vectors(1,1);
eig_vectors(:,2) = eig_vectors(:,2)/eig_vectors(1,2);

eig_vectors
```

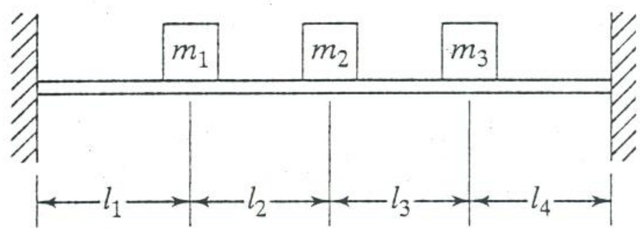
The output is

```

mass_mat =
     4     0
     0     2

stiffness_mat =
        1000        -200
        -200         200

A_mat =
   250   -50
  -100   100

natural_frequencies =
   16.6757
    8.4807

eig_vectors =
    1.0000    1.0000
   -0.5616    3.5616
```

0.3 Problem 3

Problem 3.

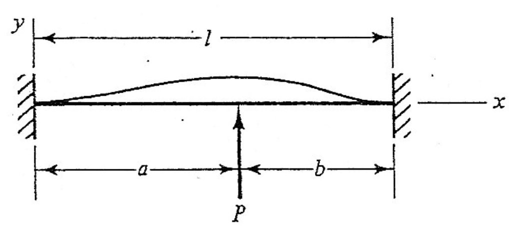
Determine the flexibility matrix of the uniform beam shown in the figure below. Disregard the mass of the beam compared to the concentrated masses fastened on the beam and assume the beam has a stiffness of EI and that all $l_i = l$.



Definitions For stiffness matrix $[K]$, element k_{ij} means: Apply unit displacement at location j and measure the force at location i. While for flexibility matrix $[a]$, its element a_{ij} means: Apply unit force at location j and measure the displacement at location i.

To solve this problem, this part of handout is used

Fixed-fixed beam*



(i)

$$y = \frac{Pb^2}{6EIl^3}[(2b-3l)x^3 + 3l(l-b)x^2] \qquad (x \le a)$$

$$y = \frac{Pb^2}{6EIl^3}\left[(2b-3l)x^3 + 3l(l-b)x^2 + \frac{l^3}{b^2}(x-a)^3\right] \qquad (x \ge a)$$

Since $[a]$ is symmetric, only lower triangle part needs to be found (or upper triangle).

$$\begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

To find a_{11}, a unit force is put at location m_1 and displacement at m_1 is measured. To find a_{21}, a unit force is put at location m_1 and displacement at m_2 is measured and so on. The formulas in the above hand out are used for this. To speed this process and make less error, a small function is written to do the computation. Here is the function and the result generated for $a_{11}, a_{21}, a_{32}, a_{22}, a_{32}, a_{33}$

Define the function to find a_ij

```
getFlexibility[x_, a_, b_] := Piecewise[{
```

$$\left\{\frac{b^2}{6\,\text{E0}\,\text{I0}\,\text{L0}^3}\left((2\,b - 3\,\text{L0})\,x^3 + 3\,\text{L0}\,(\text{L0} - b)\,x^2\right),\ x \le a\right\},$$

$$\left\{\frac{b^2}{6\,\text{E0}\,\text{I0}\,\text{L0}^3}\left((2\,b - 3\,\text{L0})\,x^3 + 3\,\text{L0}\,(\text{L0} - b)\,x^2 + \frac{\text{L0}^3}{b^2}\,(x - a)^3\right),\ x > a\right\}\}];$$

Call the function to find each element in lower triangle

In[43]:=
```
L0 = 4 L;
a = L; b = 3 L; x = L;
flex[1, 1] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[45]= $\frac{9\,\text{L}^3}{64\,\text{E0}\,\text{I0}}$

In[48]:=
```
a = L; b = 3 L; x = 2 L;
flex[2, 1] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[49]= $\frac{\text{L}^3}{6\,\text{E0}\,\text{I0}}$

In[50]:=
```
a = L; b = 3 L; x = 3 L;
flex[3, 1] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[51]= $\frac{13\,\text{L}^3}{192\,\text{E0}\,\text{I0}}$

In[52]:=
```
a = 2 L; b = 2 L; x = 2 L;
flex[2, 2] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[53]= $\frac{\text{L}^3}{3\,\text{E0}\,\text{I0}}$

In[54]:=
```
a = 2 L; b = 2 L; x = 3 L;
flex[3, 2] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[55]= $\frac{\text{L}^3}{6\,\text{E0}\,\text{I0}}$

In[56]:=
```
a = 3 L; b = L; x = 3 L;
flex[3, 3] = Assuming[x > 0, Simplify[getFlexibility[x, a, b]]]
```

Out[57]= $\frac{9\,\text{L}^3}{64\,\text{E0}\,\text{I0}}$

Therefore, using this result, the lower triangle is

$$\begin{bmatrix} \frac{9}{64} & & \\ \frac{1}{6} & \frac{1}{3} & \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix} \frac{L^3}{EI}$$

Hence by symmetry

$$[a] = \begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix} \frac{L^3}{EI}$$