HW7, Math 322, Fall 2016

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Contents

1 HW 7

1.1 Problem 5.6.1 (a)

5.6.1. **Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of**

(a) $\frac{d^2\phi}{dx^2}+(\lambda-x^2)\phi=0$ **with** $\frac{d\phi}{dx}(0)=0$ **and** $\phi(1)=0$

(b) $\frac{d^2\phi}{dx^2}+(\lambda-x)\phi=0$ **with** $\frac{d\phi}{dx}(0)=0$ **and** $\frac{d\phi}{dx}(1)+2\phi(1)=0$

***(c)** $\frac{d^2\phi}{dx^2}+\lambda\phi=0$ **with** $\phi(0)=0$ **and** $\frac{d\phi}{dx}(1)+\phi(1)=0$ **(See Exercise 5.8.10.)**

1.1.1 part (a)

$$
\begin{aligned}
\frac{d^2\phi}{dx^2}+\left(\lambda-x^2\right)\phi&=0\\
\phi'(0)&=0\\
\phi(1)&=0
\end{aligned}
$$

Putting the equation in the form

$$\frac{d^2\phi}{dx^2}-x^2\phi=-\lambda\phi$$

And comparing it to the standard Sturm-Liouville form

$$p\frac{d^2\phi}{dx^2}+p'\frac{d\phi}{dx}+q\phi=-\lambda\sigma\phi$$

Shows that

$$
\begin{aligned}
p&=1\\
q&=-x^2\\
\sigma&=1
\end{aligned}
$$

Now the Rayleigh quotient is

$$\lambda=\frac{-\left(p\phi\phi'\right)_0^1+\int_0^1 p\left(\phi'\right)^2-q\phi^2dx}{\int_0^1\sigma\phi^2dx}$$

Substituting known values, and since $\phi'(0)=0,\phi(1)=0$ the above simplifies to

$$\lambda=\frac{\int_0^1\left(\phi'\right)^2+x^2\phi^2dx}{\int_0^1\phi^2dx}$$

Now we can say that

$$\lambda_{\min}=\lambda_1\leq\frac{\int_0^1\left(\phi'\right)^2+x^2\phi^2dx}{\int_0^1\phi^2dx}\tag{1}$$

We now need a trial solution ϕ_{trial} to use in the above, which needs only to satisfy boundary conditions to use to estimate lowest $\lambda_{\min}$. The simplest such function will do. The boundary conditions are $\phi'(0)=0$,$\phi(1)=0$. We see for example that $\phi_{trial}(x)=x^2-1$ works, since $\phi'_{trial}(x)=2x$, and $\phi'_{trial}(0) = 0$ and $\phi_{trial}(1) = 1 - 1 = 0$. So will use this in (1)

$$\begin{aligned}
\lambda_{\min} = \lambda_1 &\leq \frac{\int_0^1 (2x)^2 + x^2\left(x^2-1\right)^2 dx}{\int_0^1 \left(x^2-1\right)^2 dx} \\
&= \frac{\int_0^1 (2x)^2 + x^2\left(x^4 - 2x^2 + 1\right) dx}{\int_0^1 \left(x^4 - 2x^2 + 1\right) dx} \\
&= \frac{\int_0^1 4x^2 + x^6 - 2x^4 + x^2 dx}{\int_0^1 \left(x^4 - 2x^2 + 1\right) dx} \\
&= \frac{\int_0^1 3x^2 + x^6 dx}{\int_0^1 x^4 - 2x^2 + 1 dx} \\
&= \frac{\left(x^3 + \frac{1}{7}x^7\right)_0^1}{\left(\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right)_0^1} = \frac{\left(1 + \frac{1}{7}\right)}{\left(\frac{1}{5} - \frac{2}{3} + 1\right)} \\
&= \frac{15}{7} \\
&= 2.1429
\end{aligned}$$

Hence

$$\lambda_1 \leq 2.1429$$

1.2 Problem 5.6.2

5.6.2. Consider the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$$

subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).

$$\begin{aligned}
\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi &= 0 \\
\phi'(0) &= 0 \\
\phi'(1) &= 0
\end{aligned}$$

Putting the equation in the form

$$\frac{d^2\phi}{dx^2} - x^2\phi = -\lambda\phi$$

And comparing it to the standard Sturm-Liouville form

$$p\frac{d^2\phi}{dx^2} + p'\frac{d\phi}{dx} + q\phi = -\lambda\sigma\phi$$

Shows that

$$\begin{aligned}
p &= 1 \\
q &= -x^2 \\
\sigma &= 1
\end{aligned}$$

Now the Rayleigh quotient is

$$\lambda = \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p\left(\phi'\right)^2 - q\phi^2 dx}{\int_0^1 \sigma\phi^2 dx}$$

Substituting known values, and since $\phi'(0) = 0, \phi(1) = 0$ the above simplifies to

$$\lambda = \frac{\int_0^1 (\phi')^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx}$$

Since eigenfunction ϕ can not be identically zero, the denominator in the above expression can only be positive, since the integrand is positive. So we need now to consider the numerator term only:

$$\int_0^1 (\phi')^2 dx + \int_0^1 x^2\phi^2 dx$$

For the second term, again, this can only be positive since ϕ can not be zero. For the first term, there are two cases. If ϕ' zero or not. If it is not zero, then the term is positive and we are done. This means $\lambda > 0$. if $\phi' = 0$ then $\int_0^1 (\phi')^2 dx = 0$ and also conclude $\lambda > 0$ thanks to the second term $\int_0^1 x^2\phi^2$ being positive. So we conclude that λ can only be positive.

1.3 Problem 5.6.4

Problem

Consider eigenvalue problem $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda r\phi, 0 < r < 1$ subject to B.C. $|\phi(0)| < \infty$ (you may also assume $\frac{d\phi}{dr}$ bounded). And $\frac{d\phi}{dr}(1) = 0$. (a) prove that $\lambda \geq 0$. (b) Solve the boundary value problem. You may assume eigenfunctions are known. Derive coefficients using orthogonality.

Notice: Correction was made to problem per class email. Book said to show that $\lambda > 0$ which is error changed to $\lambda \geq 0$.

1.3.1 Part (a)

From the problem we see that $p = r, q = 0, \sigma = r$. The Rayleigh quotient is

$$\begin{aligned}
\lambda &= \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p(\phi')^2 - q\phi^2 dr}{\int_0^1 \sigma\phi^2 dr} \\
&= \frac{-\left(r\phi\phi'\right)_0^1 + \int_0^1 r(\phi')^2 dr}{\int_0^1 r\phi^2 dr} \qquad (1)
\end{aligned}$$

The term $-\left(r\phi\phi'\right)_0^1$ expands to

$$-\left((1)\phi(1)\phi'(1) - (0)\phi(0)\phi'(0)\right)$$

Since $\phi'(1) = 0$, the above is zero and Equation(1) reduces to

$$\lambda = \frac{\int_0^1 r(\phi')^2 dr}{\int_0^1 r\phi^2 dr}$$

The denominator above can only be positive, as an eigenfunction ϕ can not be identically zero. For the numerator, we have to consider two cases.

case 1 If $\phi' \neq 0$ then we are done. The numerator is positive and we conclude that $\lambda > 0$.

case 2 If $\phi' = 0$ then ϕ is constant and this means $\lambda = 0$ is possible hence $\lambda \geq 0$. Now we need to show ϕ being constant is also possible. Since $\phi'(1) = 0$, then for $\phi' = 0$ to be true everywhere, it should also be $\phi'(0) = 0$ which means $\phi(0)$ is some constant. We are told that $|\phi(0)| < \infty$. Hence means $\phi(0)$ is constant is possible value (since bounded). Hence $\phi' = 0$ is possible.

Therefore $\lambda \geq 0$. QED.

1.3.2 Part (b)

The ODE is

$$\begin{aligned}
r\phi'' + \phi' + \lambda r\phi &= 0 \qquad 0 < r < 1 \qquad (1)\\
|\phi(0)| &< \infty \\
\phi'(1) &= 0
\end{aligned}$$

In standard form the ODE is $\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0$. This shows that $r = 0$ is a regular singular point. Therefore we try

$$\phi(r) = \sum_{n=0}^{\infty} a_n r^{n+\alpha}$$

Hence

$$\begin{aligned}
\phi'(r) &= \sum_{n=0}^{\infty} (n+\alpha) a_n r^{n+\alpha-1} \\
\phi''(r) &= \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n r^{n+\alpha-2}
\end{aligned}$$

Substituting back into the ODE gives

$$\begin{aligned}
r\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n r^{n+\alpha-2} + \sum_{n=0}^{\infty} (n+\alpha) a_n r^{n+\alpha-1} + \lambda r \sum_{n=0}^{\infty} a_n r^{n+\alpha} &= 0 \\
\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n r^{n+\alpha-1} + \sum_{n=0}^{\infty} (n+\alpha) a_n r^{n+\alpha-1} + \lambda \sum_{n=0}^{\infty} a_n r^{n+\alpha+1} &= 0
\end{aligned}$$

To make all powers of r the same, we subtract 2 from the power of last term, and add 2 to the index, resulting in

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n r^{n+\alpha-1} + \sum_{n=0}^{\infty} (n+\alpha) a_n r^{n+\alpha-1} + \lambda \sum_{n=2}^{\infty} a_{n-2} r^{n+\alpha-1} = 0$$

For $n = 0$ we obtain

$$\begin{aligned}
(\alpha)(\alpha-1) a_0 r^{\alpha-1} + (\alpha) a_0 r^{\alpha-1} &= 0 \\
(\alpha)(\alpha-1) a_0 + (\alpha) a_0 &= 0 \\
a_0\left(\alpha^2 - \alpha + \alpha\right) &= 0 \\
a_0 \alpha^2 &= 0
\end{aligned}$$

But $a_0 \neq 0$ (we always enforce this condition in power series solution), which implies

$$\boxed{\alpha = 0}$$

Now we look at $n = 1$, which gives

$$\begin{aligned}
(1)(1-1) a_1 r^{n+\alpha-1} + (1) a_1 r^{n+\alpha-1} &= 0 \\
a_1 &= 0
\end{aligned}$$

For $n \geq 2$, now all terms join in, and we get a recursive relation

$$\begin{aligned}
(n)(n-1) a_n r^{n-1} + (n) a_n r^{n-1} + \lambda a_{n-2} r^{n-1} &= 0 \\
(n)(n-1) a_n + (n) a_n + \lambda a_{n-2} &= 0 \\
a_n &= \frac{-\lambda a_{n-2}}{(n)(n-1) + n} \\
&= \frac{-\lambda}{n^2} a_{n-2}
\end{aligned}$$

For example, for $n = 2$, we get

$$a_2 = \frac{-\lambda}{2^2} a_0$$

All odd powers of n result in $a_n = 0$. For $n = 4$

$$a_4 = \frac{-\lambda}{4^2} a_2 = \frac{-\lambda}{4^2}\left(\frac{-\lambda}{2^2} a_0\right) = \frac{\lambda^2}{\left(2^2\right)\left(4^2\right)} a_0$$

And for $n = 6$

$$a_6 = \frac{-\lambda}{6^2} a_4 = \frac{-\lambda}{6^2} \frac{\lambda^2}{\left(2^2\right)\left(4^2\right)} a_0 = \frac{-\lambda^3}{\left(2^2\right)\left(4^2\right)\left(6^2\right)} a_0$$

And so on. The series is

$$\begin{aligned}
\phi(r) &= \sum_{n=0}^{\infty} a_n r^n \\
&= a_0 + 0 + a_2 r^2 + 0 + a_4 r^4 + 0 + a_6 r^6 + \cdots \\
&= a_0 - \frac{\lambda r^2}{2^2} a_0 + \frac{\lambda^2 r^4}{\left(2^2\right)\left(4^2\right)} a_0 - \frac{\lambda^3 r^6}{\left(2^2\right)\left(4^2\right)\left(6^2\right)} a_0 + \cdots \\
&= a_0 \left(1 - \frac{\left(\sqrt{\lambda} r\right)^2}{2^2} + \frac{\left(\sqrt{\lambda} r\right)^4}{\left(2^2\right)\left(4^2\right)} - \frac{\left(\sqrt{\lambda} r\right)^6}{\left(2^2\right)\left(4^2\right)\left(6^2\right)} + \cdots\right) \qquad (2)
\end{aligned}$$

From tables, Bessel function of first kind of order zero, has series expansion given by

$$
\begin{aligned}
J_o(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \\
&= 1 - \left(\frac{z}{2}\right)^2 + \frac{1}{(2)^2}\left(\frac{z}{2}\right)^4 - \frac{1}{((2)(3))^2}\left(\frac{z}{2}\right)^6 + \cdots \\
&= 1 - \frac{z^2}{2^2} + \frac{1}{2^2 4^2} z^4 - \frac{1}{2^2 3^2 2^6} z^6 + \cdots \\
&= 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} + \cdots
\end{aligned} \tag{3}
$$

By comparing (2),(3) we see a match between $J_o(z)$ and $\phi(r)$, if we let $z = \sqrt{\lambda} r$ we conclude that

$$
\boxed{\phi_1(r) = a_0 J_0\left(\sqrt{\lambda} r\right)}
$$

We can now normalized the above eigenfunction so that $a_0 = 1$ as mentioned in class. But it is not needed. The above is the first solution. We now need second solution. For repeated roots, the second solution will be

$$
\phi_2(r) = \phi_1(r) \ln(r) + r^\alpha \sum_{n=0}^{\infty} b_n r^n
$$

But $\alpha = 0$, hence

$$
\phi_2(r) = \phi_1(r) \ln(r) + \sum_{n=0}^{\infty} b_n r^n
$$

Hence the solution is

$$
\phi(r) = c_1 \phi_1(r) + c_2 \phi_2(r)
$$

Since $\phi(0)$ is bounded, then $c_2 = 0$ (since $\ln(0)$ not bounded at zero), and the solution becomes (where a_0 is now absorbed with the constant c_1)

$$
\begin{aligned}
\phi(r) &= c_1 \phi_1(r) \\
&= c J_0\left(\sqrt{\lambda} r\right)
\end{aligned}
$$

The boundedness condition has eliminated the second solution altogether. Now we apply the second boundary conditions $\phi'(1) = 0$ to find allowed eigenvalues. Since

$$
\phi'(r) = -c J_1\left(\sqrt{\lambda} r\right)
$$

Then $\phi'(1) = 0$ implies

$$
0 = -c J_1\left(\sqrt{\lambda}\right)
$$

The zeros of this are the values of $\sqrt{\lambda}$. Using the computer, these are the first few such values.

$$
\begin{aligned}
\sqrt{\lambda_1} &= 3.8317 \\
\sqrt{\lambda_2} &= 7.01559 \\
\sqrt{\lambda_3} &= 10.1735 \\
&\vdots
\end{aligned}
$$

or

$$
\begin{aligned}
\lambda_1 &= 14.682 \\
\lambda_2 &= 49.219 \\
\lambda_3 &= 103.5 \\
&\vdots
\end{aligned}
$$

Hence

$$
\begin{aligned}
\phi_n(r) &= c_n J_0\left(\sqrt{\lambda_n} r\right) \\
\phi(r) &= \sum_{n=1}^{\infty} \phi_n(r) \\
&= \sum_{n=1}^{\infty} c_n J_0\left(\sqrt{\lambda_n} r\right)
\end{aligned} \tag{4}
$$

To find c_n, we use orthogonality. Per class discussion, we can now assume this problem was part of initial value problem, and that at $t = 0$ we had initial condition of $f(r)$, therefore, we now write

$$f(r)=\sum_{n=1}^{\infty}c_nJ_0\left(\sqrt{\lambda_n}r\right)$$

Multiplying both sides by $J_0\left(\sqrt{\lambda_m}r\right)\sigma$ and integrating gives (where $\sigma=r$)

$$\begin{aligned}\int_0^1\sum_{n=1}^{\infty}f(r)J_0\left(\sqrt{\lambda_m}r\right)rdr &= \sum_{n=1}^{\infty}\int_0^1 c_nJ_0\left(\sqrt{\lambda_m}r\right)J_0\left(\sqrt{\lambda_n}r\right)rdr\\ &= \int_0^1 c_mJ_0^2\left(\sqrt{\lambda_m}r\right)rdr\\ &= c_m\int_0^1 J_0^2\left(\sqrt{\lambda_m}r\right)rdr\\ &= c_m\Omega\end{aligned}$$

Where Ω is some constant. Therefore

$$c_n=\frac{\int_0^1\sum_{n=1}^{\infty}f(r)J_0\left(\sqrt{\lambda_m}r\right)rdr}{\Omega}$$

And $\phi(r)$

$$\sum_{n=1}^{\infty}c_nJ_0\left(\sqrt{\lambda_n}r\right)$$

With the eigenvalues given as above, which have to be computed for each n using the computer.

1.4 Problem 5.9.1 (b)

5.9.1. **Estimate (to leading order) the large eigenvalues and corresponding eigenfunctions for**

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)+[\lambda\sigma(x)+q(x)]\phi=0$$

if the boundary conditions are

(a) $\frac{d\phi}{dx}(0)=0$ **and** $\frac{d\phi}{dx}(L)=0$

***(b)** $\phi(0)=0$ **and** $\frac{d\phi}{dx}(L)=0$

(c) $\phi(0)=0$ **and** $\frac{d\phi}{dx}(L)+h\phi(L)=0$

From textbook, equation 5.9.8, we are given that for large λ

$$\begin{aligned}\phi(x) &\approx (\sigma p)^{\frac{-1}{4}}\exp\left(\pm i\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\\ &= (\sigma p)^{\frac{-1}{4}}\left(c_1\cos\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)+c_2\sin\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\right)\end{aligned} \tag{1}$$

Where c_1,c_2 are the two constants of integration since this is second order ODE. For $\phi(0)=0$, the integral $\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt=\int_0^0\sqrt{\frac{\sigma(t)}{p(t)}}dt=0$ and the above becomes

$$\begin{aligned}0 &= \phi(0)\\ &= (\sigma p)^{\frac{-1}{4}}(c_1\cos(0)+c_2\sin(0))\\ &= c_1(\sigma p)^{\frac{-1}{4}}\end{aligned}$$

Hence $c_1=0$ and (1) reduces to

$$\phi(x)=c_2(\sigma p)^{\frac{-1}{4}}\sin\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Hence

$$\phi'(x) = c_2 (\sigma p)^{\frac{-1}{4}} \cos\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\left(\frac{d}{dx}\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$
$$= c_2 (\sigma p)^{\frac{-1}{4}} \cos\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\left(\sqrt{\lambda}\sqrt{\frac{\sigma(x)}{p(x)}}\right)$$
$$= c_2 (\sigma p)^{\frac{-1}{4}} \sqrt{\frac{\lambda\sigma}{p}} \cos\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Since $\phi'(L) = 0$ then

$$0 = c_2 (\sigma p)^{\frac{-1}{4}} \sqrt{\frac{\lambda\sigma}{p}} \cos\left(\sqrt{\lambda}\int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Which means, for non-trivial solution, that

$$\sqrt{\lambda_n}\int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt = \left(n - \frac{1}{2}\right)\pi$$

Therefore, for large λ, (i.e. large n) the estimate is

$$\sqrt{\lambda_n} = \frac{\left(n-\frac{1}{2}\right)\pi}{\int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt}$$
$$\lambda_n = \left(\frac{\left(n-\frac{1}{2}\right)\pi}{\int_0^L \sqrt{\frac{\sigma(t)}{p(t)}}dt}\right)^2$$

1.5 Problem 5.9.2

> **5.9.2. Consider**
> $$\frac{d^2\phi}{dx^2} + \lambda(1+x)\phi = 0$$
> **subject to $\phi(0) = 0$ and $\phi(1) = 0$. Roughly sketch the eigenfunctions for λ large. Take into account amplitude and period variations.**

$$\phi'' + \lambda(1+x)\phi = 0$$

Comparing the above to Sturm-Liouville form

$$(p\phi')' + q\phi = -\lambda\sigma\phi$$

Shows that

$$p = 1$$
$$q = 0$$
$$\sigma = 1 + x$$

Now, from textbook, equation 5.9.8, we are given that for large λ

$$\phi(x) \approx (\sigma p)^{\frac{-1}{4}} \exp\left(\pm i\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$
$$= (\sigma p)^{\frac{-1}{4}} \left(c_1 \cos\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) + c_2 \sin\left(\sqrt{\lambda}\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\right) \tag{1}$$

Where c_1, c_2 are the two constants of integration since this is second order ODE. For $\phi(0) = 0$, the integral $\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt = \int_0^0 \sqrt{\frac{\sigma(t)}{p(t)}}dt = 0$ and the above becomes

$$\begin{aligned}0 &= \phi(0)\\ &= (\sigma p)^{\frac{-1}{4}} (c_1 \cos(0) + c_2 \sin(0))\\ &= c_1 (\sigma p)^{\frac{-1}{4}}\end{aligned}$$

Hence $c_1 = 0$ and (1) reduces to

$$\phi(x) = c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Applying the second boundary condition $\phi(1) = 0$ on the above gives

$$\begin{aligned}0 &= \phi(1)\\ &= c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\end{aligned}$$

Hence for non-trivial solution we want, for large positive integer n

$$\begin{aligned}\sqrt{\lambda} \int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}}dt &= n\pi\\ \sqrt{\lambda} &= \frac{n\pi}{\int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}}dt}\\ &= \frac{n\pi}{\int_0^1 \sqrt{1+t}dt}\end{aligned}$$

But $\int_0^1 \sqrt{1+t}dt = 1.21895$, hence

$$\begin{aligned}\sqrt{\lambda} &= \frac{n\pi}{1.21895} = 2.5773n\\ \lambda &= 6.6424n^2\end{aligned}$$

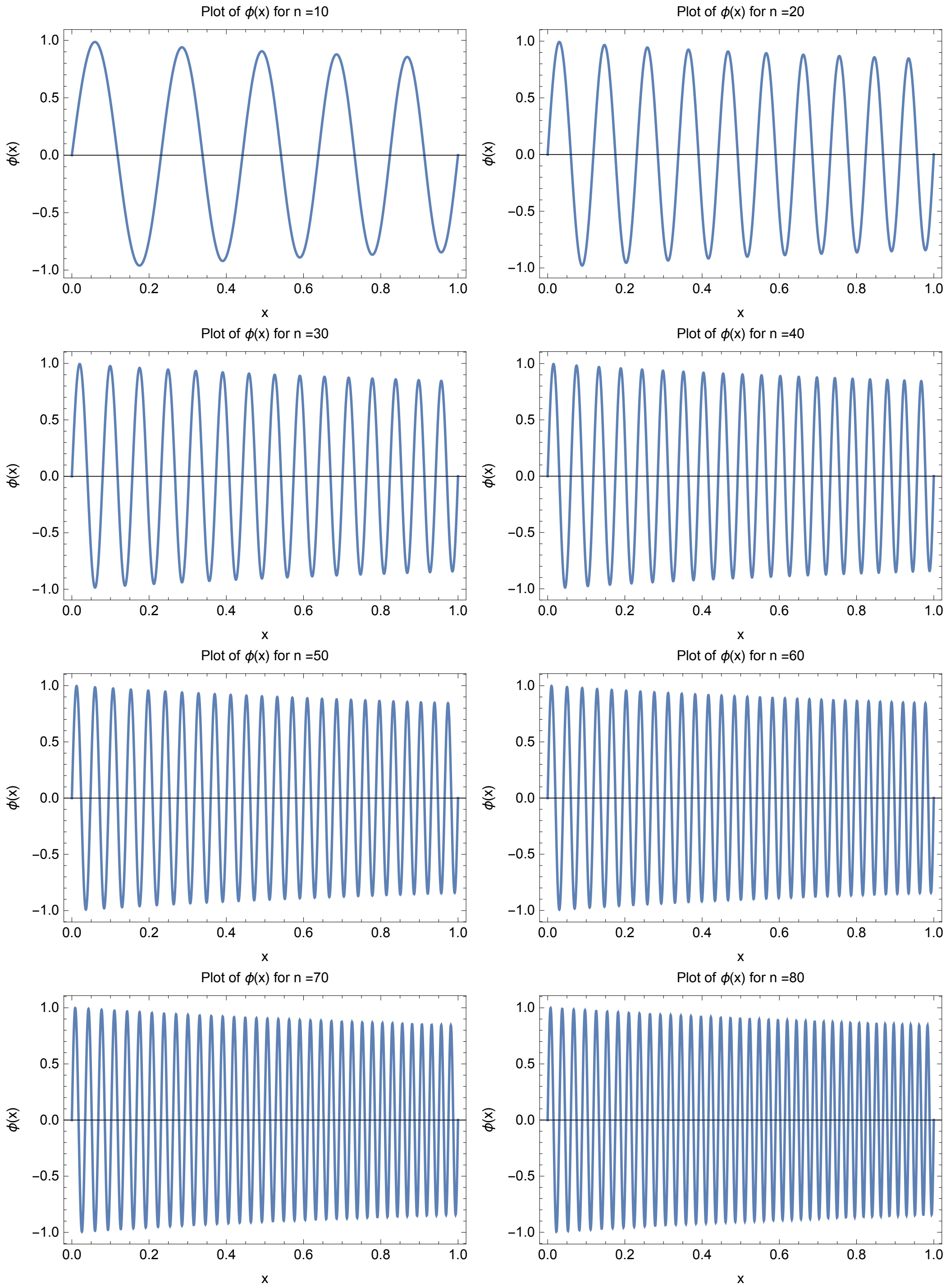
Therefore, solution for large λ is

$$\begin{aligned}\phi(x) &= c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\\ &= c_2 (\sigma p)^{\frac{-1}{4}} \sin\left(2.5773n \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\\ &= c_2 (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \int_0^x \sqrt{1+t}dt\right)\\ &= c_2 (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \left(-\frac{2}{3} + \frac{2}{3}(1+x)^{\frac{3}{2}}\right)\right)\end{aligned}$$

To plot this, let us assume $c_2 = 1$ (we have no information given to find c_2). What value of n to use? Will use different values of n in increasing order. So the following is plot of

$$\phi(x) = (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \left(-\frac{2}{3} + \frac{2}{3}(1+x)^{\frac{3}{2}}\right)\right)$$

For $x = 0 \cdots 1$ and for $n = 10, 20, 30, \cdots, 80$.


Plot of ϕ(x) for n =10
Plot of ϕ(x) for n =20
Plot of ϕ(x) for n =30
Plot of ϕ(x) for n =40
Plot of ϕ(x) for n =50
Plot of ϕ(x) for n =60
Plot of ϕ(x) for n =70
Plot of ϕ(x) for n =80
ϕ(x)
x