HW4, Math 322, Fall 2016

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December 30, 2019

Contents

1 HW 4

1.1 Problem 2.5.24

> **2.5.24. Consider the velocity u_θ at the cylinder. If the circulation is negative, show that the velocity will be larger above the cylinder than below.**

Introduction. The stream velocity $\bar{u}$ in Cartesian coordinates is

$$\begin{aligned}\bar{u} &= u\hat{\imath} + v\hat{\jmath}\\ &= \frac{\partial \Psi}{\partial y}\hat{\imath} - \frac{\partial \Psi}{\partial x}\hat{\jmath}\end{aligned} \tag{1}$$

Where Ψ is the stream function which satisfies Laplace PDE in 2D $\nabla^2\Psi = 0$. In Polar coordinates the above becomes

$$\begin{aligned}\bar{u} &= u_r\hat{r} + u_\theta\hat{\theta}\\ &= \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\hat{r} - \frac{\partial \Psi}{\partial r}\hat{\theta}\end{aligned} \tag{2}$$

The solution to $\nabla^2\Psi = 0$ was found under the following conditions

1. When r very large, or in other words, when too far away from the cylinder or the wing, the flow lines are horizontal only. This means at $r = \infty$ the y component of $\bar{u}$ in (1) is zero. This means $\frac{\partial \Psi(x,y)}{\partial x} = 0$. Therefore $\Psi\left(x, y\right) = u_0 y$ where u_0 is some constant. In polar coordinates this implies $\underline{\Psi\left(r, \theta\right) = u_0 r \sin\theta}$, since $y = r\sin\theta$.

2. The second condition is that radial component of $\bar{u}$ is zero. In other words, $\frac{1}{r}\frac{\partial \Psi}{\partial \theta} = 0$ when $r = a$, where a is the radius of the cylinder.

3. In addition to the above two main condition, there is a condition that $\Psi = 0$ at $r = 0$

Using the above three conditions, the solution to $\nabla^2\Psi = 0$ was derived in lecture Sept. 30, 2016, to be

$$\Psi\left(r, \theta\right) = c_1 \ln\left(\frac{r}{a}\right) + u_0\left(r - \frac{a^2}{r}\right)\sin\theta$$

Using the above solution, the velocity $\bar{u}$ can now be found using the definition in (2) as follows

$$\begin{aligned}\frac{1}{r}\frac{\partial \Psi}{\partial \theta} &= \frac{1}{r}u_0\left(r - \frac{a^2}{r}\right)\cos\theta\\ \frac{\partial \Psi}{\partial r} &= \frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\end{aligned}$$

Hence, in polar coordinates

$$\boxed{\bar{u} = \left(\frac{1}{r}u_0\left(r - \frac{a^2}{r}\right)\cos\theta\right)\hat{r} - \left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)\hat{\theta}} \tag{3}$$

Now the question posed can be answered. The circulation is given by

$$\Gamma = \int_0^{2\pi} u_\theta r d\theta$$

But from (3) $u_\theta = -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)$, therefore the above becomes

$$\Gamma = \int_0^{2\pi} -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right) r d\theta$$

At $r = a$ the above simplifies to

$$\begin{aligned}\Gamma &= \int_0^{2\pi} -\left(\frac{c_1}{a} + 2u_0\sin\theta\right) a d\theta\\ &= \int_0^{2\pi} -c_1 - 2au_0\sin\theta d\theta\\ &= -\int_0^{2\pi} c_1 d\theta - 2au_0\int_0^{2\pi}\sin\theta d\theta\end{aligned}$$

But $\int_0^{2\pi} \sin\theta d\theta = 0$, hence

$$\Gamma = -c_1 \int_0^{2\pi} d\theta$$
$$= -2c_1\pi$$

Since $\Gamma < 0$, then $\underline{c_1 > 0}$. Now that c_1 is known to be positive, then the velocity is calculated at $\theta = \frac{-\pi}{2}$ and then at $\theta = \frac{+\pi}{2}$ to see which is larger. Since this is calculated at $r = a$, then the radial velocity is zero and only u_θ needs to be evaluated in (3).

$\underline{\text{At } \theta = \frac{-\pi}{2}}$

$$\begin{aligned} u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{-\pi}{2}\right)\right) \\ &= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\ &= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\right) \end{aligned}$$

At $r = a$

$$\begin{aligned} u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} - 2u_0\right) \\ &= -\frac{c_1}{a} + 2u_0 \end{aligned} \tag{4}$$

$\underline{\text{At } \theta = \frac{+\pi}{2}}$

$$\begin{aligned} u_{\left(\frac{+\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\ &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\right) \end{aligned}$$

At $r = a$

$$\begin{aligned} u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} + 2u_0\right) \\ &= -\frac{c_1}{a} - 2u_0 \end{aligned} \tag{5}$$

Comparing (4),(5), and since $c_1 > 0$, then the $\underline{\text{magnitude}}$ of u_θ at $\frac{\pi}{2}$ is larger than the magnitude of u_θ at $\frac{-\pi}{2}$. Which implies the stream flows faster above the cylinder than below it.

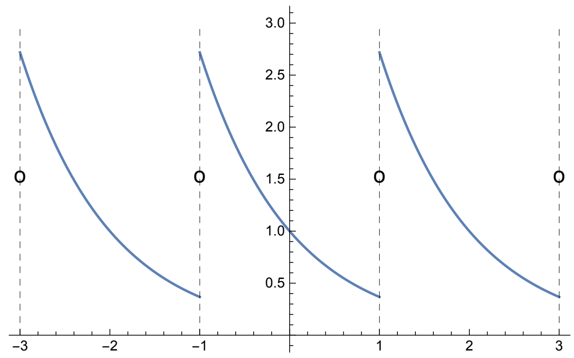
1.2 Problem 3.2.2 (b,d)

3.2.2. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) and determine the Fourier coefficients:

* (a) $f(x) = x$
(b) $f(x) = e^{-x}$
* (c) $f(x) = \sin\frac{\pi x}{L}$
(d) $f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$
(e) $f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$
* (f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$
(g) $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$

1.2.1 Part b

The following is sketch of periodic extension of e^{-x} from $x = -L \cdots L$ (for $L = 1$) for illustration. The function will converge to e^{-x} between $x = -L \cdots L$ and between $x = -3L \cdots -L$ and between $x = L \cdots 3L$ and so on. But at the jump discontinuities which occurs at $x = \cdots, -3L, -L, L, 3L, \cdots$ it will converge to the average shown as small circles in the sketch.



Periodic extension of exp(-x) from -1...1

By definitions,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx$$
$$a_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\cos\left(n\left(\frac{2\pi}{T}\right)x\right)dx$$
$$b_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\sin\left(n\left(\frac{2\pi}{T}\right)x\right)dx$$

The period here is $T = 2L$, therefore the above becomes

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

These are now evaluated for $f(x) = e^{-x}$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} e^{-x}dx = \frac{1}{2L}\left(\frac{e^{-x}}{-1}\right)_{-L}^{L} = \frac{-1}{2L}\left(e^{-x}\right)_{-L}^{L} = \frac{-1}{2L}\left(e^{-L} - e^{L}\right) = \frac{e^{L} - e^{-L}}{2L}$$

Now a_n is found

$$a_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx$$

This can be done using integration by parts. $\int udv = uv - \int vdu$. Let

$$I = \int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx$$

and $u = \cos\left(n\frac{\pi}{L}x\right), dv = e^{-x}, \rightarrow du = -\frac{n\pi}{L}\sin\left(n\frac{\pi}{L}x\right), v = -e^{-x}$, therefore

$$\begin{aligned}
I &= [uv]_{-L}^{L} - \int_{-L}^{L} vdu \\
&= \left[-e^{-x}\cos\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L} - \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx \\
&= \left[-e^{-L}\cos\left(n\frac{\pi}{L}L\right) + e^{L}\cos\left(n\frac{\pi}{L}(-L)\right)\right] - \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx \\
&= \left[-e^{-L}\cos(n\pi) + e^{L}\cos(n\pi)\right] - \frac{n\pi}{L}\int_{-L}^{L} e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx
\end{aligned}$$

Applying integration by parts again to $\int e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx$ where now $u = \sin\left(n\frac{\pi}{L}x\right), dv = e^{-x} \rightarrow du =$

$\frac{n\pi}{L}\cos\left(n\frac{\pi}{L}x\right), v=-e^{-x}$, hence the above becomes

$$
\begin{aligned}
I &= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\frac{n\pi}{L}\left(uv-\int vdu\right)\\
&= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\frac{n\pi}{L}\left(\overbrace{\left[-e^{-x}\sin\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L}}^{0}+\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\frac{n\pi}{L}\left(\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \left[-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)\right]-\left(\frac{n\pi}{L}\right)^{2}\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx
\end{aligned}
$$

But $\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx=I$ and the above becomes

$$
I=-e^{-L}\cos(n\pi)+e^{L}\cos(n\pi)-\left(\frac{n\pi}{L}\right)^{2}I
$$

Simplifying and solving for I

$$
\begin{aligned}
I+\left(\frac{n\pi}{L}\right)^{2}I &= \cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I\left(1+\left(\frac{n\pi}{L}\right)^{2}\right) &= \cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I\left(\frac{L^{2}+n^{2}\pi^{2}}{L^{2}}\right) &= \cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I &= \left(\frac{L^{2}}{L^{2}+n^{2}\pi^{2}}\right)\cos(n\pi)\left(e^{L}-e^{-L}\right)
\end{aligned}
$$

Hence a_n becomes

$$
a_n=\frac{1}{L}\left(\frac{L^{2}}{L^{2}+n^{2}\pi^{2}}\right)\cos(n\pi)\left(e^{L}-e^{-L}\right)
$$

But $\cos(n\pi)=-1^{n}$ hence

$$
a_n=(-1)^{n}\left(\frac{L}{n^{2}\pi^{2}+L^{2}}\right)\left(e^{L}-e^{-L}\right)
$$

Similarly for b_n

$$
b_n=\frac{1}{L}\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx
$$

This can be done using integration by parts. $\int udv=uv-\int vdu$. Let

$$
I=\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx
$$

and $u=\sin\left(n\frac{\pi}{L}x\right), dv=e^{-x}, \rightarrow du=\frac{n\pi}{L}\cos\left(n\frac{\pi}{L}x\right), v=-e^{-x}$, therefore

$$
\begin{aligned}
I &= [uv]_{-L}^{L}-\int_{-L}^{L}vdu\\
&= \overbrace{\left[-e^{-x}\sin\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L}}^{0}+\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx\\
&= \frac{n\pi}{L}\int_{-L}^{L}e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx
\end{aligned}
$$

Applying integration by parts again to $\int e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx$ where now $u=\cos\left(n\frac{\pi}{L}x\right), dv=e^{-x}\rightarrow du=\frac{-n\pi}{L}\sin\left(n\frac{\pi}{L}x\right), v=-e^{-x}$, hence the above becomes

$$
\begin{aligned}
I &= \frac{n\pi}{L}\left(uv-\int vdu\right)\\
&= \frac{n\pi}{L}\left(\left[-e^{-x}\cos\left(n\frac{\pi}{L}x\right)\right]_{-L}^{L}-\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{n\pi}{L}\left(-e^{-L}\cos\left(n\frac{\pi}{L}L\right)+e^{L}\cos\left(n\frac{\pi}{L}L\right)-\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{n\pi}{L}\left(\cos(n\pi)\left(e^{L}-e^{-L}\right)-\frac{n\pi}{L}\int_{-L}^{L}e^{-x}\sin\left(n\frac{\pi}{L}x\right)dx\right)
\end{aligned}
$$

But $\int_{-L}^{L} e^{-x}\cos\left(n\frac{\pi}{L}x\right)dx = I$ and the above becomes

$$I = \frac{n\pi}{L}\left(\cos(n\pi)\left(e^{L}-e^{-L}\right) - \frac{n\pi}{L}I\right)$$

Simplifying and solving for I

$$\begin{aligned}
I &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L}-e^{-L}\right) - \left(\frac{n\pi}{L}\right)^2 I\\
I + \left(\frac{n\pi}{L}\right)^2 I &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I\left(1+\left(\frac{n\pi}{L}\right)^2\right) &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I\left(\frac{L^2+n^2\pi^2}{L^2}\right) &= \frac{n\pi}{L}\cos(n\pi)\left(e^{L}-e^{-L}\right)\\
I &= \left(\frac{L^2}{L^2+n^2\pi^2}\right)\frac{n\pi}{L}\cos(n\pi)\left(e^{L}-e^{-L}\right)
\end{aligned}$$

Hence b_n becomes

$$\begin{aligned}
b_n &= \frac{1}{L}\left(\frac{L^2}{L^2+n^2\pi^2}\right)\frac{n\pi}{L}\cos(n\pi)\left(e^{L}-e^{-L}\right)\\
&= \left(\frac{n\pi}{L^2+n^2\pi^2}\right)\cos(n\pi)\left(e^{L}-e^{-L}\right)
\end{aligned}$$

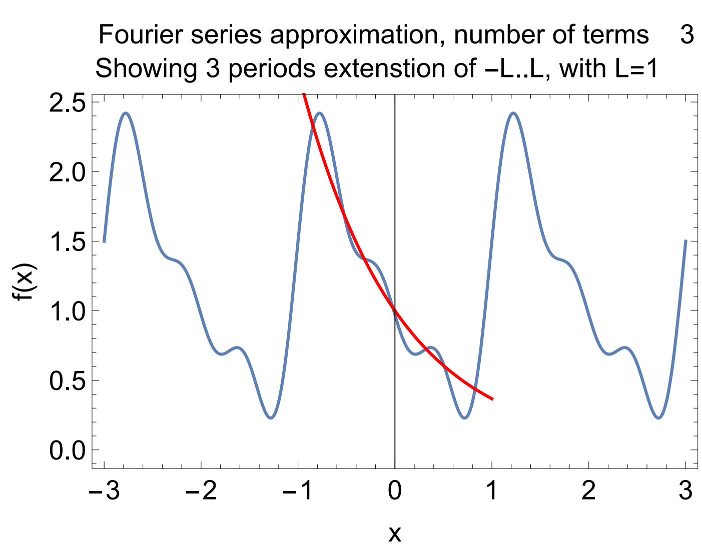
But $\cos(n\pi) = -1^n$ hence

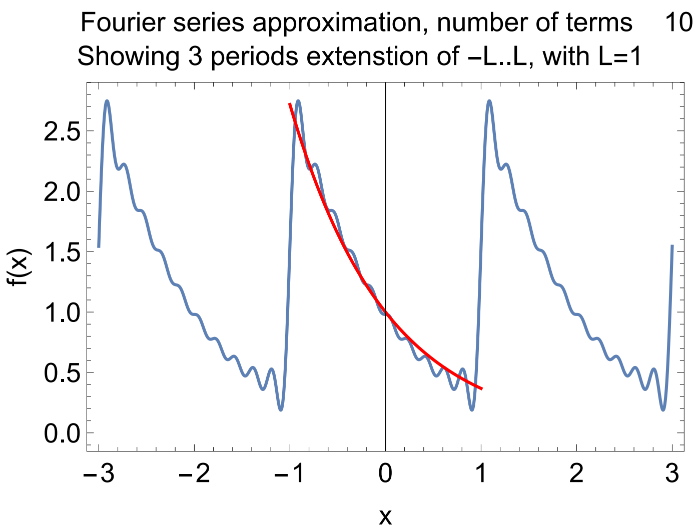
$$b_n = (-1)^n\left(\frac{n\pi}{L^2+n^2\pi^2}\right)\left(e^{L}-e^{-L}\right)$$

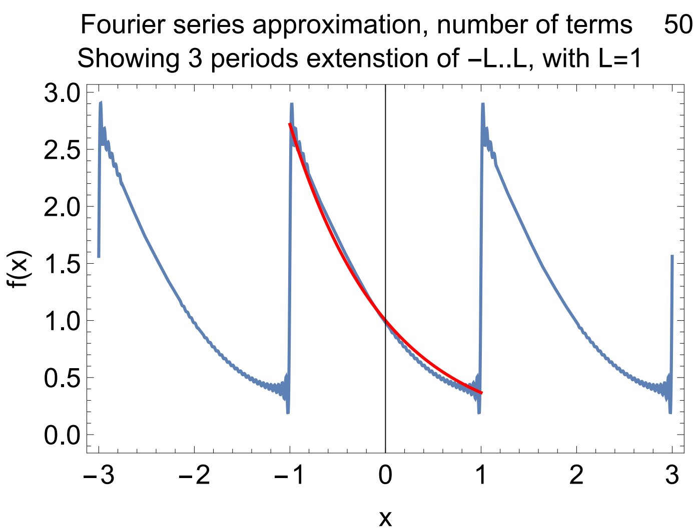
<u>Summary</u>

$$\begin{aligned}
a_0 &= \frac{e^{L}-e^{-L}}{2L}\\
a_n &= (-1)^n\left(\frac{L}{n^2\pi^2+L^2}\right)\left(e^{L}-e^{-L}\right)\\
b_n &= (-1)^n\left(\frac{n\pi}{L^2+n^2\pi^2}\right)\left(e^{L}-e^{-L}\right)\\
f(x) &\approx a_0 + \sum_{n=1}^{\infty} a_n\cos\left(n\left(\frac{2\pi}{T}\right)x\right) + b_n\sin\left(n\left(\frac{2\pi}{T}\right)x\right)\\
&\approx a_0 + \sum_{n=1}^{\infty} a_n\cos\left(n\frac{\pi}{L}x\right) + b_n\sin\left(n\frac{\pi}{L}x\right)
\end{aligned}$$

The following shows the approximation $f(x)$ for increasing number of terms. Notice the Gibbs phenomena at the jump discontinuity.

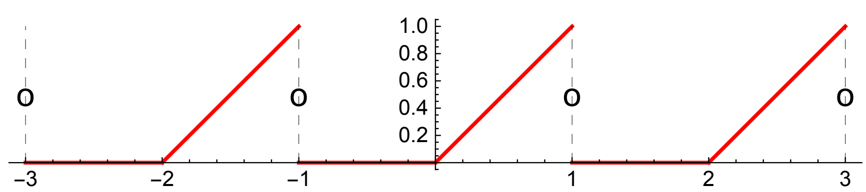






1.2.2 Part d

The following is sketch of periodic extension of $f(x)$ from $x = -L \cdots L$ (for $L = 1$) for illustration. The function will converge to $f(x)$ between $x = -L \cdots L$ and between $x = -3L \cdots -L$ and between $x = L \cdots 3L$ and so on. But at the jump discontinuities which occurs at $x = \cdots, -3L, -L, L, 3L, \cdots$ it will converge to the average $\frac{1}{2}$ shown as small circles in the sketch.



Showing 3 periods extenstion of f(x) between -L..L, with L=1

By definitions,

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(x)\,dx$$

$$a_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\cos\left(n\left(\frac{2\pi}{T}\right)x\right)dx$$

$$b_n = \frac{1}{T/2}\int_{-T/2}^{T/2} f(x)\sin\left(n\left(\frac{2\pi}{T}\right)x\right)dx$$

The period here is $T = 2L$, therefore the above becomes

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

These are now evaluated for given $f(x)$

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\
&= \frac{1}{2L}\left(\int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx\right) \\
&= \frac{1}{2L}\left(0 + \int_{0}^{L} x dx\right) \\
&= \frac{1}{2L}\left(\frac{x^2}{2}\right)_0^L \\
&= \frac{L}{4}
\end{aligned}$$

Now a_n is found

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx \\
&= \frac{1}{L}\left(\int_{-L}^{0} f(x)\cos\left(n\frac{\pi}{L}x\right)dx + \int_{0}^{L} f(x)\cos\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{1}{L}\int_{0}^{L} x\cos\left(n\frac{\pi}{L}x\right)dx
\end{aligned}$$

Integration by parts. Let $u = x, du = 1, dv = \cos\left(n\frac{\pi}{L}x\right), v = \frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}$, hence the above becomes

$$\begin{aligned}
a_n &= \frac{1}{L}\left(\overbrace{\left(\frac{n\pi}{L}x\sin\left(n\frac{\pi}{L}x\right)\right)_0^L}^{0} - \int_0^L \frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}dx\right) \\
&= \frac{1}{L}\left(-\frac{L}{n\pi}\int_0^L \sin\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{1}{L}\left(-\frac{L}{n\pi}\left(\frac{-\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right)_0^L\right) \\
&= \frac{1}{L}\left(\left(\frac{L}{n\pi}\right)^2\cos\left(n\frac{\pi}{L}x\right)_0^L\right) \\
&= \frac{L}{n^2\pi^2}\cos\left(n\frac{\pi}{L}x\right)_0^L \\
&= \frac{L}{n^2\pi^2}\left[\cos\left(n\frac{\pi}{L}L\right) - 1\right] \\
&= \frac{L}{n^2\pi^2}\left[-1^n - 1\right]
\end{aligned}$$

Now b_n is found

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx \\
&= \frac{1}{L}\left(\int_{-L}^{0} f(x)\sin\left(n\frac{\pi}{L}x\right)dx + \int_{0}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{1}{L}\int_{0}^{L} x\sin\left(n\frac{\pi}{L}x\right)dx
\end{aligned}$$

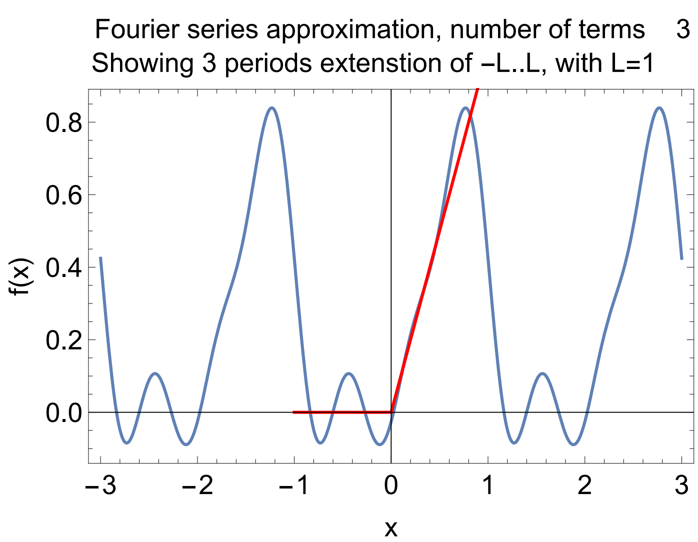
Integration by parts. Let $u = x, du = 1, dv = \sin\left(n\frac{\pi}{L}x\right), v = \frac{-\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}$, hence the above becomes

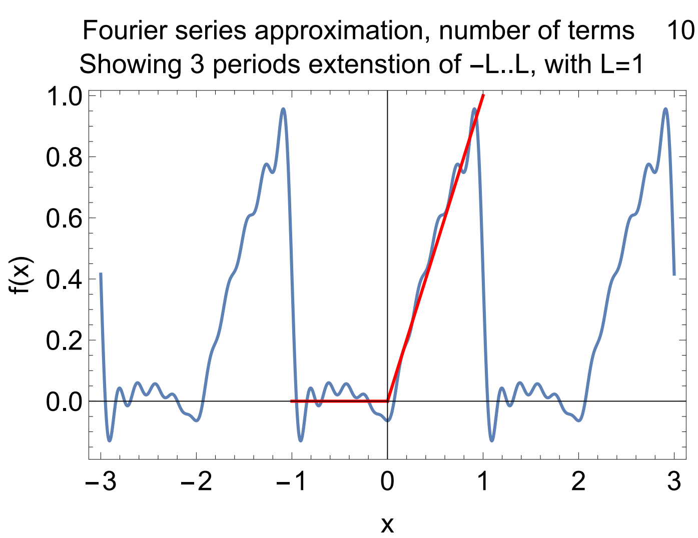
$$\begin{aligned}
b_n &= \frac{1}{L}\left(\left(-\frac{L}{n\pi}x\cos\left(n\frac{\pi}{L}x\right)\right)_0^L + \int_0^L \frac{\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}dx\right) \\
&= \frac{1}{L}\left(-\frac{L}{n\pi}\left(L\cos\left(n\frac{\pi}{L}L\right) - 0\right) + \frac{L}{n\pi}\int_0^L \cos\left(n\frac{\pi}{L}x\right)dx\right) \\
&= \frac{1}{L}\left(-\frac{L^2}{n\pi}(-1)^n + \frac{L}{n\pi}\overbrace{\left[\frac{\sin\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right]_0^L}^{0}\right) \\
&= \frac{L}{n\pi}\left(-(-1)^n\right) \\
&= (-1)^{n+1}\frac{L}{n\pi}
\end{aligned}$$

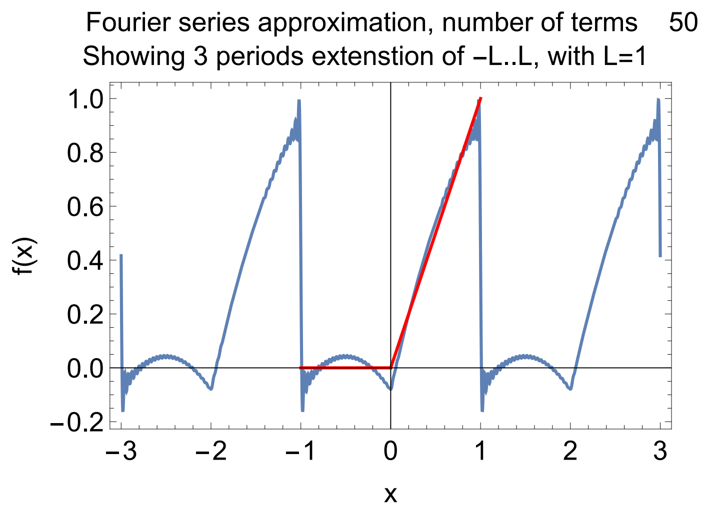
Summary

$$\begin{aligned}
a_0 &= \frac{L}{4} \\
a_n &= \frac{L}{n^2\pi^2}\left[-1^n - 1\right] \\
b_n &= (-1)^{n+1}\frac{L}{n\pi} \\
f(x) &\approx a_0 + \sum_{n=1}^{\infty} a_n\cos\left(n\left(\frac{2\pi}{T}\right)x\right) + b_n\sin\left(n\left(\frac{2\pi}{T}\right)x\right) \\
&\approx a_0 + \sum_{n=1}^{\infty} a_n\cos\left(n\frac{\pi}{L}x\right) + b_n\sin\left(n\frac{\pi}{L}x\right)
\end{aligned}$$

The following shows the approximation $f(x)$ for increasing number of terms. Notice the Gibbs phenomena at the jump discontinuity.







1.3 Problem 3.2.4

3.2.4. **Suppose that $f(x)$ is piecewise smooth. What value does the Fourier series of $f(x)$ converge to at the endpoint $x = -L$? at $x = L$?**

It will converge to the average value of the function at the end points after making periodic extensions of the function. Specifically, at $x = -L$ the Fourier series will converge to

$$\frac{1}{2}\left(f(-L) + f(L)\right)$$

And at $x = L$ it will converge to

$$\frac{1}{2}\left(f(L) + f(-L)\right)$$

Notice that if $f(L)$ has same value as $f(-L)$, then there will not be a jump discontinuity when periodic extension are made, and the above formula simply gives the value of the function at either end, since it is the same value.

1.4 Problem 3.3.2 (d)

3.3.2. **For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients.**

(a) $f(x) = \cos \pi x/L$ [Verify formula (3.3.13).]

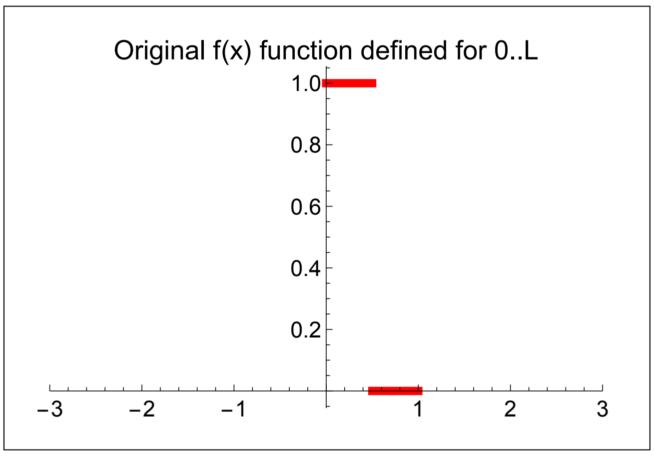
(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

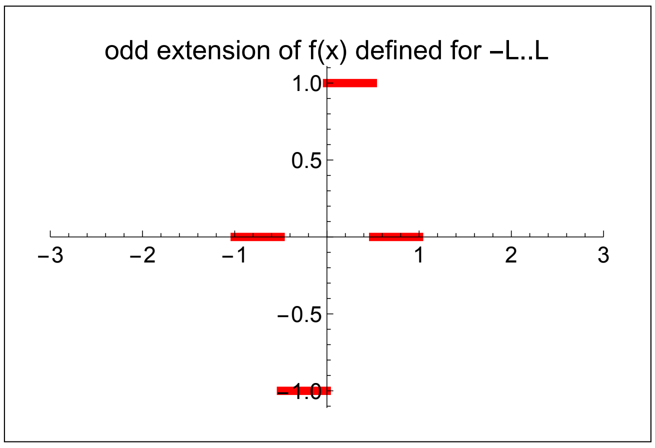
*(d) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

$$f(x) = \begin{cases} 1 & x < \frac{L}{2} \\ 0 & x > \frac{L}{2} \end{cases}$$

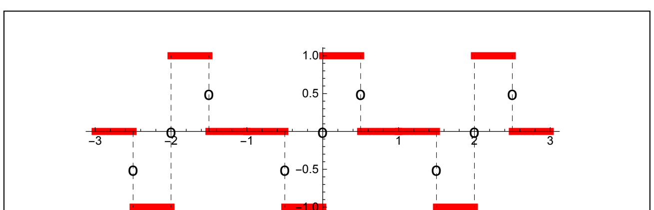
The first step is to sketch $f(x)$ over $0 \cdots L$. This is the result for $L = 1$ as an example.



The second step is to make an odd extension of $f(x)$ over $-L \cdots L$. This is the result.



The third step is to extend the above as periodic function with period $2L$ (as normally would be done) and mark the average value at the jump discontinuities. This is the result



Showing 3 periods extenstion of f(x) between -L..L, with L=1

Now the Fourier sin series is found for the above function. Since the function $f(x)$ is odd, then only b_n will exist

$$
\begin{aligned}
f(x) &\approx \sum_{n=1}^{\infty} b_n \sin\left(n\left(\frac{2\pi}{2L}\right)x\right) \\
&\approx \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}
$$

Where

$$
b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\left(\frac{2\pi}{2L}\right)x\right)dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(n\frac{\pi}{L}x\right)dx
$$

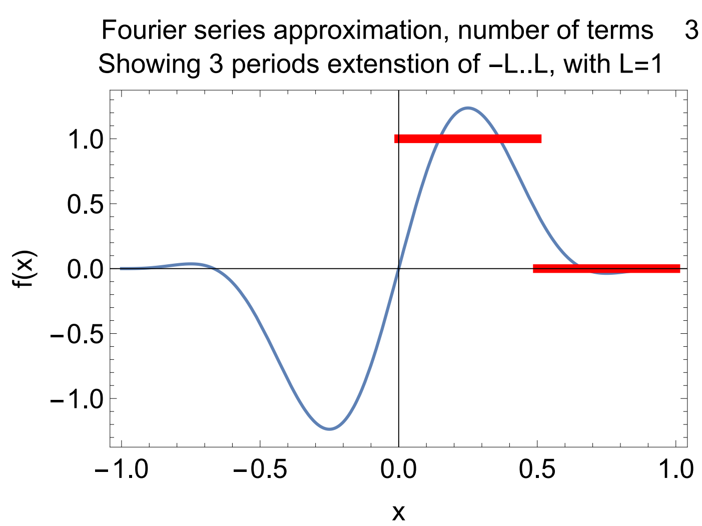
Since $f(x)\sin\left(n\frac{\pi}{L}x\right)$ is even, then the above becomes

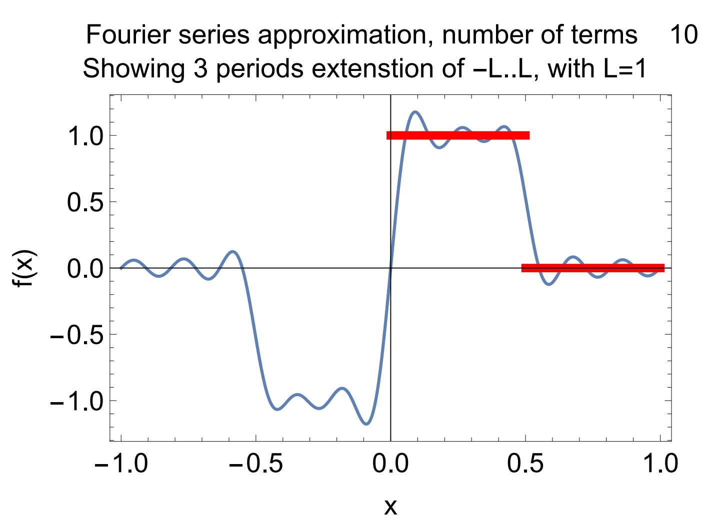
$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx\\
&= \frac{2}{L}\left(\int_0^{L/2} 1\times\sin\left(n\frac{\pi}{L}x\right)dx + \int_0^{L/2} 0\times\sin\left(n\frac{\pi}{L}x\right)dx\right)\\
&= \frac{2}{L}\int_0^{L/2}\sin\left(n\frac{\pi}{L}x\right)dx\\
&= \frac{2}{L}\left[-\frac{\cos\left(n\frac{\pi}{L}x\right)}{n\frac{\pi}{L}}\right]_0^{L/2}\\
&= \frac{-2}{n\pi}\left[\cos\left(n\frac{\pi}{L}x\right)\right]_0^{L/2}\\
&= \frac{-2}{n\pi}\left[\cos\left(n\frac{\pi}{L}\frac{L}{2}\right)-1\right]\\
&= \frac{-2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-1\right]\\
&= \frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]
\end{aligned}$$

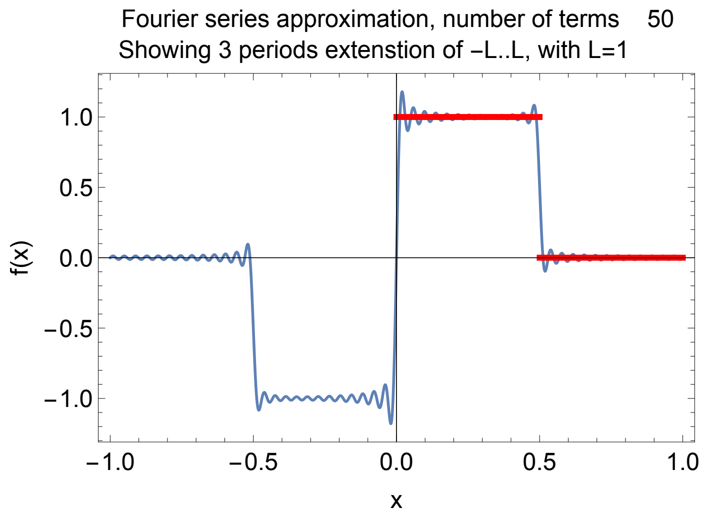
Therefore

$$f(x)\approx\sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\sin\left(n\frac{\pi}{L}x\right)$$

The following shows the approximation $f(x)$ for increasing number of terms. Notice the Gibbs phenomena at the jump discontinuity.







1.5 Problem 3.3.3 (b)

3.3.3. **For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a *finite* number of nonzero terms (at least the first two) of the Fourier sine series:**

(a) $f(x) = \cos \pi x/L$ [Use formula (3.3.13).]

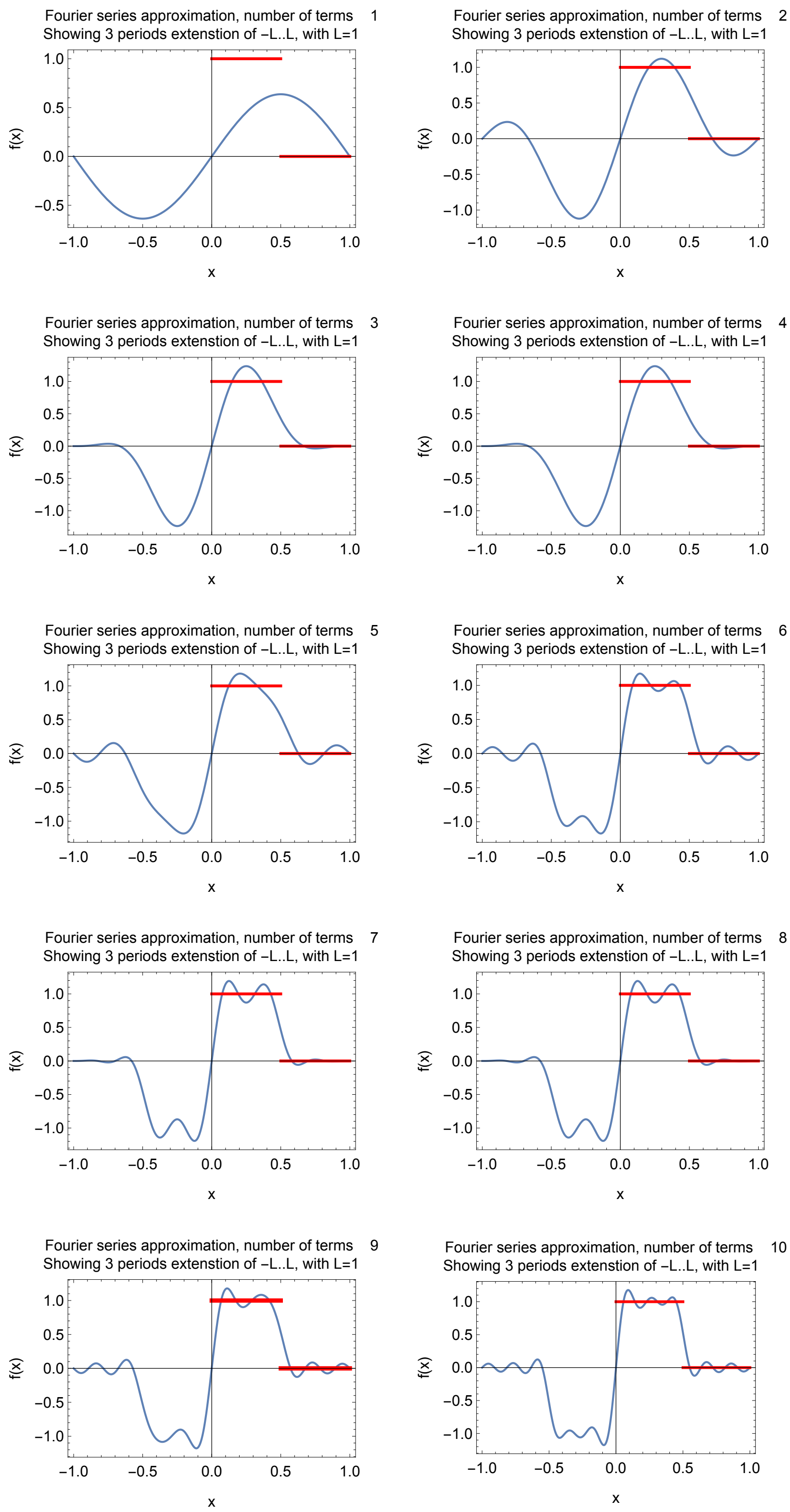
(b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = x$ [Use formula (3.3.12).]

This is the same problem as 3.3.2 part (d). But it asks to plot for $n = 1$ and $n = 2$ in the sum. The sketch of the Fourier sin series was done above in solving 3.3.2 part(d) and will not be repeated again. From above, it was found that

$$f(x) \approx \sum_{n=1}^{\infty} B_n \sin\left(n\frac{\pi}{L}x\right)$$

Where $B_n = \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$. The following is the plot for $n = 1 \cdots 10$.



1.6 Problem 3.3.8

3.3.8. (a) Determine formulas for the even extension of any $f(x)$. Compare to the formula for the even part of $f(x)$.

(b) Do the same for the odd extension of $f(x)$ and the odd part of $f(x)$.

(c) Calculate and sketch the four functions of parts (a) and (b) if

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0. \end{cases}$$

Graphically add the even and odd parts of $f(x)$. What occurs? Similarly, add the even and odd extensions. What occurs then?

1.6.1 Part (a)

The even extension of $f(x)$ is

$$\begin{cases} f(x) & x > 0 \\ f(-x) & x < 0 \end{cases}$$

But the even part of $f(x)$ is

$$\frac{1}{2}\left(f(x) + f(-x)\right)$$

1.6.2 Part (b)

The odd extension of $f(x)$ is

$$\begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}$$

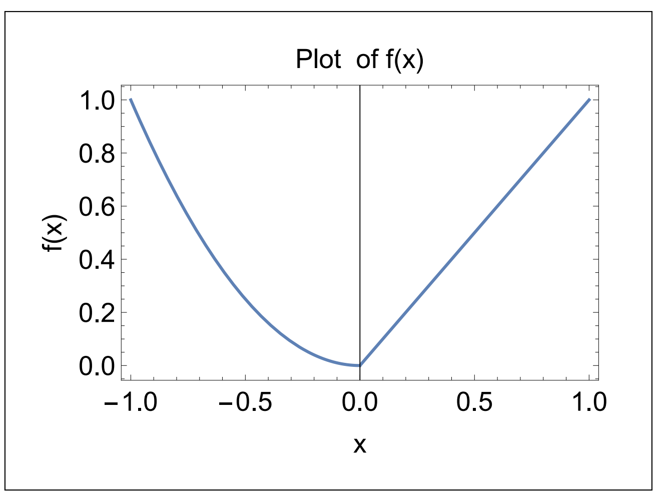
While the odd part of $f(x)$ is

$$\frac{1}{2}\left(f(x) - f(-x)\right)$$

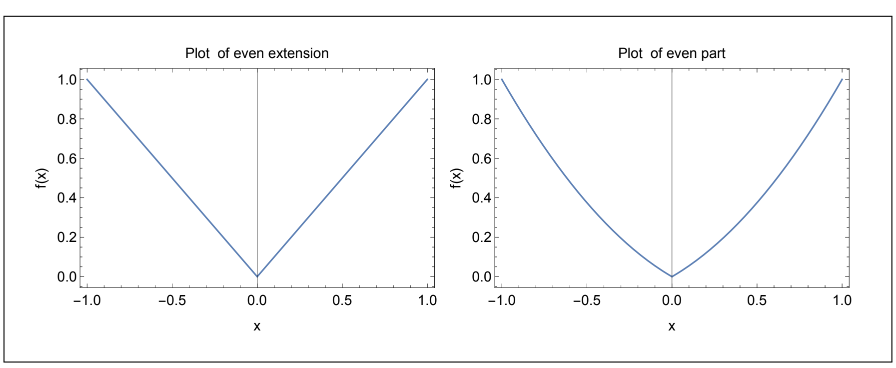
1.6.3 Part (c)

First a plot of $f(x)$ is given

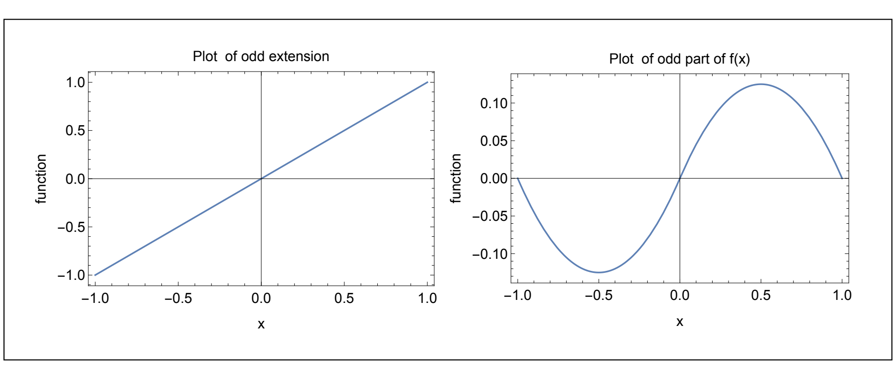
$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0 \end{cases}$$



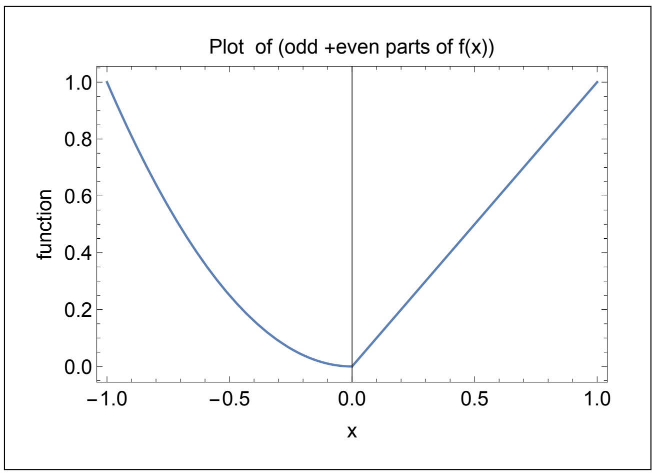
A plot of even extension and the even part for $f(x)$ Is given below



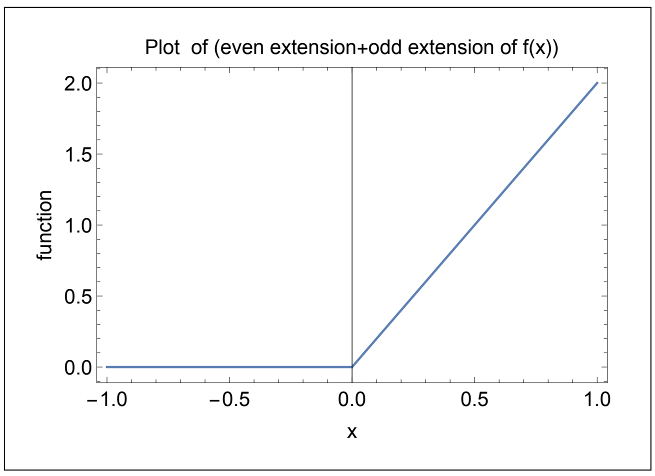
A plot of odd extension and the odd part is given below



Adding the even part and the odd part gives back the original function



Plot of adding the even extension and the odd extension is below



1.7 Problem 3.4.3

3.4.3. Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0$, $f(x_0^-) = \alpha$ and $f(x_0^+) = \beta$] and df/dx is piecewise smooth.

***(a) Determine the Fourier sine series of df/dx in terms of the Fourier cosine series coefficients of $f(x)$.**

(b) Determine the Fourier cosine series of df/dx in terms of the Fourier sine series coefficients of $f(x)$.

1.7.1 Part (a)

Fourier sin series of $f'(x)$ is given by, assuming period is $-L \cdots L$

$$f'(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)$$

Where

$$b_n = \frac{2}{L}\int_0^L f'(x)\sin\left(n\frac{\pi}{L}x\right)dx$$

Applying integration by parts. Let $f'(x) = dv, u = \sin\left(n\frac{\pi}{L}x\right)$, then $v = f(x), du = \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right)$. Since $v = f(x)$ has has jump discontinuity at x_0 as described, and assuming $x_0 > 0$, then, and using $\sin\left(n\frac{\pi}{L}x\right) = 0$ at $x = L$

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L u\,dv \\ &= \frac{2}{L}\left[\left([uv]_0^{x_0^-} + [uv]_{x_0^+}^L\right) - \int_0^L v\,du\right] \\ &= \frac{2}{L}\left(\left[\sin\left(n\frac{\pi}{L}x\right)f(x)\right]_0^{x_0^-} + \left[\sin\left(n\frac{\pi}{L}x\right)f(x)\right]_{x_0^+}^L - \frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right) \\ &= \frac{2}{L}\left(\sin\left(n\frac{\pi}{L}x_0^-\right)f\left(x_0^-\right) - \sin\left(n\frac{\pi}{L}x_0^+ f\left(x_0^+\right)\right) - \frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right) \end{aligned} \tag{1}$$

In the above, $\sin\left(n\frac{\pi}{L}x\right)=0$ and at $x=L$ was used. But

$$f\left(x_0^-\right)=\alpha$$
$$f\left(x_0^+\right)=\beta$$

And since sin is continuous, then $\sin\left(n\frac{\pi}{L}x_0^-\right)=\sin\left(n\frac{\pi}{L}x_0^+\right)=\sin\left(n\frac{\pi}{L}x_0\right)$. Equation (1) simplifies to

$$b_n=\frac{2}{L}\left(\left(\alpha-\beta\right)\sin\left(n\frac{\pi}{L}x_0\right)-\frac{n\pi}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx\right) \tag{2}$$

On the other hand, the Fourier cosine series for $f(x)$ is given by

$$f(x)\sim a_0+\sum_{n=1}^{\infty}a_n\cos\left(n\frac{\pi}{L}x\right)$$

Where

$$a_0=\frac{1}{L}\int_0^L f(x)\,dx$$
$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx$$

Therefore $\int_0^L f(x)\cos\left(n\frac{\pi}{L}x\right)dx=\frac{L}{2}a_n$. Substituting this into (2) gives

$$\begin{aligned}b_n&=\frac{2}{L}\left(\left(\alpha-\beta\right)\sin\left(n\frac{\pi}{L}x_0\right)-\frac{n\pi}{L}\left(\frac{L}{2}a_n\right)\right)\\&=\frac{2}{L}\left(\alpha-\beta\right)\sin\left(n\frac{\pi}{L}x_0\right)-\frac{2}{L}\frac{n\pi}{L}\left(\frac{L}{2}a_n\right)\end{aligned}$$

Hence

$$\boxed{b_n=\frac{2}{L}\sin\left(n\frac{\pi}{L}x_0\right)\left(\alpha-\beta\right)-\frac{n\pi}{L}a_n} \tag{3}$$

Summary the Fourier sin series of $f'(x)$ is

$$f'(x)\sim\sum_{n=1}^{\infty}b_n\sin\left(n\frac{\pi}{L}x\right)$$

With b_n given by (3). The above is in terms of a_n, which is the Fourier cosine series of $f(x)$, which is what required to show. In addition, the cos series of $f(x)$ can also be written in terms of sin series of $f'(x)$. From (3), solving for a_n

$$\begin{aligned}a_n&=\frac{L}{n\pi}b_n-\frac{2}{n\pi}\sin\left(n\frac{\pi}{L}x_0\right)\left(\alpha-\beta\right)\\f(x)&\sim a_0+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{L}{\pi}b_n-\frac{2}{\pi}\sin\left(n\frac{\pi}{L}x_0\right)\left(\alpha-\beta\right)\right)\cos\left(\frac{n\pi}{L}x\right)\end{aligned}$$

This shows more clearly that the Fourier series of $f(x)$ has order of convergence in a_n as $\frac{1}{n}$ as expected.

1.7.2 Part (b)

Fourier cos series of $f'(x)$ is given by, assuming period is $-L\cdots L$

$$f'(x)\sim\sum_{n=0}^{\infty}a_n\cos\left(n\frac{\pi}{L}x\right)$$

Where

$$\begin{aligned}a_0&=\frac{1}{L}\int_0^L f'(x)\,dx\\&=\frac{1}{L}\left(\int_0^{x_0^-}f'(x)\,dx+\int_{x_0^+}^L f'(x)\,dx\right)\\&=\frac{1}{L}\left(\left[f(x)\right]_0^{x_0^-}+\left[f(x)\right]_{x_0^+}^L\right)\\&=\frac{1}{L}\left(\left[\alpha-f(0)\right]+\left[f(L)-\beta\right]\right)\\&=\frac{\left(\alpha-\beta\right)}{L}+\frac{f(0)+f(L)}{L}\end{aligned}$$

And for $n>0$

$$a_n=\frac{2}{L}\int_0^L f'(x)\cos\left(n\frac{\pi}{L}x\right)dx$$

Applying integration by parts. Let $f'(x) = dv, u = \cos\left(n\frac{\pi}{L}x\right)$, then $v = f(x), du = \frac{-n\pi}{L}\sin\left(\frac{n\pi}{L}x\right)$. Since $v = f(x)$ has has jump discontinuity at x_0 as described, then

$$
\begin{aligned}
a_n &= \frac{2}{L}\int_0^L u dv \\
&= \frac{2}{L}\left[\left([uv]_0^{x_0^-} + [uv]_{x_0^+}^L\right) - \int_0^L v du\right] \\
&= \frac{2}{L}\left(\left[\cos\left(n\frac{\pi}{L}x\right)f(x)\right]_0^{x_0^-} + \left[\cos\left(n\frac{\pi}{L}x\right)f(x)\right]_{x_0^+}^L + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right) \\
&= \frac{2}{L}\left(\cos\left(n\frac{\pi}{L}x_0^-\right)f\left(x_0^-\right) - f(0) + \cos(n\pi)f(L) - \cos\left(n\frac{\pi}{L}x_0^+\right)f\left(x_0^+\right) + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right) \quad (1)
\end{aligned}
$$

But

$$
\begin{aligned}
f\left(x_0^-\right) &= \alpha \\
f\left(x_0^+\right) &= \beta
\end{aligned}
$$

And since cos is continuous, then $\cos\left(n\frac{\pi}{L}x_0^-\right) = \cos\left(n\frac{\pi}{L}x_0^+\right) = \cos\left(n\frac{\pi}{L}x_0\right)$, therefore (1) becomes

$$
a_n = \frac{2}{L}\left(\cos(n\pi)f(L) - f(0) + \cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right) \quad (2)
$$

<u>On the other hand</u>, the Fourier *sin* series for $f(x)$ is given by

$$
f(x) \sim \sum_{n=0}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)
$$

Where

$$
b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx
$$

Therefore $\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx = \frac{L}{2}b_n$. Substituting this into (2) gives

$$
\begin{aligned}
a_n &= \frac{2}{L}\left(\cos(n\pi)f(L) - f(0) + \cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}\frac{L}{2}b_n\right) \\
&= \frac{2}{L}\cos(n\pi)f(L) - \frac{2}{L}f(0) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{2}{L}\frac{n\pi}{2}b_n \\
&= \frac{2}{L}\left((-1^n)f(L) - f(0)\right) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}b_n
\end{aligned}
$$

Hence

$$
\boxed{a_n = \frac{2}{L}\left((-1^n)f(L) - f(0)\right) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}b_n} \quad (3)
$$

<u>Summary</u> the Fourier cos series of $f'(x)$ is

$$
\begin{aligned}
f'(x) &\sim \sum_{n=0}^{\infty} a_n \cos\left(n\frac{\pi}{L}x\right) \\
a_0 &= \frac{(\alpha - \beta)}{L} + \frac{f(0) + f(L)}{L} \\
a_n &= \frac{2}{L}\left((-1^n)f(L) - f(0)\right) + \frac{2}{L}\cos\left(n\frac{\pi}{L}x_0\right)(\alpha - \beta) + \frac{n\pi}{L}b_n
\end{aligned}
$$

The above is in terms of b_n, which is the Fourier *sin* series of $f(x)$, which is what required to show.

1.8 Problem 3.4.9

*3.4.9 Consider the heat equation with a known source $q(x,t)$:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \text{ with } u(0,t) = 0 \text{ and } u(L,t) = 0.$$

Assume that $q(x,t)$ (for each $t > 0$) is a piecewise smooth function of x. Also assume that u and $\partial u/\partial x$ are continuous functions of x (for $t > 0$) and $\partial^2 u/\partial x^2$ and $\partial u/\partial t$ are piecewise smooth. Thus,

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\frac{n\pi x}{L}.$$

What ordinary differential equation does $b_n(t)$ satisfy? Do not solve this differential equation.

The PDE is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t) \tag{1}$$

Since the boundary conditions are homogenous Dirichlet conditions, then the solution can be written down as

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(n\frac{\pi}{L}x\right)$$

Since the solution is assumed to be continuous with continuous derivative, then term by term differentiation is allowed w.r.t. x

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} n\frac{\pi}{L} b_n(t)\cos\left(n\frac{\pi}{L}x\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) \tag{2}$$

Also using assumption that $\frac{\partial u}{\partial t}$ is smooth, then

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) \tag{3}$$

Substituting (2,3) into (1) gives

$$\sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + q(x,t) \tag{4}$$

Expanding $q(x,t)$ as Fourier sin series in x. Hence

$$q(x,t) = \sum_{n=1}^{\infty} q_n(t)\sin\left(\frac{n\pi}{L}x\right)$$

Where now $q_n(t)$ are time dependent given by (by orthogonality)

$$q_n(t) = \frac{2}{L}\int_0^L q(x,t)\sin\left(\frac{n\pi}{L}x\right)$$

Hence (4) becomes

$$\sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -\sum_{n=1}^{\infty} k\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + \sum_{n=1}^{\infty} q(t)_n \sin\left(\frac{n\pi}{L}x\right)$$

Applying orthogonality the above reduces to one term only

$$\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -k\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + q(t)_n\sin\left(\frac{n\pi}{L}x\right)$$

Dividing by $\sin\left(n\frac{\pi}{L}x\right) \neq 0$

$$\frac{db_n(t)}{dt} = -k\left(\frac{n\pi}{L}\right)^2 b_n(t) + q_n(t)$$

$$\frac{db_n(t)}{dt} + k\left(\frac{n\pi}{L}\right)^2 B_n(t) = q_n(t) \tag{5}$$

The above is the ODE that needs to be solved for $b_n(t)$. It is first order inhomogeneous ODE. The question asks to stop here.

1.9 Problem 3.4.11

3.4.11. Consider the *nonhomogeneous* heat equation (with a steady heat source):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x).$$

Solve this equation with the initial condition

$$u(x,0) = f(x)$$

and the boundary conditions

$$u(0,t) = 0 \text{ and } u(L,t) = 0.$$

Assume that a continuous solution exists (with continuous derivatives). [*Hints*: Expand the solution as a Fourier sine series (i.e., use the method of eigenfunction expansion). Expand $g(x)$ as a Fourier sine series. Solve for the Fourier sine series of the solution. Justify all differentiations with respect to x.]

The PDE is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g(x) \tag{1}$$

Since the boundary conditions are homogenous Dirichlet conditions, then the solution can be written down as

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin\left(n\frac{\pi}{L}x\right)$$

Since the solution is assumed to be continuous with continuous derivative, then term by term differentiation is allowed w.r.t. x

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} n\frac{\pi}{L} b_n(t)\cos\left(n\frac{\pi}{L}x\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) \tag{2}$$

Also using assumption that $\frac{\partial u}{\partial t}$ is smooth, then

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) \tag{3}$$

Substituting (2,3) into (1) gives

$$\sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -k\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + g(x) \tag{4}$$

Using hint given in the problem, which is to expand $g(x)$ as Fourier sin series. Hence

$$g(x) = \sum_{n=1}^{\infty} g_n\sin\left(\frac{n\pi}{L}x\right)$$

Where

$$g_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)$$

Hence (4) becomes

$$\sum_{n=1}^{\infty}\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -\sum_{n=1}^{\infty}k\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + \sum_{n=1}^{\infty} g_n\sin\left(\frac{n\pi}{L}x\right)$$

Applying orthogonality the above reduces to one term only

$$\frac{db_n(t)}{dt}\sin\left(n\frac{\pi}{L}x\right) = -k\left(\frac{n\pi}{L}\right)^2 b_n(t)\sin\left(n\frac{\pi}{L}x\right) + g_n\sin\left(\frac{n\pi}{L}x\right)$$

Dividing by $\sin\left(n\frac{\pi}{L}x\right) \neq 0$

$$\frac{db_n(t)}{dt} = -k\left(\frac{n\pi}{L}\right)^2 b_n(t) + g_n$$

$$\frac{db_n(t)}{dt} + k\left(\frac{n\pi}{L}\right)^2 b_n(t) = g_n \tag{5}$$

This is of the form $y' + ay = g_n$, where $a = k\left(\frac{n\pi}{L}\right)^2$. This is solved using an integration factor $\mu = e^{at}$, where $\frac{d}{dt}\left(e^{at}y\right) = e^{at}g_n$, giving the solution

$$y(t) = \frac{1}{\mu}\int \mu g_n dt + \frac{c}{\mu}$$

Hence the solution to (5) is

$$\begin{aligned}
b_n(t)\, e^{k\left(\frac{n\pi}{L}\right)^2 t} &= \int e^{k\left(\frac{n\pi}{L}\right)^2 t} g_n dt + c \\
b_n(t)\, e^{k\left(\frac{n\pi}{L}\right)^2 t} &= \frac{L^2 e^{k\left(\frac{n\pi}{L}\right)^2 t}}{kn^2\pi^2} g_n + c \\
b_n(t) &= \frac{L^2}{kn^2\pi^2} g_n + ce^{-k\left(\frac{n\pi}{L}\right)^2 t}
\end{aligned}$$

Where c above is constant of integration. Hence the solution becomes

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n(t) \sin\left(n\frac{\pi}{L}x\right) \\
&= \sum_{n=1}^{\infty} \left(\frac{L^2}{kn^2\pi^2} g_n + ce^{-k\left(\frac{n\pi}{L}\right)^2 t}\right) \sin\left(n\frac{\pi}{L}x\right)
\end{aligned}$$

At $t = 0$, $u(x,0) = f(x)$, therefore

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{L^2}{kn^2\pi^2} g_n + c\right) \sin\left(n\frac{\pi}{L}x\right)$$

Therefore

$$\frac{L^2}{kn^2\pi^2} g_n + c = \frac{2}{L}\int_0^L f(x) \sin\left(n\frac{\pi}{L}x\right) dx$$

Solving for c gives

$$c = \frac{2}{L}\int_0^L f(x) \sin\left(n\frac{\pi}{L}x\right) dx - \frac{L^2}{kn^2\pi^2} g_n$$

This completes the solution. Everything is now known. Summary

$$\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} b_n(t) \sin\left(n\frac{\pi}{L}x\right) \\
b_n(t) &= \left(\frac{L^2}{kn^2\pi^2} g_n + ce^{-k\left(\frac{n\pi}{L}\right)^2 t}\right) \\
g_n &= \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) \\
c &= \frac{2}{L}\int_0^L f(x) \sin\left(n\frac{\pi}{L}x\right) dx - \frac{L^2}{kn^2\pi^2} g_n
\end{aligned}$$