# HW7, Math 322, Fall 2016

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December 30, 2019

## Contents

# 1 HW 7

## 1.1 Problem 5.6.1 (a)

**5.6.1.** **Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of**

**(a)** $\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$ **with** $\frac{d\phi}{dx}(0) = 0$ **and** $\phi(1) = 0$

**(b)** $\frac{d^2\phi}{dx^2} + (\lambda - x)\phi = 0$ **with** $\frac{d\phi}{dx}(0) = 0$ **and** $\frac{d\phi}{dx}(1) + 2\phi(1) = 0$

***(c)** $\frac{d^2\phi}{dx^2} + \lambda\phi = 0$ **with** $\phi(0) = 0$ **and** $\frac{d\phi}{dx}(1) + \phi(1) = 0$ **(See Exercise 5.8.10.)**

### 1.1.1 part (a)

$$\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi = 0$$
$$\phi'(0) = 0$$
$$\phi(1) = 0$$

Putting the equation in the form

$$\frac{d^2\phi}{dx^2} - x^2\phi = -\lambda\phi$$

And comparing it to the standard Sturm-Liouville form

$$p\frac{d^2\phi}{dx^2} + p'\frac{d\phi}{dx} + q\phi = -\lambda\sigma\phi$$

Shows that

$$p = 1$$
$$q = -x^2$$
$$\sigma = 1$$

Now the Rayleigh quotient is

$$\lambda = \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p\left(\phi'\right)^2 - q\phi^2 dx}{\int_0^1 \sigma\phi^2 dx}$$

Substituting known values, and since $\phi'(0) = 0, \phi(1) = 0$ the above simplifies to

$$\lambda = \frac{\int_0^1 \left(\phi'\right)^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx}$$

Now we can say that

$$\lambda_{\min} = \lambda_1 \leq \frac{\int_0^1 \left(\phi'\right)^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx} \tag{1}$$

We now need a trial solution $\phi_{trial}$ to use in the above, which needs only to satisfy boundary conditions to use to estimate lowest $\lambda_{\min}$. The simplest such function will do. The boundary conditions are $\phi'(0) = 0, \phi(1) = 0$. We see for example that $\phi_{trial}(x) = x^2 - 1$ works, since $\phi'_{trial}(x) = 2x$, and $\phi'_{trial}(0) = 0$ and $\phi_{trial}(1) = 1 - 1 = 0$. So will use this in (1)

$$\begin{aligned}
\lambda_{\min} = \lambda_1 &\leq \frac{\int_0^1 (2x)^2 + x^2\left(x^2-1\right)^2 dx}{\int_0^1 \left(x^2-1\right)^2 dx} \\
&= \frac{\int_0^1 (2x)^2 + x^2\left(x^4 - 2x^2 + 1\right) dx}{\int_0^1 \left(x^4 - 2x^2 + 1\right) dx} \\
&= \frac{\int_0^1 4x^2 + x^6 - 2x^4 + x^2 dx}{\int_0^1 \left(x^4 - 2x^2 + 1\right) dx} \\
&= \frac{\int_0^1 3x^2 + x^6 dx}{\int_0^1 x^4 - 2x^2 + 1 dx} \\
&= \frac{\left(x^3 + \frac{1}{7}x^7\right)_0^1}{\left(\frac{1}{5}x^5 - \frac{2}{3}x^3 + x\right)_0^1} = \frac{\left(1 + \frac{1}{7}\right)}{\left(\frac{1}{5} - \frac{2}{3} + 1\right)} \\
&= \frac{15}{7} \\
&= 2.1429
\end{aligned}$$

Hence

$$\lambda_1 \leq 2.1429$$

## 1.2 Problem 5.6.2

**5.6.2. Consider the eigenvalue problem**

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$$

**subject to $\frac{d\phi}{dx}(0) = 0$ and $\frac{d\phi}{dx}(1) = 0$. Show that $\lambda > 0$ (be sure to show that $\lambda \neq 0$).**

$$\begin{aligned}
\frac{d^2\phi}{dx^2} + \left(\lambda - x^2\right)\phi &= 0 \\
\phi'(0) &= 0 \\
\phi'(1) &= 0
\end{aligned}$$

Putting the equation in the form

$$\frac{d^2\phi}{dx^2} - x^2\phi = -\lambda\phi$$

And comparing it to the standard Sturm-Liouville form

$$p\frac{d^2\phi}{dx^2} + p'\frac{d\phi}{dx} + q\phi = -\lambda\sigma\phi$$

Shows that

$$p = 1$$
$$q = -x^2$$
$$\sigma = 1$$

Now the Rayleigh quotient is

$$\lambda = \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p\left(\phi'\right)^2 - q\phi^2 dx}{\int_0^1 \sigma\phi^2 dx}$$

Substituting known values, and since $\phi'(0) = 0, \phi(1) = 0$ the above simplifies to

$$\lambda = \frac{\int_0^1 \left(\phi'\right)^2 + x^2\phi^2 dx}{\int_0^1 \phi^2 dx}$$

Since eigenfunction $\phi$ can not be identically zero, the denominator in the above expression can only be positive, since the integrand is positive. So we need now to consider the numerator term only:

$$\int_0^1 \left(\phi'\right)^2 dx + \int_0^1 x^2\phi^2 dx$$

For the second term, again, this can only be positive since $\phi$ can not be zero. For the first term, there are two cases. If $\phi'$ zero or not. If it is not zero, then the term is positive and we are done. This means $\lambda > 0$. if $\phi' = 0$ then $\int_0^1 \left(\phi'\right)^2 dx = 0$ and also conclude $\lambda > 0$ thanks to the second term $\int_0^1 x^2\phi^2$ being positive. So we conclude that $\lambda$ can only be positive.

## 1.3 Problem 5.6.4

Problem

Consider eigenvalue problem $\frac{d}{dr}\left(r\frac{d\phi}{dr}\right) = -\lambda r\phi, 0 < r < 1$ subject to B.C. $\left|\phi(0)\right| < \infty$ (you may also assume $\frac{d\phi}{dr}$ bounded). And $\frac{d\phi}{dr}(1) = 0$. (a) prove that $\lambda \geq 0$. (b) Solve the boundary value problem. You may assume eigenfunctions are known. Derive coefficients using orthogonality.

Notice: Correction was made to problem per class email. Book said to show that $\lambda > 0$ which is error changed to $\lambda \geq 0$.

### 1.3.1 Part (a)

From the problem we see that $p = r, q = 0, \sigma = r$. The Rayleigh quotient is

$$\begin{aligned} \lambda &= \frac{-\left(p\phi\phi'\right)_0^1 + \int_0^1 p\left(\phi'\right)^2 - q\phi^2 dr}{\int_0^1 \sigma\phi^2 dr} \\ &= \frac{-\left(r\phi\phi'\right)_0^1 + \int_0^1 r\left(\phi'\right)^2 dr}{\int_0^1 r\phi^2 dr} \end{aligned} \tag{1}$$

The term $-\left(r\phi\phi'\right)_0^1$ expands to

$$-\left((1)\,\phi\,(1)\,\phi'\,(1) - (0)\,\phi\,(0)\,\phi'\,(0)\right)$$

Since $\phi'(1) = 0$, the above is zero and Equation(1) reduces to

$$\lambda = \frac{\int_0^1 r\left(\phi'\right)^2 dr}{\int_0^1 r\phi^2 dr}$$

The denominator above can only be positive, as an eigenfunction $\phi$ can not be identically zero. For the numerator, we have to consider two cases.

<u>case 1</u> If $\phi' \neq 0$ then we are done. The numerator is positive and we conclude that $\lambda > 0$.

<u>case 2</u> If $\phi' = 0$ then $\phi$ is constant and this means $\lambda = 0$ is possible hence $\lambda \geq 0$. Now we need to show $\phi$ being constant is also possible. Since $\phi'(1) = 0$, then for $\phi' = 0$ to be true everywhere, it should also be $\phi'(0) = 0$ which means $\phi(0)$ is some constant. We are told that $\left|\phi(0)\right| < \infty$. Hence means $\phi(0)$ is constant is possible value (since bounded). Hence $\phi' = 0$ is possible.

Therefore $\lambda \geq 0$. QED.

### 1.3.2 Part (b)

The ODE is

$$\begin{aligned} r\phi'' + \phi' + \lambda r\phi &= 0 \qquad 0 < r < 1 \\ \left|\phi(0)\right| &< \infty \\ \phi'(1) &= 0 \end{aligned} \tag{1}$$

In standard form the ODE is $\phi'' + \frac{1}{r}\phi' + \lambda\phi = 0$. This shows that $r = 0$ is a regular singular point. Therefore we try

$$\phi(r) = \sum_{n=0}^{\infty} a_n r^{n+\alpha}$$

Hence

$$\begin{aligned} \phi'(r) &= \sum_{n=0}^{\infty} (n+\alpha)\, a_n r^{n+\alpha-1} \\ \phi''(r) &= \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1)\, a_n r^{n+\alpha-2} \end{aligned}$$

Substituting back into the ODE gives

$$r\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n r^{n+\alpha-2}+\sum_{n=0}^{\infty}(n+\alpha)a_n r^{n+\alpha-1}+\lambda r\sum_{n=0}^{\infty}a_n r^{n+\alpha}=0$$
$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n r^{n+\alpha-1}+\sum_{n=0}^{\infty}(n+\alpha)a_n r^{n+\alpha-1}+\lambda\sum_{n=0}^{\infty}a_n r^{n+\alpha+1}=0$$

To make all powers of $r$ the same, we subtract 2 from the power of last term, and add 2 to the index, resulting in

$$\sum_{n=0}^{\infty}(n+\alpha)(n+\alpha-1)a_n r^{n+\alpha-1}+\sum_{n=0}^{\infty}(n+\alpha)a_n r^{n+\alpha-1}+\lambda\sum_{n=2}^{\infty}a_{n-2} r^{n+\alpha-1}=0$$

For $n=0$ we obtain

$$\begin{aligned}(\alpha)(\alpha-1)a_0 r^{\alpha-1}+(\alpha)a_0 r^{\alpha-1}&=0\\(\alpha)(\alpha-1)a_0+(\alpha)a_0&=0\\a_0\left(\alpha^2-\alpha+\alpha\right)&=0\\a_0\alpha^2&=0\end{aligned}$$

But $a_0\neq 0$ (we always enforce this condition in power series solution), which implies

$$\boxed{\alpha=0}$$

Now we look at $n=1$, which gives

$$\begin{aligned}(1)(1-1)a_1 r^{n+\alpha-1}+(1)a_1 r^{n+\alpha-1}&=0\\a_1&=0\end{aligned}$$

For $n\geq 2$, now all terms join in, and we get a recursive relation

$$\begin{aligned}(n)(n-1)a_n r^{n-1}+(n)a_n r^{n-1}+\lambda a_{n-2}r^{n-1}&=0\\(n)(n-1)a_n+(n)a_n+\lambda a_{n-2}&=0\\a_n&=\frac{-\lambda a_{n-2}}{(n)(n-1)+n}\\&=\frac{-\lambda}{n^2}a_{n-2}\end{aligned}$$

For example, for $n=2$, we get

$$a_2=\frac{-\lambda}{2^2}a_0$$

All odd powers of $n$ result in $a_n=0$. For $n=4$

$$a_4=\frac{-\lambda}{4^2}a_2=\frac{-\lambda}{4^2}\left(\frac{-\lambda}{2^2}a_0\right)=\frac{\lambda^2}{\left(2^2\right)\left(4^2\right)}a_0$$

And for $n=6$

$$a_6=\frac{-\lambda}{6^2}a_4=\frac{-\lambda}{6^2}\frac{\lambda^2}{\left(2^2\right)\left(4^2\right)}a_0=\frac{-\lambda^3}{\left(2^2\right)\left(4^2\right)\left(6^2\right)}a_0$$

And so on. The series is

$$
\begin{aligned}
\phi(r) &= \sum_{n=0}^{\infty} a_n r^n \\
&= a_0 + 0 + a_2 r^2 + 0 + a_4 r^4 + 0 + a_6 r^6 + \cdots \\
&= a_0 - \frac{\lambda r^2}{2^2} a_0 + \frac{\lambda^2 r^4}{(2^2)(4^2)} a_0 - \frac{\lambda^3 r^6}{(2^2)(4^2)(6^2)} a_0 + \cdots \\
&= a_0 \left( 1 - \frac{\left(\sqrt{\lambda} r\right)^2}{2^2} + \frac{\left(\sqrt{\lambda} r\right)^4}{(2^2)(4^2)} - \frac{\left(\sqrt{\lambda} r\right)^6}{(2^2)(4^2)(6^2)} + \cdots \right) \qquad (2)
\end{aligned}
$$

From tables, Bessel function of first kind of order zero, has series expansion given by

$$
\begin{aligned}
J_o(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \\
&= 1 - \left(\frac{z}{2}\right)^2 + \frac{1}{(2)^2}\left(\frac{z}{2}\right)^4 - \frac{1}{((2)(3))^2}\left(\frac{z}{2}\right)^6 + \cdots \\
&= 1 - \frac{z^2}{2^2} + \frac{1}{2^2 4^2} z^4 - \frac{1}{2^2 3^2 2^6} z^6 + \cdots \\
&= 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} + \cdots \qquad (3)
\end{aligned}
$$

By comparing (2),(3) we see a match between $J_o(z)$ and $\phi(r)$, if we let $z = \sqrt{\lambda} r$ we conclude that

$$
\boxed{\phi_1(r) = a_0 J_0\left(\sqrt{\lambda} r\right)}
$$

We can now normalized the above eigenfunction so that $a_0 = 1$ as mentioned in class. But it is not needed. The above is the first solution. We now need second solution. For repeated roots, the second solution will be

$$
\phi_2(r) = \phi_1(r)\ln(r) + r^{\alpha} \sum_{n=0}^{\infty} b_n r^n
$$

But $\alpha = 0$, hence

$$
\phi_2(r) = \phi_1(r)\ln(r) + \sum_{n=0}^{\infty} b_n r^n
$$

Hence the solution is

$$
\phi(r) = c_1\phi_1(r) + c_2\phi_2(r)
$$

Since $\phi(0)$ is bounded, then $c_2 = 0$ (since $\ln(0)$ not bounded at zero), and the solution becomes (where $a_0$ is now absorbed with the constant $c_1$)

$$
\begin{aligned}
\phi(r) &= c_1\phi_1(r) \\
&= cJ_0\left(\sqrt{\lambda} r\right)
\end{aligned}
$$

The boundedness condition has eliminated the second solution altogether. Now we apply the second boundary conditions $\phi'(1) = 0$ to find allowed eigenvalues. Since

$$
\phi'(r) = -cJ_1\left(\sqrt{\lambda} r\right)
$$

Then $\phi'(1) = 0$ implies

$$0 = -cJ_1\left(\sqrt{\lambda}\right)$$

The zeros of this are the values of $\sqrt{\lambda}$. Using the computer, these are the first few such values.

$$\begin{aligned}\sqrt{\lambda_1} &= 3.8317\\ \sqrt{\lambda_2} &= 7.01559\\ \sqrt{\lambda_3} &= 10.1735\\ &\vdots\end{aligned}$$

or

$$\begin{aligned}\lambda_1 &= 14.682\\ \lambda_2 &= 49.219\\ \lambda_3 &= 103.5\\ &\vdots\end{aligned}$$

Hence

$$\begin{aligned}\phi_n(r) &= c_n J_0\left(\sqrt{\lambda_n}r\right) \qquad (4)\\ \phi(r) &= \sum_{n=1}^{\infty}\phi_n(r)\\ &= \sum_{n=1}^{\infty} c_n J_0\left(\sqrt{\lambda_n}r\right)\end{aligned}$$

To find $c_n$, we use orthogonality. Per class discussion, we can now assume this problem was part of initial value problem, and that at $t = 0$ we had initial condition of $f(r)$, therefore, we now write

$$f(r) = \sum_{n=1}^{\infty} c_n J_0\left(\sqrt{\lambda_n}r\right)$$

Multiplying both sides by $J_0\left(\sqrt{\lambda_m}r\right)\sigma$ and integrating gives (where $\sigma = r$)

$$\begin{aligned}\int_0^1 \sum_{n=1}^{\infty} f(r) J_0\left(\sqrt{\lambda_m}r\right) r dr &= \sum_{n=1}^{\infty}\int_0^1 c_n J_0\left(\sqrt{\lambda_m}r\right) J_0\left(\sqrt{\lambda_n}r\right) r dr\\ &= \int_0^1 c_m J_0^2\left(\sqrt{\lambda_m}r\right) r dr\\ &= c_m\int_0^1 J_0^2\left(\sqrt{\lambda_m}r\right) r dr\\ &= c_m\Omega\end{aligned}$$

Where $\Omega$ is some constant. Therefore

$$c_n = \frac{\int_0^1 \sum_{n=1}^{\infty} f(r) J_0\left(\sqrt{\lambda_m}r\right) r dr}{\Omega}$$

And $\phi(r)$

$$\sum_{n=1}^{\infty} c_n J_0\left(\sqrt{\lambda_n}r\right)$$

With the eigenvalues given as above, which have to be computed for each $n$ using the computer.

## 1.4 Problem 5.9.1 (b)

**5.9.1.** Estimate (to leading order) the large eigenvalues and corresponding eigenfunctions for

$$\frac{d}{dx}\left(p(x)\frac{d\phi}{dx}\right)+[\lambda\sigma(x)+q(x)]\phi=0$$

if the boundary conditions are

(a) $\frac{d\phi}{dx}(0)=0$ and $\frac{d\phi}{dx}(L)=0$

*(b) $\phi(0)=0$ and $\frac{d\phi}{dx}(L)=0$

(c) $\phi(0)=0$ and $\frac{d\phi}{dx}(L)+h\phi(L)=0$

From textbook, equation 5.9.8, we are given that for large $\lambda$

$$\begin{aligned}\phi(x) &\approx (\sigma p)^{\frac{-1}{4}}\exp\left(\pm i\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\\ &=(\sigma p)^{\frac{-1}{4}}\left(c_1\cos\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)+c_2\sin\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\right)\end{aligned}\tag{1}$$

Where $c_1, c_2$ are the two constants of integration since this is second order ODE. For $\phi(0)=0$, the integral $\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt=\int_0^0\sqrt{\frac{\sigma(t)}{p(t)}}dt=0$ and the above becomes

$$\begin{aligned}0&=\phi(0)\\ &=(\sigma p)^{\frac{-1}{4}}(c_1\cos(0)+c_2\sin(0))\\ &=c_1(\sigma p)^{\frac{-1}{4}}\end{aligned}$$

Hence $c_1=0$ and (1) reduces to

$$\phi(x)=c_2(\sigma p)^{\frac{-1}{4}}\sin\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Hence

$$\begin{aligned}\phi'(x)&=c_2(\sigma p)^{\frac{-1}{4}}\cos\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\left(\frac{d}{dx}\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\\ &=c_2(\sigma p)^{\frac{-1}{4}}\cos\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\left(\sqrt{\lambda}\sqrt{\frac{\sigma(x)}{p(x)}}\right)\\ &=c_2(\sigma p)^{\frac{-1}{4}}\sqrt{\frac{\lambda\sigma}{p}}\cos\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\end{aligned}$$

Since $\phi'(L)=0$ then

$$0=c_2(\sigma p)^{\frac{-1}{4}}\sqrt{\frac{\lambda\sigma}{p}}\cos\left(\sqrt{\lambda}\int_0^L\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Which means, for non-trivial solution, that

$$\sqrt{\lambda_n}\int_0^L\sqrt{\frac{\sigma(t)}{p(t)}}dt=\left(n-\frac{1}{2}\right)\pi$$

Therefore, for large $\lambda$, (i.e. large $n$) the estimate is

$$\sqrt{\lambda_n}=\frac{\left(n-\frac{1}{2}\right)\pi}{\int_0^L\sqrt{\frac{\sigma(t)}{p(t)}}dt}$$

$$\lambda_n=\left(\frac{\left(n-\frac{1}{2}\right)\pi}{\int_0^L\sqrt{\frac{\sigma(t)}{p(t)}}dt}\right)^2$$

### 1.5 Problem 5.9.2

**5.9.2.** **Consider**

$$\frac{d^2\phi}{dx^2}+\lambda(1+x)\phi=0$$

**subject to $\phi(0)=0$ and $\phi(1)=0$. Roughly sketch the eigenfunctions for $\lambda$ large. Take into account amplitude and period variations.**

$$\phi''+\lambda(1+x)\phi=0$$

Comparing the above to Sturm-Liouville form

$$\left(p\phi'\right)'+q\phi=-\lambda\sigma\phi$$

Shows that

$$\begin{aligned}p&=1\\q&=0\\\sigma&=1+x\end{aligned}$$

Now, from textbook, equation 5.9.8, we are given that for large $\lambda$

$$\begin{aligned}\phi(x)&\approx(\sigma p)^{\frac{-1}{4}}\exp\left(\pm i\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\\&=(\sigma p)^{\frac{-1}{4}}\left(c_1\cos\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)+c_2\sin\left(\sqrt{\lambda}\int_0^x\sqrt{\frac{\sigma(t)}{p(t)}}dt\right)\right)\qquad(1)\end{aligned}$$

Where $c_1,c_2$ are the two constants of integration since this is second order ODE. For $\phi(0)=0$, the integral $\int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt = \int_0^0 \sqrt{\frac{\sigma(t)}{p(t)}}dt = 0$ and the above becomes

$$\begin{aligned} 0 &= \phi(0) \\ &= \left(\sigma p\right)^{\frac{-1}{4}} \left(c_1 \cos(0) + c_2 \sin(0)\right) \\ &= c_1 \left(\sigma p\right)^{\frac{-1}{4}} \end{aligned}$$

Hence $c_1 = 0$ and (1) reduces to

$$\phi(x) = c_2 \left(\sigma p\right)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right)$$

Applying the second boundary condition $\phi(1) = 0$ on the above gives

$$\begin{aligned} 0 &= \phi(1) \\ &= c_2 \left(\sigma p\right)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) \end{aligned}$$

Hence for non-trivial solution we want, for large positive integer $n$

$$\begin{aligned} \sqrt{\lambda} \int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}}dt &= n\pi \\ \sqrt{\lambda} &= \frac{n\pi}{\int_0^1 \sqrt{\frac{\sigma(t)}{p(t)}}dt} \\ &= \frac{n\pi}{\int_0^1 \sqrt{1+t}dt} \end{aligned}$$

But $\int_0^1 \sqrt{1+t}dt = 1.21895$, hence

$$\begin{aligned} \sqrt{\lambda} &= \frac{n\pi}{1.21895} = 2.5773n \\ \lambda &= 6.6424n^2 \end{aligned}$$

Therefore, solution for large $\lambda$ is

$$\begin{aligned} \phi(x) &= c_2 \left(\sigma p\right)^{\frac{-1}{4}} \sin\left(\sqrt{\lambda} \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) \\ &= c_2 \left(\sigma p\right)^{\frac{-1}{4}} \sin\left(2.5773n \int_0^x \sqrt{\frac{\sigma(t)}{p(t)}}dt\right) \\ &= c_2 (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \int_0^x \sqrt{1+t}dt\right) \\ &= c_2 (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \left(-\frac{2}{3} + \frac{2}{3}(1+x)^{\frac{3}{2}}\right)\right) \end{aligned}$$

To plot this, let us assume $c_2 = 1$ (we have no information given to find $c_2$). What value of $n$ to use? Will use different values of $n$ in increasing order. So the following is plot of

$$\phi(x) = (1+x)^{\frac{-1}{4}} \sin\left(2.5773n \left(-\frac{2}{3} + \frac{2}{3}(1+x)^{\frac{3}{2}}\right)\right)$$

For $x = 0 \cdots 1$ and for $n = 10, 20, 30, \cdots, 80$.