HW6, Math 322, Fall 2016

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1 HW 6

1.1 Problem 5.3.2

5.3.2. Consider

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u + \beta \frac{\partial u}{\partial t}.$$

(a) Give a brief physical interpretation. What signs must α and β have to be physical?

(b) Allow ρ, α, β to be functions of x. Show that separation of variables works only if $\beta = c\rho$, where c is a constant.

(c) If $\beta = c\rho$, show that the spatial equation is a Sturm-Liouville differential equation. Solve the time equation.

1.1.1 Part (a)

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial t^2} + \alpha u + \beta \frac{\partial u}{\partial t}$$

The PDE equation represents the vertical displacement $u(x,t)$ of the string as a function of time and horizontal position. This is 1D wave equation. The term $\beta \frac{\partial u}{\partial t}$ represents the damping force (can be due to motion of the string in air or fluid). The damping coefficient β must be negative to make $\beta \frac{\partial u}{\partial t}$ opposite to direction of motion. Damping force is proportional to velocity and acts opposite to direction of motion.

The term αu represents the stiffness in the system. This is a restoring force, and acts also opposite to direction of motion and is proportional to current displacement from equilibrium position. Hence $\alpha < 0$ also.

1.1.2 Part (b)

Let $u = X(x)T(t)$. Substituting this into the above PDE gives

$$\rho T'' X = T_0 X'' T + \alpha X T + \beta T' X$$

Dividing by $XT \neq 0$

$$\rho \frac{T''}{T} = T_0 \frac{X''}{X} + \alpha + \beta \frac{T'}{T}$$

$$\rho \frac{T''}{T} - \beta \frac{T'}{T} = T_0 \frac{X''}{X} + \alpha$$

To make each side depends on one variable only, we move $\rho(x), \beta(x)$ to the right side since these depends on x. Then dividing by $\rho(x)$ gives

$$\frac{T''}{T} - \frac{\beta}{\rho} \frac{T'}{T} = T_0 \frac{X''}{\rho X} + \frac{\alpha}{\rho}$$

If $\frac{\beta(x)}{\rho(x)} = c$ is constant, then we see the equations have now been separated, since $\frac{\beta(x)}{\rho(x)}$ do not depend on x any more and the above becomes

$$\frac{T''}{T} - c\frac{T'}{T} = T_0 \frac{X''}{\rho X} + \frac{\alpha(x)}{\rho(x)}$$

Now we can say that both side is equal to some constant $-\lambda$ giving the two ODE's

$$\frac{T''}{T} - c\frac{T'}{T} = -\lambda$$

$$T_0 \frac{X''}{\rho X} + \frac{\alpha}{\rho} = -\lambda$$

Or

$$T'' - cT' + \lambda T = 0$$

$$X'' + X\left(\frac{\alpha}{T_0} + \lambda \frac{\rho}{T_0}\right) = 0$$

1.1.3 Part (c)

From above, the spatial ODE is

$$X'' + X\left(\frac{\alpha}{T_0} + \lambda\frac{\rho}{T_0}\right) = 0 \tag{1}$$

Comparing to regular Sturm Liouville (RSL) form, which is

$$\frac{d}{dx}\left(pX'\right) + qX + \lambda\sigma X = 0$$
$$pX'' + p'X' + \left(q + \lambda\sigma\right)X = 0 \tag{2}$$

Comparing (1) and (2) we see that

$$\begin{aligned} p &= 1 \\ q &= \frac{\alpha}{T_0} \\ \sigma &= \frac{\rho}{T_0} \end{aligned}$$

To solve the time ODE $T'' - cT' + \lambda T = 0$, since this is second order linear with constant coefficients, then the characteristic equation is

$$\begin{aligned} r^2 - cr + \lambda &= 0 \\ r &= \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} \\ &= \frac{c}{2} \pm \frac{\sqrt{c^2 - 4\lambda}}{2} \end{aligned}$$

Hence the two solutions are

$$\begin{aligned} T_1(t) &= e^{\left(\frac{c}{2} + \frac{\sqrt{c^2-4\lambda}}{2}\right)t} \\ T_2(t) &= e^{\left(\frac{c}{2} - \frac{\sqrt{c^2-4\lambda}}{2}\right)t} \end{aligned}$$

The general solution is linear combination of the above two solution, therefore final solution is

$$T(t) = c_1 e^{\left(\frac{c}{2} + \frac{\sqrt{c^2-4\lambda}}{2}\right)t} + c_2 e^{\left(\frac{c}{2} - \frac{\sqrt{c^2-4\lambda}}{2}\right)t}$$

Where c_1, c_2 are arbitrary constants of integration.

1.2 Problem 5.3.3

***5.3.3. Consider the non-Sturm-Liouville differential equation**

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + [\lambda\beta(x) + \gamma(x)]\phi = 0.$$

Multiply this equation by $H(x)$. Determine $H(x)$ such that the equation may be reduced to the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + [\lambda\sigma(x) + q(x)]\phi = 0.$$

Given $\alpha(x), \beta(x)$, and $\gamma(x)$, what are $p(x), \sigma(x)$, and $q(x)$?

$$\frac{d^2\phi}{dx^2} + \alpha(x)\frac{d\phi}{dx} + \left(\lambda\beta(x) + \gamma(x)\right)\phi = 0$$

Multiplying by $H(x)$ gives

$$H(x)\phi''(x) + H(x)\alpha(x)\phi'(x) + H(x)\left(\lambda\beta(x) + \gamma(x)\right)\phi = 0 \tag{1}$$

Comparing (1) to Sturm Liouville form, which is

$$\frac{d}{dx}\left(p\phi'\right) + q\phi + \lambda\sigma\phi = 0$$
$$p(x)\phi''(x) + p'(x)\phi'(x) + \left(q + \lambda\sigma\right)\phi(x) = 0 \tag{2}$$

Then we need to satisfy

$$\begin{aligned} H(x) &= P(x) \\ H(x)\alpha(x) &= P'(x) \end{aligned}$$

Therefore, by combining the above, we obtain one ODE equation to solve for $H(x)$

$$H'(x) = H(x)\,\alpha(x)$$

This is first order separable ODE. $\frac{H'}{H} = \alpha$ or $\ln|H| = \int \alpha dx + c$ or

$$H = Ae^{\int \alpha(x)dx}$$

Where A is some constant. By comparing (1),(2) again, we see that

$$q + \lambda\sigma = \lambda\beta(x)\,H(x) + \gamma(x)\,H(x)$$

Summary of solution

$$\begin{aligned}
\sigma(x) &= \beta(x)\,H(x)\\
q(x) &= \gamma(x)\,H(x)\\
P(x) &= H(x)\\
H(x) &= Ae^{\int \alpha(x)dx}
\end{aligned}$$

QED

1.3 Problem 5.3.9

5.3.9. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = 0, \quad \text{and} \quad \phi(b) = 0. \quad (5.3.10)$$

(a) Show that multiplying by $1/x$ puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

(b) Show that $\lambda \geq 0$.

***(c) Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.**

(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the nth eigenfunction has $n-1$ zeros.

$$x^2\phi'' + x\phi' + \lambda\phi = 0 \tag{1}$$
$$\phi(1) = 0$$
$$\phi(b) = 0$$

1.3.1 Part (a)

Multiplying (1) by $\frac{1}{x}$ where $x \neq 0$ gives

$$x\phi'' + \phi' + \frac{\lambda}{x}\phi = 0 \tag{2}$$

Comparing (2) to Sturm-Liouville form

$$p\phi'' + p'\phi' + (q + \lambda\sigma)\,\phi = 0 \tag{3}$$

Then

$$\begin{aligned}
p &= x\\
q &= 0\\
\sigma &= \frac{1}{x}
\end{aligned}$$

And since the given boundary conditions also satisfy the Sturm-Liouville boundary conditions, then (2) is a regular Sturm-Liouville ODE.

1.3.2 Part(b)

Using equation 5.3.8 in page 160 of text (called Raleigh quotient), which applies to regular Sturm-Liouville ODE, which relates the eigenvalues to the eigenfunctions

$$\lambda = \frac{-\left[p\phi\phi'\right]_{x=1}^{x=b} + \int_1^b p\left(\phi'\right)^2 - q\phi^2 dx}{\int_1^b \phi^2\sigma dx} \tag{5.3.8}$$

$$= \frac{-\left[p(b)\phi(b)\phi'(b) - p(1)\phi(1)\phi'(b)\right] + \int_1^b p\left(\phi'\right)^2 - q\phi^2 dx}{\int_1^b \phi^2\sigma dx}$$

Using $p = x, q = 0, \sigma = \frac{1}{x}$ and using $\phi(1) = 0, \phi(b) = 0$, then the above simplifies to

$$\lambda = \frac{-\int_1^b p\left(\phi'\right)^2 dx}{\int_1^b \frac{\phi^2}{x} dx}$$

The integrands in the numerator and denominator can not be negative, since they are squared quantities, and also since $x > 0$ as the domain starts from $x = 1$, then RHS above can not be negative. This means the eigenvalue λ can not be negative. It can only be $\lambda \geq 0$. QED.

1.3.3 Part(c)

The possible values of $\lambda > 0$ are determined by trying to solve the ODE and seeing which λ produces non-trivial solutions given the boundary conditions. The ODE to solve is (1) above. Here it is again

$$x^2\phi'' + x\phi' + \lambda\phi = 0 \tag{1}$$

We know $\lambda \geq 0$, so we do not need to check for negative λ.

Case $\lambda = 0$.

Equation (1) becomes

$$\begin{aligned} x^2\phi'' + x\phi' &= 0 \\ x\phi'' + \phi' &= 0 \\ \frac{d}{dx}\left(x\phi'\right) &= 0 \end{aligned}$$

Hence $x\phi' = c_1$ where c_1 is constant. Therefore $\frac{d}{dx}\phi = \frac{c_1}{x}$ or

$$\begin{aligned} \phi &= c_1 \int \frac{1}{x} dx + c_2 \\ &= c_1 \ln|x| + c_2 \end{aligned}$$

At $x = 1$, $\phi(1) = 0$, hence

$$0 = c_1 \ln(1) + c_2$$

But $\ln(1) = 0$, therefore $c_2 = 0$. The solution now becomes

$$\phi = c_1 \ln|x|$$

At the right end, $x = b, \phi(b) = 0$, therefore

$$0 = c_1 \ln b$$

But since $b > 1$ the above implies that $c_1 = 0$. This gives trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$

$$x^2\phi'' + x\phi' + \lambda\phi = 0$$

This is non-constant coefficients, linear, second order ODE. Let $\phi(x) = x^p$. Equation (1) becomes

$$\begin{aligned} x^2 p\left(p-1\right)x^{p-2} + xpx^{p-1} + \lambda x^p &= 0 \\ p\left(p-1\right)x^p + px^p + \lambda x^p &= 0 \end{aligned}$$

Dividing by $x^p \neq 0$ gives the characteristic equation

$$\begin{aligned} p\left(p-1\right) + p + \lambda &= 0 \\ p^2 - p + p + \lambda &= 0 \\ p^2 &= -\lambda \end{aligned}$$

Since $\lambda \geq 0$ then p is complex. Therefore the roots are

$$p = \pm i\sqrt{\lambda}$$

Therefore the two solutions (eigenfunctions) are

$$\phi_1(x) = x^{i\sqrt{\lambda}}$$
$$\phi_2(x) = x^{-i\sqrt{\lambda}}$$

To more easily use standard form of solution, the standard trick is to rewrite these solution in exponential form

$$\phi_1(x) = e^{i\sqrt{\lambda}\ln x}$$
$$\phi_2(x) = e^{-i\sqrt{\lambda}\ln x}$$

The general solution to (1) is linear combination of these two solutions, therefore

$$\phi(x) = c_1 e^{i\sqrt{\lambda}\ln x} + c_2 e^{-i\sqrt{\lambda}\ln x} \tag{2}$$

Since $\lambda > 0$ then the above can be written using trig functions as

$$\phi(x) = c_1 \cos\left(\sqrt{\lambda}\ln x\right) + c_2 \sin\left(\sqrt{\lambda}\ln x\right)$$

We are now ready to check for allowed values of λ by applying B.C's. The first B.C. gives

$$\begin{aligned} 0 &= c_1 \cos\left(\sqrt{\lambda}\ln 1\right) + c_2 \sin\left(\sqrt{\lambda}\ln 1\right) \\ &= c_1 \cos(0) + c_2 \sin(0) \\ &= c_1 \end{aligned}$$

Hence the solution now simplifies to

$$\phi(x) = c_2 \sin\left(\sqrt{\lambda}\ln x\right)$$

Applying the second B.C. gives

$$0 = c_2 \sin\left(\sqrt{\lambda}\ln b\right)$$

For non-trivial solution we want

$$\begin{aligned} \sqrt{\lambda}\ln b &= n\pi \qquad n = 1,2,3,\cdots \\ \sqrt{\lambda} &= \frac{n\pi}{\ln b} \\ \lambda_n &= \left(\frac{n\pi}{\ln b}\right)^2 \qquad n = 1,2,3,\cdots \end{aligned}$$

Therefore, there are infinite numbers of eigenvalues. The smallest is when $n = 1$ given by

$$\lambda_1 = \left(\frac{\pi}{\ln b}\right)^2$$

1.3.4 Part (d)

From Equation 5.3.6, page 159 in textbook, the eigenfunction are orthogonal with weight function $\sigma(x)$

$$\int_a^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = 0 \qquad n \neq m$$

In this problem, the weight $\sigma = \frac{1}{x}$ and the solution (eigenfuctions) were found above to be

$$\phi_n(x) = \sin\left(\sqrt{\lambda_n}\ln x\right)$$

Now we can verify the orthogonality

$$\int_1^b \phi_n(x)\phi_m(x)\sigma(x)\,dx = \int_{x=1}^{x=b} \sin\left(\frac{n\pi}{\ln b}\ln x\right)\sin\left(\frac{m\pi}{\ln b}\ln x\right)\frac{1}{x}dx$$

Using the substitution $z = \ln x$, then $\frac{dz}{dx} = \frac{1}{x}$. When $x = 1, z = \ln 1 = 0$ and when $x = b, z = \ln b$, then the above integral becomes

$$\begin{aligned} I &= \int_{z=0}^{z=\ln b} \sin\left(\frac{n\pi}{\ln b}z\right)\sin\left(\frac{m\pi}{\ln b}z\right)\frac{dz}{dx}dx \\ &= \int_0^{\ln b} \sin\left(\frac{n\pi}{\ln b}z\right)\sin\left(\frac{m\pi}{\ln b}z\right)dz \end{aligned}$$

But $\sin\left(\frac{n\pi}{\ln b}z\right)$ and $\sin\left(\frac{m\pi}{\ln b}z\right)$ are orthogonal functions (now with weight 1). Hence the above gives 0 when $n \neq m$ using standard orthogonality of the $\sin$ functions we used before many times. QED.

1.3.5 Part(e)

The n^{th} eigenfunction is

$$\phi_n(x) = \sin\left(\frac{n\pi}{\ln b}\ln x\right)$$

Here, the zeros are inside the interval, not counting the end points $x = 1$ and $x = b$.

$$\left(\frac{n\pi}{\ln b}\ln x\right)\Big|_{x=1} = \left(\frac{n\pi}{\ln b}0\right) = 0$$

And

$$\begin{aligned}\left(\frac{n\pi}{\ln b}\ln x\right)\Big|_{x=b} &= \frac{n\pi}{\ln b}\ln b \\ &= n\pi\end{aligned}$$

Hence for $n = 1$, The domain of $\phi_1(x)$ is $0 \cdots \pi$. And there are no zeros inside this for sin function not counting the end points. For $n = 2$, the domain is $0 \cdots 2\pi$ and sin has one zero inside this (at π), not counting end points. And for $n = 3$, the domain is $0 \cdots 3\pi$ and sin has two zeros inside this (at $\pi, 2\pi$), not counting end points. And so on. Hence $\phi_n(x)$ has $n - 1$ zeros not counting the end points.

1.4 Problem 5.5.1 (b,d,g)

5.5.1. A Sturm-Liouville eigenvalue problem is called self-adjoint if

$$p\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b = 0$$

since then $\int_a^b [uL(v) - vL(u)]\ dx = 0$ for any two functions u and v satisfying the boundary conditions. Show that the following yield self-adjoint problems.

(a) $\phi(0) = 0$ and $\phi(L) = 0$

(b) $\frac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

(c) $\frac{d\phi}{dx}(0) - h\phi(0) = 0$ and $\frac{d\phi}{dx}(L) = 0$

(d) $\phi(a) = \phi(b)$ and $p(a)\frac{d\phi}{dx}(a) = p(b)\frac{d\phi}{dx}(b)$

(e) $\phi(a) = \phi(b)$ and $\frac{d\phi}{dx}(a) = \frac{d\phi}{dx}(b)$ [self-adjoint only if $p(a) = p(b)$]

(f) $\phi(L) = 0$ and [in the situation in which $p(0) = 0$] $\phi(0)$ bounded and $\lim_{x\to 0} p(x)\frac{d\phi}{dx} = 0$

***(g) Under what conditions is the following self-adjoint (if p is constant)?**

$$\begin{aligned}\phi(L) + \alpha\phi(0) + \beta\frac{d\phi}{dx}(0) &= 0 \\ \frac{d\phi}{dx}(L) + \gamma\phi(0) + \delta\frac{d\phi}{dx}(0) &= 0\end{aligned}$$

The Sturm-Liouville ODE is

$$\frac{d}{dx}\left(p\phi'\right) + q\phi = -\lambda\sigma\phi$$

Or in operator form, defining $L \equiv \frac{d}{dx}\left(p\frac{d}{dx}\right) + q$, becomes

$$L\left[\phi\right] = -\lambda\sigma\phi$$

The operator L is self adjoined when

$$\int_a^b uL[v]\,dx = \int_a^b vL[u]\,dx$$

For the above to work out, we need to show that

$$p\left(uv' - vu'\right)\Big|_a^b = 0$$

And this is what we will do now.

1.4.1 Part(b)

Here $a=0$ and $b=L$.

$$\begin{aligned} p\left(uv'-vu'\right)\Big|_a^b &= p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_0^L \\ &= \left[p\left(L\right)\left(u\left(L\right)\frac{dv}{dx}\left(L\right)-v\left(L\right)\frac{du}{dx}\left(L\right)\right)-p\left(0\right)\left(u\left(0\right)\frac{dv}{dx}\left(0\right)-v\left(0\right)\frac{du}{dx}\left(0\right)\right)\right] \end{aligned}$$

Substituting $u(L)=v(L)=0$ and $\frac{dv}{dx}(0)=\frac{du}{dx}(0)=0$ into the above (since there are the B.C. given) gives

$$\begin{aligned} p\left(uv'-vu'\right)\Big|_a^b &= \left[p\left(L\right)\left(0\times\frac{dv}{dx}\left(L\right)-0\times\frac{du}{dx}\left(L\right)\right)-p\left(0\right)\left(u\left(0\right)\times 0-v\left(0\right)\times 0\right)\right] \\ &= [0-0] \\ &= 0 \end{aligned}$$

1.4.2 Part (d)

$$\begin{aligned} p\left(uv'-vu'\right)\Big|_a^b &= p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_b^a \\ &= \left[p\left(a\right)\left(u\left(a\right)v'\left(a\right)-v\left(a\right)u'\left(a\right)\right)-p\left(b\right)\left(u\left(b\right)v'\left(b\right)-v\left(b\right)u'\left(b\right)\right)\right] \\ &= p\left(a\right)u\left(a\right)v'\left(a\right)-p\left(a\right)v\left(a\right)u'\left(a\right)-p\left(b\right)u\left(b\right)v'\left(b\right)+p\left(b\right)v\left(b\right)u'\left(b\right) \qquad (1) \end{aligned}$$

We are given that $u(a)=u(b)$ and $v(a)=v(b)$ and $p(a)u'(a)=p(b)u'(b)$ and $p(a)v'(a)=p(b)v'(b)$.

We start by replacing $u(a)$ by $u(a)$ and replacing $v(a)$ by $v(b)$ in (1), this gives

$$\begin{aligned} p\left(uv'-vu'\right)\Big|_a^b &= p\left(a\right)u\left(b\right)v'\left(a\right)-p\left(a\right)v\left(b\right)u'\left(a\right)-p\left(b\right)u\left(b\right)v'\left(b\right)+p\left(b\right)v\left(b\right)u'\left(b\right) \\ &= u\left(b\right)\left(p\left(a\right)v'\left(a\right)-p\left(b\right)v'\left(b\right)\right)+v\left(b\right)\left(p\left(b\right)u'\left(b\right)-p\left(a\right)u'\left(a\right)\right) \end{aligned}$$

Now using $p(a)u'(a)=p(b)u'(b)$ and $p(a)v'(a)=p(b)v'(b)$ in the above gives

$$\begin{aligned} p\left(uv'-vu'\right)\Big|_a^b &= u\left(b\right)\left(p\left(b\right)v'\left(b\right)-p\left(b\right)v'\left(b\right)\right)+v\left(b\right)\left(p\left(b\right)u'\left(b\right)-p\left(b\right)u'\left(b\right)\right) \\ &= u\left(b\right)\left(0\right)+v\left(b\right)\left(0\right) \\ &= 0-0 \\ &= 0 \end{aligned}$$

1.4.3 Part (g)

p is constant. Hence

$$\begin{aligned} p\left(uv'-vu'\right)\Big|_0^L &= p\left(u\frac{dv}{dx}-v\frac{du}{dx}\right)\Big|_0^L \\ &= p\left[\left(u\left(L\right)v'\left(L\right)-v\left(L\right)u'\left(L\right)\right)-\left(u\left(0\right)v'\left(0\right)-v\left(0\right)u'\left(0\right)\right)\right] \qquad (1) \end{aligned}$$

We are given that

$$u\left(L\right)+\alpha u\left(0\right)+\beta u'\left(0\right)=0 \qquad (2)$$
$$u'\left(L\right)+\gamma u\left(0\right)+\delta u'\left(0\right)=0 \qquad (3)$$

And

$$v\left(L\right)+\alpha v\left(0\right)+\beta v'\left(0\right)=0 \qquad (4)$$
$$v'\left(L\right)+\gamma v\left(0\right)+\delta v'\left(0\right)=0 \qquad (5)$$

From (2),

$$u\left(L\right)=-\alpha u\left(0\right)-\beta u'\left(0\right)$$

From (3)

$$u'\left(L\right)=-\gamma u\left(0\right)-\delta u'\left(0\right)$$

From (4)

$$v\left(L\right)=-\alpha v\left(0\right)-\beta v'\left(0\right)$$

From (5)

$$v'\left(L\right)=-\gamma v\left(0\right)-\delta v'\left(0\right)$$

Using these 4 relations in equation (1) gives (where p is removed out, since it is constant, to simplify the equations)

$$
\begin{aligned}
(uv' - vu')|_0^L &= u(L)\,v'(L) - v(L)\,u'(L) - u(0)\,v'(0) + v(0)\,u'(0) \\
&= \left(-\alpha u(0) - \beta u'(0)\right)\left(-\gamma v(0) - \delta v'(0)\right) \\
&\quad - \left(-\alpha v(0) - \beta v'(0)\right)\left(-\gamma u(0) - \delta u'(0)\right) \\
&\quad - u(0)\,v'(0) + v(0)\,u'(0)
\end{aligned}
$$

Simplifying

$$
\begin{aligned}
(uv' - vu')|_0^L &= \alpha u(0)\,\gamma v(0) + \alpha u(0)\,\delta v'(0) + \beta u'(0)\,\gamma v(0) + \beta u'(0)\,\delta v'(0) \\
&\quad - \left(\alpha v(0)\,\gamma u(0) + \alpha v(0)\,\delta u'(0) + \beta v'(0)\,\gamma u(0) + \beta v'(0)\,\delta u'(0)\right) \\
&\quad - u(0)\,v'(0) + v(0)\,u'(0) \\
&= \alpha u(0)\,\gamma v(0) + \alpha u(0)\,\delta v'(0) + \beta u'(0)\,\gamma v(0) + \beta u'(0)\,\delta v'(0) \\
&\quad - \alpha v(0)\,\gamma u(0) - \alpha v(0)\,\delta u'(0) - \beta v'(0)\,\gamma u(0) - \beta v'(0)\,\delta u'(0) - u(0)\,v'(0) + v(0)\,u'(0)
\end{aligned}
$$

Collecting

$$
\begin{aligned}
(uv' - vu')|_0^L &= \alpha\delta\,(u(0)\,v'(0) - v(0)\,u'(0)) \\
&\quad + \beta\delta\,(u'(0)\,v'(0) - v'(0)\,u'(0)) \\
&\quad + \alpha\gamma\,(u(0)\,v(0) - v(0)\,u(0)) \\
&\quad + \beta\gamma\,(u'(0)\,v(0) - v'(0)\,u(0)) \\
&\quad - u(0)\,v'(0) + v(0)\,u'(0) \\
&= \alpha\delta\,(u(0)\,v'(0) - v(0)\,u'(0)) + \beta\gamma\,(u'(0)\,v(0) - v'(0)\,u(0)) - (u(0)\,v'(0) - v(0)\,u'(0)) \\
&= \alpha\delta\,(u(0)\,v'(0) - v(0)\,u'(0)) - \beta\gamma\,(v'(0)\,u(0) - u'(0)\,v(0)) - (u(0)\,v'(0) - v(0)\,u'(0))
\end{aligned}
$$

Let $u(0)\,v'(0) - v(0)\,u'(0) = \Delta$ then we see that the above is just

$$
\begin{aligned}
(uv' - vu')|_0^L &= \alpha\delta\,(\Delta) - \beta\gamma\,(\Delta) - (\Delta) \\
&= \Delta\left(\alpha\delta - \beta\gamma - 1\right)
\end{aligned}
$$

Hence, for $(uv' - vu')|_0^L = 0$, we need

$$\alpha\delta - \beta\gamma - 1 = 0$$

1.5 Problem 5.5.3

> **5.5.3. Consider the eigenvalue problem $L(\phi) = -\lambda\sigma(x)\phi$, subject to a given set of homogeneous boundary conditions. Suppose that**
>
> $$\int_a^b [uL(v) - vL(u)]\;dx = 0$$
>
> **for all functions u and v satisfying the same set of boundary conditions. Prove that eigenfunctions corresponding to different eigenvalues are orthogonal (with what weight?).**

We are given that

$$\int_a^b uL[v] - vL[u]\,dx = 0 \tag{1}$$

But

$$L[v] = -\lambda_v \sigma(x)\,v \tag{2}$$

$$L[u] = -\lambda_u \sigma(x)\,u \tag{3}$$

Where $\sigma(x)$ is the weight function of the corresponding Sturm-Liouville ODE that u, v are its solution eigenfunctions. Substituting (2,3) into (1) gives

$$
\begin{aligned}
\int_a^b u\,(-\lambda_v \sigma(x)\,v) - v\,(-\lambda_u \sigma(x)\,u)\,dx &= 0 \\
\int_a^b -\lambda_v \sigma(x)\,uv + \lambda_u \sigma(x)\,uv dx &= 0 \\
(\lambda_u - \lambda_v)\int_a^b \sigma(x)\,uv dx &= 0
\end{aligned}
$$

Since u, v are different eigenfunctions, then the $\lambda_u - \lambda_v \neq 0$ as these are different eigenvalues. (There is one eigenfunction corresponding to each eigenvalue). Therefore the above says that

$$\int_a^b \sigma(x)\, u(x)\, v(x)\, dx = 0$$

Hence different eigenfunctions $u(x), v(x)$ are orthogonal to each others. The weight is $\sigma(x)$.

1.6 Problem 5.5.8

5.5.8. Consider a fourth-order linear differential operator,

$$L = \frac{d^4}{dx^4}.$$

(a) Show that $uL(v) - vL(u)$ is an exact differential.

(b) Evaluate $\int_0^1 [uL(v) - vL(u)]\ dx$ in terms of the boundary data for any functions u and v.

(c) Show that $\int_0^1 [uL(v) - vL(u)]\ dx = 0$ if u and v are any two functions satisfying the boundary conditions

$$\begin{aligned} \phi(0) &= 0 \quad & \phi(1) &= 0 \\ \frac{d\phi}{dx}(0) &= 0 \quad & \frac{d^2\phi}{dx^2}(1) &= 0. \end{aligned}$$

(d) Give another example of boundary conditions such that

$$\int_0^1 [uL(v) - vL(u)]\ dx = 0.$$

(e) For the eigenvalue problem [using the boundary conditions in part (c)]

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0,$$

show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weighting function?

$$L = \frac{d^4}{dx^4}$$

1.6.1 Part (a)

$$\begin{aligned} uL[v] - vL[u] &= u\frac{d^4 v}{dx^4} - v\frac{d^4 u}{dx^4} \\ &= uv^{(4)} - vu^{(4)} \end{aligned}$$

We want to obtain expression of form $\frac{d}{dx}()$ such that it comes out to be $uv^{(4)} - vu^{(4)}$. If we can do this, then it is exact differential. Now, since

$$\frac{d}{dx}(uv''' - u'v'') = u'v''' + uv^{(4)} - u''v'' - u'v''' \tag{1}$$

And

$$\frac{d}{dx}(vu''' - v'u'') = v'u''' + vu^{(4)} - v''u'' - v'u''' \tag{2}$$

Then (1)-(2) gives

$$\begin{aligned} \frac{d}{dx}(uv''' - u'v'') - \frac{d}{dx}(vu''' - v'u'') &= \left(u'v''' + uv^{(4)} - u''v'' - u'v'''\right) - \left(v'u''' + vu^{(4)} - v''u'' - v'u'''\right) \\ &= u'v''' + uv^{(4)} - u''v'' - u'v''' - v'u''' - vu^{(4)} + v''u'' + v'u''' \\ &= uv^{(4)} - vu^{(4)} \end{aligned}$$

Hence we found that

$$\begin{aligned} \frac{d}{dx}(uv''' - u'v'' - vu''' + v'u'') &= uv^{(4)} - vu^{(4)} \\ &= uL[v] - vL[u] \end{aligned}$$

Therefore $uL[v] - vL[u]$ is exact differential.

1.6.2 Part (b)

$$
\begin{aligned}
I &= \int_a^b uL[v] - vL[u]\,dx \\
&= \int_a^b \frac{d}{dx}\left(uv''' - u'v'' - vu''' + v'u''\right) dx \\
&= uv''' - u'v'' - vu''' + v'u''\big|_a^b \\
&= u(b)\,v'''(b) - u'(b)\,v''(b) - v(b)\,u'''(b) + v'(b)\,u''(b) \\
&- \left(u(a)\,v'''(a) - u'(a)\,v''(a) - v(a)\,u'''(a) + v'(a)\,u''(a)\right)
\end{aligned}
$$

Or

$I = u(b)\,v'''(b) - u'(b)\,v''(b) - v(b)\,u'''(b) + v'(b)\,u''(b) - u(a)\,v'''(a) + u'(a)\,v''(a) + v(a)\,u'''(a) - v'(a)\,u''(a)$

1.6.3 Part (c)

From part(b),

$$
I = \int_0^1 uL[v] - vL[u]\,dx = uv''' - u'v'' - vu''' + v'u''\big|_0^1 \tag{1}
$$

Since we are given that

$$
\begin{aligned}
\phi(0) &= 0 \\
\phi'(0) &= 0 \\
\phi(1) &= 0 \\
\phi''(1) &= 0
\end{aligned}
$$

The above will give

$$
\begin{aligned}
u(0) = v(0) &= 0 \\
u'(0) = v'(0) &= 0 \\
u(1) = v(1) &= 0 \\
u''(1) = v''(1) &= 0
\end{aligned}
$$

Substituting these into (1) gives

$$
\begin{aligned}
\int_0^1 uL[v] - vL[u]\,dx = u(1)\,v'''(1) - u'(1)\,v''(1) - v(1)\,u'''(1) + v'(1)\,u''(1) \\
- u(0)\,v'''(0) + u'(0)\,v''(0) + v(0)\,u'''(0) - v'(0)\,u''(0)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_0^1 uL[v] - vL[u]\,dx &= \left(0 \times v'''(1)\right) - 0 - \left(0 \times u'''(1)\right) + 0 - \left(0 \times v'''(0)\right) + 0 + \left(0 \times u'''(0)\right) - 0 \\
&= 0
\end{aligned}
$$

1.6.4 Part (d)

Any boundary conditions which makes $uv''' - u'v'' - vu''' + v'u''\big|_0^1 = 0$ will do. For example,

$$
\begin{aligned}
\phi(0) &= 0 \\
\phi'(0) &= 0 \\
\phi(1) &= 0 \\
\phi'(1) &= 0
\end{aligned}
$$

The above will give

$$\begin{aligned} u(0) &= v(0) = 0 \\ u'(0) &= v'(0) = 0 \\ u(1) &= v(1) = 0 \\ u'(1) &= v'(1) = 0 \end{aligned}$$

Substituting these into (1) gives

$$\begin{aligned} \int_0^1 uL[v] - vL[u]\,dx &= u(1)v'''(1) - u'(1)v''(1) - v(1)u'''(1) + v'(1)u''(1) \\ &\quad - u(0)v'''(0) + u'(0)v''(0) + v(0)u'''(0) - v'(0)u''(0) \\ &= (0\times v'''(1)) - (0\times v''(1)) - (0\times u'''(1)) + (0\times u''(1)) \\ &\quad - (0\times v'''(0)) + (0\times v''(0)) + (0\times u'''(0)) - (0\times u''(0)) \\ &= 0 \end{aligned}$$

1.6.5 Part (e)

Given

$$\frac{d^4}{dx^4}\phi + \lambda e^x \phi = 0$$

Therefore

$$L\left[\phi\right] = -\lambda e^x \phi$$

Therefore, for eigenfunctions u, v we have

$$\begin{aligned} L[u] &= -\lambda_u e^x u \\ L[v] &= -\lambda_v e^x v \end{aligned}$$

Where λ_u, λ_v are the eigenvalues associated with eigenfunctions u, v and they are not the same. Hence now we can write

$$\begin{aligned} 0 &= \int_0^1 uL[v] - vL[u]\,dx \\ &= \int_0^1 u(-\lambda_v e^x v) - v(-\lambda_u e^x u)\,dx \\ &= \int_0^1 -\lambda_v e^x uv + \lambda_u e^x uv dx \\ &= \int_0^1 (\lambda_u - \lambda_v)(e^x uv)\,dx \\ &= (\lambda_u - \lambda_v)\int_0^1 (e^x uv)\,dx \end{aligned}$$

Since $\lambda_u - \lambda_v \neq 0$ then

$$\int_0^1 (e^x uv)\,dx = 0$$

Hence u, v are orthogonal to each others with weight function e^x.

1.7 Problem 5.5.10

5.5.10. **(a) Show that (5.5.22) yields (5.5.23) if at least one of the boundary conditions is of the regular Sturm-Liouville type.**
(b) Do part (a) if one boundary condition is of the singular type.

1.7.1 Part(a)

Equation 5.5.22 is

$$p\left(\phi_1\phi_2' - \phi_2\phi_1'\right) = \text{constant} \tag{5.5.22}$$

Looking at boundary conditions at one end, say at $x = a$ (left end), and let the boundary conditions there be

$$\beta_1\phi(a) + \beta_2\phi'(a) = 0$$

Therefore for eigenfunctions ϕ_1, ϕ_2 we obtain

$$\beta_1 \phi_1 (a) + \beta_2 \phi_1' (a) = 0 \tag{1}$$

$$\beta_1 \phi_2 (a) + \beta_2 \phi_2' (a) = 0 \tag{2}$$

From (1),

$$\phi_1' (a) = -\frac{\beta_1}{\beta_2} \phi_1 (a) \tag{3}$$

From (2)

$$\phi_2' (a) = -\frac{\beta_1}{\beta_2} \phi_2 (a) \tag{4}$$

Substituting (3,4) into $\phi_1 \phi_2' - \phi_2 \phi_1'$ gives, at end point a, the following

$$\begin{aligned} \phi_1 (a) \phi_2' (a) - \phi_2 (a) \phi_1' (a) &= \phi_1 (a) \left(-\frac{\beta_1}{\beta_2} \phi_2 (a) \right) - \phi_2 (a) \left(-\frac{\beta_1}{\beta_2} \phi_1 (a) \right) \\ &= -\frac{\beta_1}{\beta_2} \phi_2 (a) \phi_1 (a) + \frac{\beta_1}{\beta_2} \phi_2 (a) \phi_1 (a) \\ &= 0 \end{aligned}$$

In the above, we evaluated $\phi_1 \phi_2' - \phi_2 \phi_1'$ at one end point, and found it to be zero. But $\phi_1 \phi_2' - \phi_2 \phi_1'$ is the Wronskian $W(x)$. It is known that if $W(x) = 0$ at just one point, then it is zero at all points in the range. Hence we conclude that

$$\phi_1 \phi_2' - \phi_2 \phi_1' = 0$$

For all x. This also means the eigenfunctions ϕ_1, ϕ_2 are linearly dependent. This gives equation 5.5.23. QED.

1.7.2 Part(b)

Equation 5.5.22 is

$$p \left(\phi_1 \phi_2' - \phi_2 \phi_1' \right) = \text{constant} \tag{5.5.22}$$

At one end, say end $x = a$, is where the singularity exist. This means $p(a) = 0$. Now to show that $p \left(\phi_1 \phi_2' - \phi_2 \phi_1' \right) = 0$ at $x = a$, we just need to show that $\phi_1 \phi_2' - \phi_2 \phi_1'$ is bounded. Since in that case, we will have $0 \times A = 0$, where A is some value which is $\phi_1 \phi_2' - \phi_2 \phi_1'$. But boundary conditions at $x = 1$ must be $\phi(a) < \infty$ and also $\phi'(a) < \infty$. This is always the case at the end where $p = 0$.

Then let $\phi(a) = c_1$ and $\phi'(a) = c_2$, where c_1, c_2 are some constants. Then we write

$$\begin{aligned} \phi_1 (a) &= c_1 \\ \phi_1' (a) &= c_2 \\ \phi_2 (a) &= c_1 \\ \phi_2' (a) &= c_2 \end{aligned}$$

Hence it follows immediately that

$$\begin{aligned} \phi_1 \phi_2' - \phi_2 \phi_1' &= c_1 c_2 - c_2 c_1 \\ &= 0 \end{aligned}$$

Hence we showed that $\phi_1 \phi_2' - \phi_2 \phi_1'$ is bounded. Then $p \left(\phi_1 \phi_2' - \phi_2 \phi_1' \right) = 0$. QED.