HW3, Math 322, Fall 2016

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Contents

0.1 Problem 2.5.1(e) (problem 1)

2.5.1. Solve Laplace's equation inside a rectangle $0 \le x \le L,\ 0 \le y \le H$, with the following boundary conditions:

*(a)	$\frac{\partial u}{\partial x}(0,y)=0,$	$\frac{\partial u}{\partial x}(L,y)=0,$	$u(x,0)=0,$	$u(x,H)=f(x)$
(b)	$\frac{\partial u}{\partial x}(0,y)=g(y),$	$\frac{\partial u}{\partial x}(L,y)=0,$	$u(x,0)=0,$	$u(x,H)=0$
*(c)	$\frac{\partial u}{\partial x}(0,y)=0,$	$u(L,y)=g(y),$	$u(x,0)=0,$	$u(x,H)=0$
(d)	$u(0,y)=g(y),$	$u(L,y)=0,$	$\frac{\partial u}{\partial y}(x,0)=0,$	$u(x,H)=0$
*(e)	$u(0,y)=0,$	$u(L,y)=0,$	$u(x,0)-\frac{\partial u}{\partial y}(x,0)=0,$	$u(x,H)=f(x)$

Let $u\left(x,y\right) = X(x)\,Y(x)$. Substituting this into the PDE $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and simplifying gives

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Each side depends on different independent variable and they are equal, therefore they must be equal to same constant.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \pm\lambda$$

Since the boundary conditions along the x direction are the homogeneous ones, $-\lambda$ is selected in the above. Two ODE's (1,2) are obtained as follows

$$X'' + \lambda X = 0 \tag{1}$$

With the boundary conditions

$$X(0) = 0$$
$$X(L) = 0$$

And

$$Y'' - \lambda Y = 0 \tag{2}$$

With the boundary conditions

$$Y(0) = Y'(0)$$
$$Y(H) = f(x)$$

In all these cases λ will turn out to be positive. This is shown for this problem only and not be repeated again. The solution to (1) is

$$X = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$$

Case $\lambda < 0$

$$X = A\cosh\left(\sqrt{\lambda}x\right) + B\sinh\left(\sqrt{\lambda}x\right)$$

At $x = 0$, the above gives $0 = A$. Hence $X = B\sinh\left(\sqrt{\lambda}x\right)$. At $x = L$ this gives $X = B\sinh\left(\sqrt{\lambda}L\right)$. But $\sinh\left(\sqrt{\lambda}L\right) = 0$ only at 0 and $\sqrt{\lambda}L \neq 0$, therefore $B = 0$ and this leads to trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

Case $\lambda = 0$

$$X = Ax + B$$

Hence at $x = 0$ this gives $0 = B$ and the solution becomes $X = B$. At $x = L$, $B = 0$. Hence the trivial solution. $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$

Solution is

$$X = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

At $x=0$ this gives $0=A$ and the solution becomes $X=B\sin\left(\sqrt{\lambda}x\right)$. At $x=L$

$$0=B\sin\left(\sqrt{\lambda}L\right)$$

For non-trivial solution $\sin\left(\sqrt{\lambda}L\right)=0$ or $\sqrt{\lambda}L=n\pi$ where $n=1,2,3,\cdots$, therefore

$$\lambda_n=\left(\frac{n\pi}{L}\right)^2 \qquad n=1,2,3,\cdots$$

Eigenfunctions are

$$X_n\left(x\right)=B_n\sin\left(\frac{n\pi}{L}x\right) \qquad n=1,2,3,\cdots \tag{3}$$

For the Y ODE, the solution is

$$\begin{aligned}Y_n&=C_n\cosh\left(\frac{n\pi}{L}y\right)+D_n\sinh\left(\frac{n\pi}{L}y\right)\\ Y_n'&=C_n\frac{n\pi}{L}\sinh\left(\frac{n\pi}{L}y\right)+D_n\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)\end{aligned}$$

Applying B.C. at $y=0$ gives

$$\begin{aligned}Y\left(0\right)&=Y'\left(0\right)\\ C_n\cosh\left(0\right)&=D_n\frac{n\pi}{L}\cosh\left(0\right)\\ C_n&=D_n\frac{n\pi}{L}\end{aligned}$$

The eigenfunctions Y_n are

$$\begin{aligned}Y_n&=D_n\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+D_n\sinh\left(\frac{n\pi}{L}y\right)\\ &=D_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)\end{aligned}$$

Now the complete solution is produced

$$\begin{aligned}u_n\left(x,y\right)&=Y_nX_n\\ &=D_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)B_n\sin\left(\frac{n\pi}{L}x\right)\end{aligned}$$

Let $D_nB_n=B_n$ since a constant. (no need to make up a new symbol).

$$u_n\left(x,y\right)=B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Sum of eigenfunctions is the solution, hence

$$u\left(x,y\right)=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

The nonhomogeneous boundary condition is now resolved. At $y=H$

$$u\left(x,H\right)=f\left(x\right)$$

Therefore

$$f\left(x\right)=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating gives

$$\begin{aligned}\int_0^L f\left(x\right)\sin\left(\frac{m\pi}{L}x\right)dx&=\int_0^L\sin\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)\sin\left(\frac{n\pi}{L}x\right)dx\\ &=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)\int_0^L\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx\\ &=B_m\left(\frac{m\pi}{L}\cosh\left(\frac{m\pi}{L}H\right)+\sinh\left(\frac{m\pi}{L}H\right)\right)\frac{L}{2}\end{aligned}$$

Hence

$$B_n=\frac{2}{L}\frac{\int_0^L f\left(x\right)\sin\left(\frac{n\pi}{L}x\right)dx}{\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}H\right)+\sinh\left(\frac{n\pi}{L}H\right)\right)} \tag{4}$$

This completes the solution. In summary

$$u\left(x,y\right)=\sum_{n=1}^{\infty}B_n\left(\frac{n\pi}{L}\cosh\left(\frac{n\pi}{L}y\right)+\sinh\left(\frac{n\pi}{L}y\right)\right)\sin\left(\frac{n\pi}{L}x\right)$$

With B_n given by (4). The following are some plots of the solution above for different $f\left(x\right)$.

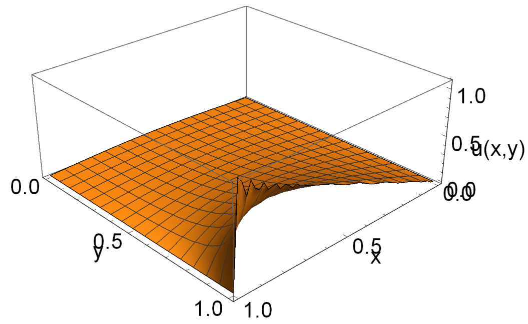


Figure 1: Solution using $f(x) = x, L = 1, H = 1$

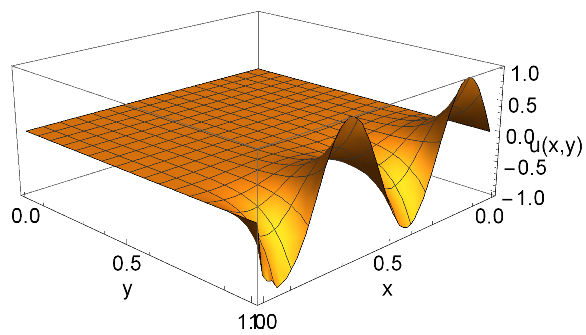


Figure 2: Solution using $f(x) = \sin(12x), L = 1, H = 1$

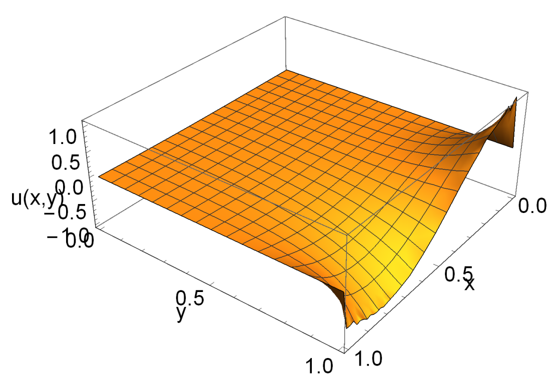


Figure 3: Solution using $f(x) = \cos(4x), L = 1, H = 1$

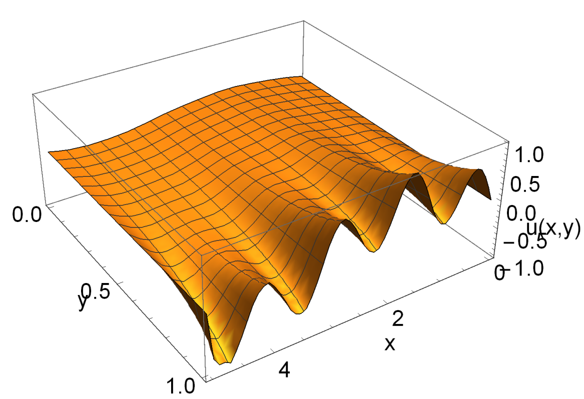


Figure 4: Solution using $f(x) = \sin(3x)\cos(2x), L = 5, H = 1$

0.2 Problem 2.5.2 (problem 2)

2.5.2. Consider $u(x,y)$ satisfying Laplace's equation inside a rectangle ($0 < x < L$, $0 < y < H$) subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0,y)=0 \quad \frac{\partial u}{\partial y}(x,0)=0$$
$$\frac{\partial u}{\partial x}(L,y)=0 \quad \frac{\partial u}{\partial y}(x,H)=f(x).$$

*(a) *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a).

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition

$$u(x,y,0)=g(x,y).$$

0.2.1 part (a)

At steady state, there will be no heat energy flowing across the boundaries. Which implies the flux is zero. Three of the boundaries are already insulated and hence the flux is zero at those boundaries as given. Therefore, the flux should also be zero at the top boundary at steady state.

By definition, the flux is $\bar{\phi} = -k\bar{\nabla} u \cdot \hat{n}$. (Direction of flux vector is from hot to cold). At the top boundary, this becomes

$$\phi = -k\frac{\partial u}{\partial y}(x,H) \tag{1}$$

Therefore, For the condition of a solution, total flux on the boundary is zero, or

$$\int_0^L \phi dx = 0$$

Using (1) in the above gives

$$-k\int_0^L \frac{\partial u}{\partial y}(x,H)\,dx = 0$$
$$\int_0^L \frac{\partial u}{\partial y}(x,H)\,dx = 0$$

But $\frac{\partial u}{\partial y}(x,H) = f(x)$ and the above becomes

$$\boxed{\int_0^L f(x)\,dx = 0}$$

0.2.2 Part (b)

Using separation of variables results in the following two ODE's

$$X'' + \lambda X = 0$$
$$X'(0) = 0$$
$$X'(L) = 0$$

And

$$Y'' - \lambda Y = 0$$
$$Y'(0) = 0$$
$$Y'(L) = f(x)$$

The solution to the $X(x)$ ODE has been obtained before as

$$X_n = A_0 + A_n \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 1,2,3,\cdots$$
$$X_n = A_n \cos\left(\sqrt{\lambda_n}x\right) \qquad n = 0,1,2,3,\cdots \tag{1}$$

Where $\lambda_n = \left(\frac{n\pi}{L}\right)^2$. In this ODE $\lambda = 0$ is applicable as well as $\lambda > 0$. (As found in last HW).

Now the $Y(y)$ ODE is solved (for same set of eigenvalues). For $\lambda = 0$ the ODE becomes $Y'' = 0$ and solution is $Y = Cy + D$. Hence $Y' = C$ and since $Y'(0) = 0$ then $C = 0$. Hence the solution is $Y = C_0$, where C_0 is some new constant. For $\lambda > 0$, the solution is

$$\begin{aligned} Y_n &= C_n \cosh\left(\sqrt{\lambda_n} y\right) + D_n \sinh\left(\sqrt{\lambda_n} y\right) \qquad n = 1,2,3,\cdots \\ Y_n' &= C_n \sqrt{\lambda_n} \sinh\left(\sqrt{\lambda_n} y\right) + D_n \sqrt{\lambda_n} \cosh\left(\sqrt{\lambda_n} y\right) \end{aligned}$$

At $y = 0$

$$\begin{aligned} 0 &= Y_n'(0) \\ &= D_n \sqrt{\lambda_n} \qquad n = 1,2,3,\cdots \end{aligned}$$

Since $\lambda_n > 0$ for $n = 1,2,3,\cdots$ then $D_n = 0$ and the $Y(y)$ solution becomes

$$\begin{aligned} Y_n &= C_0 + C_n \cosh\left(\sqrt{\lambda_n} y\right) \qquad n = 1,2,3,\cdots \\ Y_n &= C_n \cosh\left(\sqrt{\lambda_n} y\right) \qquad n = 0,1,2,3,\cdots \end{aligned} \tag{2}$$

Combining (1) and (2) gives

$$\begin{aligned} u_n(x,y) &= X_n Y_n \\ &= A_n \cos\left(\sqrt{\lambda_n} x\right) C_n \cosh\left(\sqrt{\lambda_n} y\right) \qquad n = 0,1,2,3,\cdots \\ &= A_n \cos\left(\sqrt{\lambda_n} x\right) \cosh\left(\sqrt{\lambda_n} y\right) \qquad n = 0,1,2,3,\cdots \end{aligned}$$

Where $A_n C_n$ above was combined and renamed to A_n (No need to add new symbol). Hence by superposition the solution becomes

$$u(x,y) = \sum_{n=0}^{\infty} A_n \cos\left(\sqrt{\lambda_n} x\right) \cosh\left(\sqrt{\lambda_n} y\right)$$

Since $\lambda_0 = 0$ and $\cos\left(\sqrt{\lambda_0} x\right) \cosh\left(\sqrt{\lambda_0} y\right) = 1$, the above can be also be written as

$$u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) \cosh\left(\frac{n\pi}{L} y\right) \tag{3}$$

At $y = H$, it is given that $\frac{\partial u}{\partial y}(x,H) = f(x)$. But

$$\frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) \frac{n\pi}{L} \sinh\left(\frac{n\pi}{L} y\right)$$

At $y = H$ the above becomes

$$f(x) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) \frac{n\pi}{L} \sinh\left(\frac{n\pi}{L} H\right) \tag{4}$$

To <u>verify part (a)</u> by integrating both sides

$$\begin{aligned} \int_0^L f(x)\,dx &= \int_0^L \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) \frac{n\pi}{L} \sinh\left(\frac{n\pi}{L} H\right) dx \\ &= \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \sinh\left(\frac{n\pi}{L} H\right) \int_0^L \cos\left(\frac{n\pi}{L} x\right) dx \end{aligned}$$

But $\int_0^L \cos\left(\frac{n\pi}{L} x\right) dx = 0$, hence

$$\int_0^L f(x)\,dx = 0$$

The verification is completed. Now back to (4) and multiplying by $\cos\left(\frac{m\pi}{L} x\right)$ and integrating

$$\begin{aligned} \int_0^L f(x) \cos\left(\frac{m\pi}{L} x\right) dx &= \int_0^L \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} x\right) \sqrt{\lambda_n} \sinh\left(\frac{n\pi}{L} H\right) dx \\ &= \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{L} H\right) \int_0^L \cos\left(\frac{n\pi}{L} x\right) \sqrt{\lambda_n} dx \\ &= A_m \sinh\left(\frac{m\pi}{L} H\right) \frac{L}{2} \end{aligned}$$

Hence

$$A_n = \frac{2}{L} \frac{\int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx}{\sinh\left(\frac{n\pi}{L} H\right)} \qquad n = 1,2,3,\cdots$$

Therefore the solution now becomes (from (3))

$$u\left(x,y\right)=A_0+\sum_{n=1}^{\infty}\left(\frac{2}{L}\frac{\int_0^L f\left(x\right)\cos\left(\frac{n\pi}{L}x\right)dx}{\sinh\left(\frac{n\pi}{L}H\right)}\right)\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)$$

Only A_0 remains to be found. This is done in next part.

0.2.3 Part (c)

Since at steady state, total energy is the same as initial energy. Initial temperature is given as $g\left(x,y\right)$, therefore initial thermal energy is found by integrating over the whole domain. This is 2D, therefore

$$\int\int\rho c g\left(x,y\right)dA=\rho c\int_0^L\int_0^H g\left(x,y\right)dydx$$

Setting the above to $\rho c\int_0^L\int_0^H u\left(x,y\right)dydx$ found in last part, gives one equation with one unknown, which is A_0 to solve for. Hence

$$\begin{aligned}
\rho c\int_0^L\int_0^H g\left(x,y\right)dydx &= \rho c\int_0^L\int_0^H A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx\\
\int_0^L\int_0^H g\left(x,y\right)dydx &= \int_0^L\int_0^H A_0 dydx+\int_0^L\int_0^H\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx\\
\int_0^L\int_0^H g\left(x,y\right)dydx &= A_0HL+\sum_{n=1}^{\infty}A_n\int_0^L\int_0^H\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx \qquad (5)
\end{aligned}$$

But

$$\int_0^L\int_0^H\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)dydx=\int_0^H\cosh\left(\frac{n\pi}{L}y\right)\left(\int_0^L\cos\left(\frac{n\pi}{L}x\right)dx\right)dy$$

Where $\int_0^L\cos\left(\frac{n\pi}{L}x\right)dx=0$. Hence the whole sum vanish. Therefore (5) reduces to

$$\begin{aligned}
\int_0^L\int_0^H g\left(x,y\right)dydx &= A_0HL\\
A_0 &= \frac{1}{HL}\int_0^L\int_0^H g\left(x,y\right)dydx
\end{aligned}$$

Summary The complete solution is

$$u\left(x,y\right)=\left(\frac{1}{HL}\int_0^L\int_0^H g\left(x,y\right)dydx\right)+\sum_{n=1}^{\infty}\left(\frac{2}{L}\frac{\int_0^L f\left(x\right)\cos\left(\frac{n\pi}{L}x\right)dx}{\sinh\left(\frac{n\pi}{L}H\right)}\right)\cos\left(\frac{n\pi}{L}x\right)\cosh\left(\frac{n\pi}{L}y\right)$$

The following are some plots of the solution.

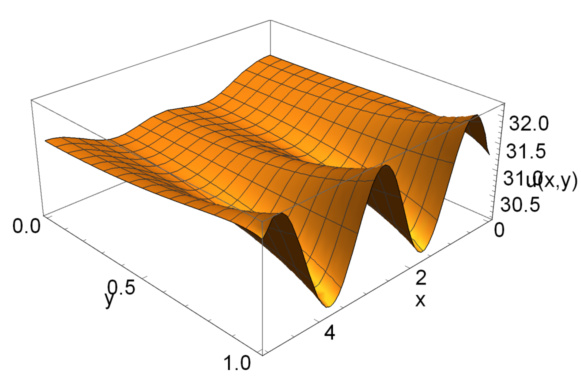


Figure 5: Solution using $g(x,y)=xy, f(x)=\sin(3x), L=5, H=1$

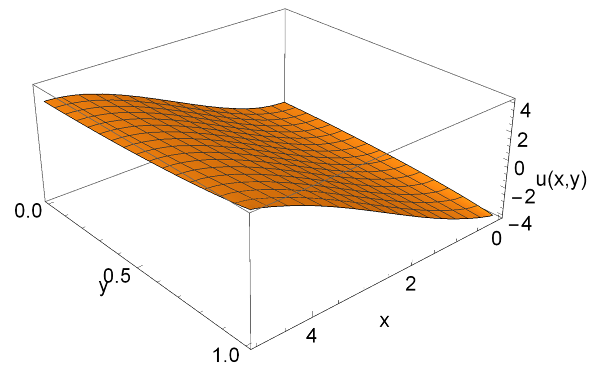


Figure 6: Solution using $g(x,y) = \sin(y)\cos(xy), f(x) = x, L = 5, H = 1$

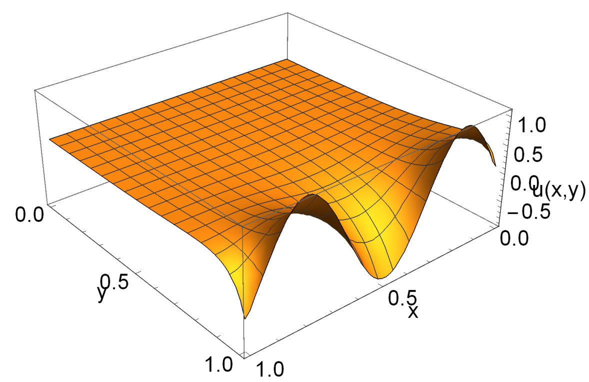


Figure 7: Solution using $g(x,y) = y\sin(y)\cos(xy), f(x) = \sin(10x), L = 1, H = 1$

0.3 Problem 2.5.5(c,d) (problem 3)

2.5.5. Solve Laplace's equation inside the quarter-circle of radius 1 ($0 \le \theta \le \pi/2$, $0 \le r \le 1$) subject to the boundary conditions

* (a) $\frac{\partial u}{\partial \theta}(r,0) = 0$, $\quad u\left(r,\frac{\pi}{2}\right) = 0$, $\quad u(1,\theta) = f(\theta)$

(b) $\frac{\partial u}{\partial \theta}(r,0) = 0$, $\quad \frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{2}\right) = 0$, $\quad u(1,\theta) = f(\theta)$

* (c) $u(r,0) = 0$, $\quad u\left(r,\frac{\pi}{2}\right) = 0$, $\quad \frac{\partial u}{\partial r}(1,\theta) = f(\theta)$

(d) $\frac{\partial u}{\partial \theta}(r,0) = 0$, $\quad \frac{\partial u}{\partial \theta}\left(r,\frac{\pi}{2}\right) = 0$, $\quad \frac{\partial u}{\partial r}(1,\theta) = g(\theta)$

Show that the solution [part (d)] exists only if $\int_0^{\pi/2} g(\theta)\, d\theta = 0$. Explain this condition physically.

0.3.1 Part c

The Laplace PDE in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{A}$$

With boundary conditions

$$\begin{aligned} u(r,0) &= 0 \\ u\left(r,\frac{\pi}{2}\right) &= 0 \\ u(1,\theta) &= f(\theta) \end{aligned} \tag{B}$$

Assuming the solution can be written as

$$u(r,\theta) = R(r)\Theta(\theta)$$

And substituting this assumed solution back into the (A) gives

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0$$

Dividing the above by $R\Theta \neq 0$ gives

$$
\begin{aligned}
r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} &= 0\\
r^2\frac{R''}{R} + r\frac{R'}{R} &= -\frac{\Theta''}{\Theta}
\end{aligned}
$$

Since each side depends on different independent variable and they are equal, they must be equal to same constant. say λ.

$$
r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda
$$

This results in the following two ODE's. The boundaries conditions in (B) are also transferred to each ODE. This gives

$$
\begin{aligned}
\Theta'' + \lambda\Theta &= 0\\
\Theta(0) &= 0 \qquad (1)\\
\Theta\left(\frac{\pi}{2}\right) &= 0
\end{aligned}
$$

And

$$
\begin{aligned}
r^2R'' + rR' - \lambda R &= 0 \qquad (2)\\
|R(0)| &< \infty
\end{aligned}
$$

Starting with (1). Consider the Case $\lambda < 0$. The solution in this case will be

$$
\Theta = A\cosh\left(\sqrt{\lambda}\theta\right) + B\sinh\left(\sqrt{\lambda}\theta\right)
$$

Applying first B.C. gives $A = 0$. The solution becomes $\Theta = B\sinh\left(\sqrt{\lambda}\theta\right)$. Applying second B.C. gives

$$
0 = B\sinh\left(\sqrt{\lambda}\frac{\pi}{2}\right)
$$

But sinh is zero only when $\sqrt{\lambda}\frac{\pi}{2} = 0$ which is not the case here. Therefore $B = 0$ and hence trivial solution. Hence $\lambda < 0$ is not an eigenvalue.

Case $\lambda = 0$ The ODE becomes $\Theta'' = 0$ with solution $\Theta = A\theta + B$. First B.C. gives $0 = B$. The solution becomes $\Theta = A\theta$. Second B.C. gives $0 = A\frac{\pi}{2}$, hence $A = 0$ and trivial solution. Therefore $\lambda = 0$ is not an eigenvalue.

Case $\lambda > 0$ The ODE becomes $\Theta'' + \lambda\Theta = 0$ with solution

$$
\Theta = A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right)
$$

The first B.C. gives $0 = A$. The solution becomes

$$
\Theta = B\sin\left(\sqrt{\lambda}\theta\right)
$$

And the second B.C. gives

$$
0 = B\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)
$$

For non-trivial solution $\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0$ or $\sqrt{\lambda}\frac{\pi}{2} = n\pi$ for $n = 1, 2, 3, \cdots$. Hence the eigenvalues are

$$
\begin{aligned}
\sqrt{\lambda_n} &= 2n\\
\lambda_n &= 4n^2 \qquad n = 1, 2, 3, \cdots
\end{aligned}
$$

And the eigenfunctions are

$$
\boxed{\Theta_n(\theta) = B_n\sin(2n\theta) \qquad n = 1, 2, 3, \cdots} \qquad (3)
$$

Now the R ODE is solved. There is one case to consider, which is $\lambda > 0$ based on the above. The ODE is

$$
\begin{aligned}
r^2R'' + rR' - \lambda_n R &= 0\\
r^2R'' + rR' - 4n^2R &= 0 \qquad n = 1, 2, 3, \cdots
\end{aligned}
$$

This is Euler ODE. Let $R(r) = r^p$. Then $R' = pr^{p-1}$ and $R'' = p\left(p-1\right)r^{p-2}$. This gives

$$\begin{aligned}
r^2\left(p\left(p-1\right)r^{p-2}\right) + r\left(pr^{p-1}\right) - 4n^2r^p &= 0\\
\left(\left(p^2-p\right)r^p\right) + pr^p - 4n^2r^p &= 0\\
r^pp^2 - pr^p + pr^p - 4n^2r^p &= 0\\
p^2 - 4n^2 &= 0\\
p &= \pm 2n
\end{aligned}$$

Hence the solution is

$$R(r) = Cr^{2n} + D\frac{1}{r^{2n}}$$

Applying the condition that $|R(0)| < \infty$ implies $D = 0$, and the solution becomes

$$R_n(r) = C_nr^{2n} \qquad n = 1,2,3,\cdots \tag{4}$$

Using (3,4) the solution $u_n(r,\theta)$ is

$$\begin{aligned}
u_n(r,\theta) &= R_n\Theta_n\\
&= C_nr^{2n}B_n\sin(2n\theta)\\
&= B_nr^{2n}\sin(2n\theta)
\end{aligned}$$

Where C_nB_n was combined into one constant B_n. (No need to introduce new symbol). The final solution is

$$\begin{aligned}
u(r,\theta) &= \sum_{n=1}^{\infty}u_n(r,\theta)\\
&= \sum_{n=1}^{\infty}B_nr^{2n}\sin(2n\theta)
\end{aligned}$$

Now the nonhomogeneous condition is applied to find B_n.

$$\frac{\partial}{\partial r}u(r,\theta) = \sum_{n=1}^{\infty}B_n(2n)r^{2n-1}\sin(2n\theta)$$

Hence $\frac{\partial}{\partial r}u(1,\theta) = f(\theta)$ becomes

$$f(\theta) = \sum_{n=1}^{\infty}2B_nn\sin(2n\theta)$$

Multiplying by $\sin(2m\theta)$ and integrating gives

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}f(\theta)\sin(2m\theta)\,d\theta &= \int_0^{\frac{\pi}{2}}\sin(2m\theta)\sum_{n=1}^{\infty}2B_nn\sin(2n\theta)\,d\theta\\
&= \sum_{n=1}^{\infty}2nB_n\int_0^{\frac{\pi}{2}}\sin(2m\theta)\sin(2n\theta)\,d\theta
\end{aligned} \tag{5}$$

When $n = m$ then

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\sin(2m\theta)\sin(2n\theta)\,d\theta &= \int_0^{\frac{\pi}{2}}\sin^2(2n\theta)\,d\theta\\
&= \int_0^{\frac{\pi}{2}}\left(\frac{1}{2}-\frac{1}{2}\cos 4n\theta\right)d\theta\\
&= \frac{1}{2}\left[\theta\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{\sin 4n\theta}{4n}\right]_0^{\frac{\pi}{2}}\\
&= \frac{\pi}{4} - \left(\frac{1}{8n}\left(\sin\frac{4n}{2}\pi\right) - \sin(0)\right)
\end{aligned}$$

And since n is integer, then $\sin\frac{4n}{2}\pi = \sin 2n\pi = 0$ and the above becomes $\frac{\pi}{4}$.

Now for the case when $n \neq m$ using $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ then

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin(2m\theta)\sin(2n\theta)\,d\theta &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos(2m\theta - 2n\theta) - \cos(2m\theta + 2n\theta))\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2m\theta - 2n\theta)\,d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2m\theta + 2n\theta)\,d\theta \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos((2m-2n)\theta)\,d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos((2m+2n)\theta)\,d\theta \\
&= \frac{1}{2}\left[\frac{\sin((2m-2n)\theta)}{(2m-2n)}\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\frac{\sin((2m+2n)\theta)}{(2m+2n)}\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{4(m-n)}[\sin((2m-2n)\theta)]_0^{\frac{\pi}{2}} - \frac{1}{4(m+n)}[\sin((2m+2n)\theta)]_0^{\frac{\pi}{2}} \\
&= \frac{1}{4(m-n)}\left[\sin\left((2m-2n)\frac{\pi}{2}\right) - 0\right] - \frac{1}{4(m+n)}\left[\sin\left((2m+2n)\frac{\pi}{2}\right) - 0\right]
\end{aligned}$$

Since $2m - 2n\frac{\pi}{2} = \pi(m-n)$ which is integer multiple of π and also $(2m+2n)\frac{\pi}{2}$ is integer multiple of π then the whole term above becomes zero. Therefore (5) becomes

$$\int_0^{\frac{\pi}{2}} f(\theta)\sin(2m\theta)\,d\theta = 2mB_m\frac{\pi}{4}$$

Hence

$$\boxed{B_n = \frac{2}{\pi n}\int_0^{\frac{\pi}{2}} f(\theta)\sin(2n\theta)\,d\theta}$$

Summary: the final solution is

$$\begin{aligned}
u(r,\theta) &= \sum_{n=1}^{\infty} B_n\left(r^{2n}\sin(2n\theta)\right) \\
B_n &= \frac{2}{\pi n}\int_0^{\frac{\pi}{2}} f(\theta)\sin(2n\theta)\,d\theta
\end{aligned}$$

The following are some plots of the solution

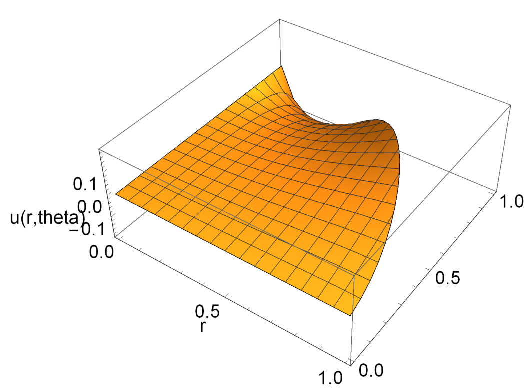


Figure 8: Solution using $f(\theta) = \theta\sin(3\theta)$

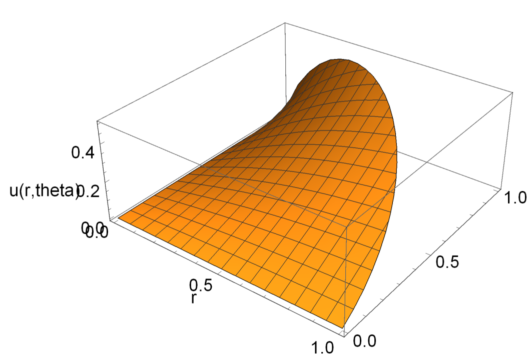


Figure 9: Solution using $f(\theta) = \theta$

0.3.2 Part (d)

The Laplace PDE in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0$$

With boundary conditions

$$\begin{aligned}u(r,0)&=0\\ u\left(r,\frac{\pi}{2}\right)&=0\\ u(1,\theta)&=f(\theta)\end{aligned}$$

Assuming the solution is

$$u(r,\theta)=R(r)\Theta(\theta)$$

Substituting this back into the PDE gives

$$r^2R''\Theta+rR'\Theta+R\Theta''=0$$

Dividing by $R\Theta\neq 0$ gives

$$\begin{aligned}r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}&=0\\ r^2\frac{R''}{R}+r\frac{R'}{R}&=-\frac{\Theta''}{\Theta}\end{aligned}$$

Since each side depends on different independent variable and they are equal, they must be equal to same constant. say λ.

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{\Theta''}{\Theta}=\lambda$$

This results in two ODE's with the following boundary conditions

$$\begin{aligned}\Theta''+\lambda\Theta&=0\\ \Theta'(0)&=0\\ \Theta'\left(\frac{\pi}{2}\right)&=0\end{aligned}\tag{1}$$

And

$$\begin{aligned}r^2R''+rR'-\lambda R&=0\\ |R(0)|&<\infty\end{aligned}\tag{2}$$

Starting with (1). Consider Case $\lambda<0$ The solution will be

$$\Theta=A\cosh\left(\sqrt{\lambda}\theta\right)+B\sinh\left(\sqrt{\lambda}\theta\right)$$

And

$$\Theta'=A\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\theta\right)+B\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\theta\right)$$

Applying first B.C. gives $0=B\sqrt{\lambda}$, therefore $B=0$ and the solution becomes $A\cosh\left(\sqrt{\lambda}\theta\right)$ and $\Theta'=A\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\theta\right)$. Applying second B.C. gives $0=A\sqrt{\lambda}\sinh\left(\sqrt{\lambda}\frac{\pi}{2}\right)$. But $\sinh\left(\sqrt{\lambda}\frac{\pi}{2}\right)\neq 0$ since $\lambda\neq 0$, therefore $A=0$ and the trivial solution results. Hence $\lambda<0$ is not an eigenvalue.

Case $\lambda=0$ The ODE becomes

$$\Theta''=0$$

With solution

$$\Theta=A\theta+B$$

And $\Theta'=A$. First B.C. gives $0=A$. Hence $\Theta=B$. Second B.C. produces no result and the solution is constant. Hence

$$\boxed{\Theta=C_0}$$

Where C_0 is constant. Therefore $\lambda=0$ is an eigenvalue.

Case $\lambda>0$ The ODE becomes $\Theta''+\lambda\Theta=0$ with solution

$$\begin{aligned}\Theta&=A\cos\left(\sqrt{\lambda}\theta\right)+B\sin\left(\sqrt{\lambda}\theta\right)\\ \Theta'&=-A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right)+B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\theta\right)\end{aligned}$$

The first B.C. gives $0 = B\sqrt{\lambda}$ or $B = 0$. The solution becomes

$$\Theta = A\cos\left(\sqrt{\lambda}\theta\right)$$

And $\Theta' = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right)$. The second B.C. gives

$$0 = -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right)$$

For non-trivial solution $\sin\left(\sqrt{\lambda}\frac{\pi}{2}\right) = 0$ or $\sqrt{\lambda}\frac{\pi}{2} = n\pi$ for $n = 1,2,3,\cdots$. Hence the eigenvalues are

$$\begin{aligned} \sqrt{\lambda_n} &= 2n \\ \lambda_n &= 4n^2 \qquad n = 1,2,3,\cdots \end{aligned}$$

And the eigenfunction is

$$\boxed{\Theta_n(\theta) = A_n\cos(2n\theta) \qquad n = 1,2,3,\cdots} \tag{3}$$

Now the R ODE is solved. The ODE is

$$r^2R'' + rR' - \lambda R = 0$$

Case $\lambda = 0$

The ODE becomes $r^2R'' + rR' = 0$. Let $v(r) = R'(r)$ and the ODE becomes

$$r^2v' + rv = 0$$

Dividing by $r \neq 0$

$$v'(r) + \frac{1}{r}v(r) = 0$$

Using integrating factor $e^{\int\frac{1}{r}dr} = e^{\ln r} = r$. Hence

$$\frac{d}{dr}(rv) = 0$$

Hence

$$\begin{aligned} rv &= A \\ v(r) &= \frac{A}{r} \end{aligned}$$

But since $v(r) = R'(r)$ then $R' = \frac{c_1}{r}$. The solution to this ODE Is

$$R(r) = \int\frac{A}{r}dr + B$$

Therefore, for $\lambda = 0$ the solution is

$$\boxed{R(r) = A\ln|r| + B \qquad r \neq 0}$$

Since

$$\lim_{r\to 0}|R(r)| < \infty$$

Then $A = 0$ and the solution is just a constant

$$R(r) = B_0$$

Case $\lambda > 0$ The ODE is

$$r^2R'' + rR' - 4n^2R = 0 \qquad n = 1,2,3,\cdots$$

The Let $R(r) = r^p$. Then $R' = pr^{p-1}$ and $R'' = p\left(p-1\right)r^{p-2}$. This gives

$$\begin{aligned} r^2\left(p\left(p-1\right)r^{p-2}\right) + r\left(pr^{p-1}\right) - 4n^2r^p &= 0 \\ \left(\left(p^2-p\right)r^p\right) + pr^p - 4n^2r^p &= 0 \\ r^pp^2 - pr^p + pr^p - 4n^2r^p &= 0 \\ p^2 - 4n^2 &= 0 \\ p &= \pm 2n \end{aligned}$$

Hence the solution is

$$R(r) = Cr^{2n} + D\frac{1}{r^{2n}}$$

The condition that

$$\lim_{r\to 0}|R(r)| < \infty$$

Implies $D = 0$, Hence the solution becomes

$$R_n(r) = C_n r^2 n \qquad n = 1, 2, 3, \cdots \tag{4}$$

Now the solutions are combined. For $\lambda = 0$ the solution is

$$u_0(r, \theta) = C_0 B_0$$

Which can be combined to one constant B_0. Hence

$$\boxed{u_0 = B_0} \tag{5}$$

And for $\lambda > 0$ the solution is

$$\begin{aligned} u_n(r, \theta) &= R_n \Theta_n \\ &= C_n r^{2n} (A_n \cos(2n\theta)) \\ &= B_n r^{2n} \cos(2n\theta) \end{aligned}$$

Where $C_n A_n$ are combined into one constant B_n. Hence

$$u_n(r, \theta) = \sum_{n=1}^{\infty} B_n r^{2n} \cos(2n\theta) \tag{6}$$

Equation (5) and (6) can be combined into one this now includes eigenfunctions for both $\lambda = 0$ and $\lambda > 0$

$$u(r, \theta) = B_0 + \sum_{n=1}^{\infty} B_n r^{2n} \cos(2n\theta) \tag{7}$$

Where B_0 represent the products of the eigenfunctions for R and Θ for $\lambda = 0$. Now the nonhomogeneous condition is applied to find B_n.

$$\frac{\partial}{\partial r} u(r, \theta) = \sum_{n=1}^{\infty} B_n (2n) r^{2n-1} \cos(2n\theta)$$

Hence $\frac{\partial}{\partial r} u(1, \theta) = g(\theta)$ becomes

$$g(\theta) = \sum_{n=1}^{\infty} 2B_n n \cos(2n\theta) \tag{8}$$

Multiplying by $\cos(2m\theta)$ and integrating gives

$$\begin{aligned} \int_0^{\frac{\pi}{2}} g(\theta) \cos(2m\theta)\, d\theta &= \int_0^{\frac{\pi}{2}} \cos(2m\theta) \sum_{n=1}^{\infty} 2B_n n \cos(2n\theta)\, d\theta \\ &= \sum_{n=1}^{\infty} 2nB_n \int_0^{\frac{\pi}{2}} \cos(2m\theta) \cos(2n\theta)\, d\theta \end{aligned} \tag{9}$$

As in the last part, the integral on right gives $\frac{\pi}{4}$ when $n = m$ and zero otherwise, hence

$$\int_0^{\frac{\pi}{2}} g(\theta) \cos(2n\theta)\, d\theta = 2nB_n \frac{\pi}{4}$$

$$B_n = \frac{2}{\pi n} \int_0^{\frac{\pi}{2}} g(\theta) \cos(2n\theta)\, d\theta \qquad n = 1, 2, 3, \cdots$$

Therefore the <u>final solution is</u> from (7) and (9)

$$\begin{aligned} u(r, \theta) &= B_0 + \sum_{n=1}^{\infty} B_n r^{2n} \cos(2n\theta) \\ &= B_0 + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n} \int_0^{\frac{\pi}{2}} g(\theta) \cos(2m\theta)\, d\theta \right) r^{2n} \cos(2n\theta) \end{aligned}$$

The unknown constant B_0 can be found if given the initial temperature as was done in problem 2.5.2 part (c). To answer the last part. Using (8) and integrating

$$\begin{aligned} \int_0^{\frac{\pi}{2}} g(\theta)\, d\theta &= \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} 2nB_n \cos(2n\theta)\, d\theta \\ &= \sum_{n=1}^{\infty} 2nB_n \int_0^{\frac{\pi}{2}} \cos(2n\theta)\, d\theta \end{aligned}$$

But

$$\int_0^{\frac{\pi}{2}} \cos(2n\theta)\,d\theta = \left[\frac{\sin(2n\theta)}{2n}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{2n}\left(\sin\frac{2n}{2}\pi - 0\right)$$
$$= \frac{1}{2n}(\sin n\pi - 0)$$
$$= 0$$

Since n is an integer. This condition physically means the same as in part (b) problem 2.5.2. Which is, since at steady state the flux must be zero on all boundaries, and $g(\theta)$ represents the flux over the surface of the quarter circle, then the integral of the flux must be zero. This means there is no thermal energy flowing across the boundary.

0.4 Problem 2.5.8(b) (problem 4)

> **2.5.8. Solve Laplace's equation inside a circular annulus ($a < r < b$) subject to the boundary conditions**
>
> *** (a) $u(a,\theta) = f(\theta)$, $u(b,\theta) = g(\theta)$**
>
> **(b) $\frac{\partial u}{\partial r}(a,\theta) = 0$, $u(b,\theta) = g(\theta)$**
>
> **(c) $\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$, $\frac{\partial u}{\partial r}(b,\theta) = g(\theta)$**
>
> **If there is a solvability condition, state it and explain it physically.**

The Laplace PDE in polar coordinates is

$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \tag{A}$$

With

$$\frac{\partial u}{\partial r}(a,\theta) = 0$$
$$u(b,\theta) = g(\theta) \tag{B}$$

Assuming the solution can be written as

$$u(r,\theta) = R(r)\Theta(\theta)$$

And substituting this assumed solution back into the (A) gives

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0$$

Dividing the above by $R\Theta$ gives

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$
$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta}$$

Since each side depends on different independent variable and they are equal, they must be equal to same constant. say λ.

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

This results in the following two ODE's. The boundaries conditions in (B) are also transferred to each ODE. This results in

$$\Theta'' + \lambda\Theta = 0 \tag{1}$$
$$\Theta(-\pi) = \Theta(\pi)$$
$$\Theta'(-\pi) = \Theta'(\pi)$$

And

$$r^2R'' + rR' - \lambda R = 0 \tag{2}$$
$$R'(a) = 0$$

Starting with (1) Case $\lambda < 0$ The solution is

$$\Theta(\theta) = A\cosh\left(\sqrt{\lambda}\theta\right) + B\sinh\left(\sqrt{\lambda}\theta\right)$$

First B.C. gives

$$\begin{aligned}
\Theta(-\pi) &= \Theta(\pi) \\
A\cosh\left(-\sqrt{\lambda}\pi\right) + B\sinh\left(-\sqrt{\lambda}\pi\right) &= A\cosh\left(\sqrt{\lambda}\pi\right) + B\sinh\left(\sqrt{\lambda}\pi\right) \\
A\cosh\left(\sqrt{\lambda}\pi\right) - B\sinh\left(\sqrt{\lambda}\pi\right) &= A\cosh\left(\sqrt{\lambda}\pi\right) + B\sinh\left(\sqrt{\lambda}\pi\right) \\
2B\sinh\left(\sqrt{\lambda}\pi\right) &= 0
\end{aligned}$$

But $\sinh\left(\sqrt{\lambda}\pi\right) = 0$ only at zero and $\lambda \neq 0$, hence $B = 0$ and the solution becomes

$$\begin{aligned}
\Theta(\theta) &= A\cosh\left(\sqrt{\lambda}\theta\right) \\
\Theta'(\theta) &= A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\theta\right)
\end{aligned}$$

Applying the second B.C. gives

$$\begin{aligned}
\Theta'(-\pi) &= \Theta'(\pi) \\
A\sqrt{\lambda}\cosh\left(-\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) \\
A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) &= A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) \\
2A\sqrt{\lambda}\cosh\left(\sqrt{\lambda}\pi\right) &= 0
\end{aligned}$$

But $\cosh\left(\sqrt{\lambda}\pi\right) \neq 0$ hence $A = 0$. Therefore trivial solution and $\lambda < 0$ is not an eigenvalue.

Case $\lambda = 0$ The solution is $\Theta = A\theta + B$. Applying the first B.C. gives

$$\begin{aligned}
\Theta(-\pi) &= \Theta(\pi) \\
-A\pi + B &= \pi A + B \\
2\pi A &= 0 \\
A &= 0
\end{aligned}$$

And the solution becomes $\Theta = B_0$. A constant. Hence $\lambda = 0$ is an eigenvalue.

Case $\lambda > 0$

The solution becomes

$$\begin{aligned}
\Theta &= A\cos\left(\sqrt{\lambda}\theta\right) + B\sin\left(\sqrt{\lambda}\theta\right) \\
\Theta' &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\theta\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\theta\right)
\end{aligned}$$

Applying first B.C. gives

$$\begin{aligned}
\Theta(-\pi) &= \Theta(\pi) \\
A\cos\left(-\sqrt{\lambda}\pi\right) + B\sin\left(-\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) \\
A\cos\left(\sqrt{\lambda}\pi\right) - B\sin\left(\sqrt{\lambda}\pi\right) &= A\cos\left(\sqrt{\lambda}\pi\right) + B\sin\left(\sqrt{\lambda}\pi\right) \\
2B\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (3)
\end{aligned}$$

Applying second B.C. gives

$$\begin{aligned}
\Theta'(-\pi) &= \Theta'(\pi) \\
-A\sqrt{\lambda}\sin\left(-\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(-\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) \\
A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) + B\sqrt{\lambda}\cos\left(\sqrt{\lambda}\pi\right) \\
A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) &= -A\sqrt{\lambda}\sin\left(\sqrt{\lambda}\pi\right) \\
2A\sin\left(\sqrt{\lambda}\pi\right) &= 0 \qquad (4)
\end{aligned}$$

Equations (3,4) can be both zero only if $A = B = 0$ which gives trivial solution, or when $\sin\left(\sqrt{\lambda}\pi\right) = 0$.

Therefore taking $\sin\left(\sqrt{\lambda}\pi\right) = 0$ gives a non-trivial solution. Hence

$$\begin{aligned} \sqrt{\lambda}\pi &= n\pi \qquad n = 1, 2, 3, \cdots \\ \lambda_n &= n^2 \qquad n = 1, 2, 3, \cdots \end{aligned}$$

Hence the solution for Θ is

$$\Theta = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\theta) + B_n \sin(n\theta) \tag{5}$$

Now the R equation is solved

The case for $\lambda = 0$ gives

$$\begin{aligned} r^2 R'' + rR' &= 0 \\ R'' + \frac{1}{r} R' &= 0 \qquad r \neq 0 \end{aligned}$$

As was done in last problem, the solution to this is

$$R(r) = A \ln |r| + C$$

Since $r > 0$ no need to keep worrying about $|r|$ and is removed for simplicity. Applying the B.C. gives

$$R' = A \frac{1}{r}$$

Evaluating at $r = a$ gives

$$0 = A \frac{1}{a}$$

Hence $A = 0$, and the solution becomes

$$R(r) = C_0$$

Which is a constant.

Case $\lambda > 0$ The ODE in this case is

$$r^2 R'' + rR' - n^2 R = 0 \qquad n = 1, 2, 3, \cdots$$

Let $R = r^p$, the above becomes

$$\begin{aligned} r^2 p\left(p-1\right) r^{p-2} + rpr^{p-1} - n^2 r^p &= 0 \\ p\left(p-1\right) r^p + pr^p - n^2 r^p &= 0 \\ p\left(p-1\right) + p - n^2 &= 0 \\ p^2 &= n^2 \\ p &= \pm n \end{aligned}$$

Hence the solution is

$$R_n(r) = Cr^n + D\frac{1}{r^n} \qquad n = 1, 2, 3, \cdots$$

Applying the boundary condition $R'(a) = 0$ gives

$$\begin{aligned} R_n'(r) &= nC_n r^{n-1} - nD_n \frac{1}{r^{n+1}} \\ 0 &= R_n'(a) \\ &= nC_n a^{n-1} - nD_n \frac{1}{a^{n+1}} \\ &= nC_n a^{2n} - nD_n \\ &= C_n a^{2n} - D_n \\ D_n &= C_n a^{2n} \end{aligned}$$

The solution becomes

$$\begin{aligned} R_n(r) &= C_n r^n + C_n a^{2n} \frac{1}{r^n} \qquad n = 1, 2, 3, \cdots \\ &= C_n \left(r^n + \frac{a^{2n}}{r^n}\right) \end{aligned}$$

Hence the complete solution for $R(r)$ is

$$R(r) = C_0 + \sum_{n=1}^{\infty} C_n \left(r^n + \frac{a^{2n}}{r^n}\right) \tag{6}$$

Using (5),(6) gives

$$\begin{aligned}
u_n(r,\theta) &= R_n\Theta_n \\
u(r,\theta) &= \left[C_0 + \sum_{n=1}^{\infty} C_n\left(r^n + \frac{a^{2n}}{r^n}\right)\right]\left[A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta)\right] \\
&= D_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta)\,C_n\left(r^n + \frac{a^{2n}}{r^n}\right) + \sum_{n=1}^{\infty} B_n\sin(n\theta)\,C_n\left(r^n + \frac{a^{2n}}{r^n}\right)
\end{aligned}$$

Where $D_0 = C_0A_0$. To simplify more, A_nC_n is combined to A_n and B_nC_n is combined to B_n. The full solution is

$$u(r,\theta) = D_0 + \sum_{n=1}^{\infty} A_n\left(r^n + \frac{a^{2n}}{r^n}\right)\cos(n\theta) + \sum_{n=1}^{\infty} B_n\left(r^n + \frac{a^{2n}}{r^n}\right)\sin(n\theta)$$

The final nonhomogeneous B.C. is applied.

$$\begin{aligned}
u(b,\theta) &= g(\theta) \\
g(\theta) &= D_0 + \sum_{n=1}^{\infty} A_n\left(b^n + \frac{a^{2n}}{b^n}\right)\cos(n\theta) + \sum_{n=1}^{\infty} B_n\left(b^n + \frac{a^{2n}}{b^n}\right)\sin(n\theta)
\end{aligned}$$

For $n = 0$, integrating both sides give

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\,d\theta &= \int_{-\pi}^{\pi} D_0\,d\theta \\
D_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta
\end{aligned}$$

For $n > 0$, multiplying both sides by $\cos(m\theta)$ and integrating gives

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\cos(m\theta)\,d\theta = &\int_{-\pi}^{\pi} D_0\cos(m\theta)\,d\theta \\
&+ \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} A_n\left(b^n + \frac{a^{2n}}{b^n}\right)\cos(m\theta)\cos(n\theta)\,d\theta \\
&+ \int_{-\pi}^{\pi}\sum_{n=1}^{\infty} B_n\left(b^n + \frac{a^{2n}}{b^n}\right)\cos(m\theta)\sin(n\theta)\,d\theta
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\cos(m\theta)\,d\theta = &\int_{-\pi}^{\pi} D_0\cos(m\theta)\,d\theta \\
&+ \sum_{n=1}^{\infty} A_n\left(b^n + \frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta \\
&+ \sum_{n=1}^{\infty} B_n\left(b^n + \frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi}\cos(m\theta)\sin(n\theta)\,d\theta \qquad (7)
\end{aligned}$$

But

$$\begin{aligned}
\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta &= \pi \qquad n = m \neq 0 \\
\int_{-\pi}^{\pi}\cos(m\theta)\cos(n\theta)\,d\theta &= 0 \qquad n \neq m
\end{aligned}$$

And

$$\int_{-\pi}^{\pi}\cos(m\theta)\sin(n\theta)\,d\theta = 0$$

And

$$\int_{-\pi}^{\pi} D_0\cos(m\theta)\,d\theta = 0$$

Then (7) becomes

$$\begin{aligned}
\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta &= \pi A_n\left(b^n + \frac{a^{2n}}{b^n}\right) \\
A_n &= \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta}{b^n + \frac{a^{2n}}{b^n}} \qquad (8)
\end{aligned}$$

Again, multiplying both sides by $\sin(m\theta)$ and integrating gives

$$\int_{-\pi}^{\pi} g(\theta)\sin(m\theta)\,d\theta = \int_{-\pi}^{\pi} D_0 \sin(m\theta)\,d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \left(b^n + \frac{a^{2n}}{b^n}\right)\sin(m\theta)\cos(n\theta)\,d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} B_n \left(b^n + \frac{a^{2n}}{b^n}\right)\sin(m\theta)\sin(n\theta)\,d\theta$$

Hence

$$\int_{-\pi}^{\pi} g(\theta)\sin(m\theta)\,d\theta = \int_{-\pi}^{\pi} D_0 \sin(m\theta)\,d\theta + \sum_{n=1}^{\infty} A_n \left(b^n + \frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi} \sin(m\theta)\cos(n\theta)\,d\theta + \sum_{n=1}^{\infty} B_n \left(b^n + \frac{a^{2n}}{b^n}\right)\int_{-\pi}^{\pi} \sin(m\theta)\sin(n\theta)\,d\theta \tag{9}$$

But

$$\int_{-\pi}^{\pi} \sin(m\theta)\sin(n\theta)\,d\theta = \pi \qquad n = m \neq 0$$
$$\int_{-\pi}^{\pi} \sin(m\theta)\sin(n\theta)\,d\theta = 0 \qquad n \neq m$$

And

$$\int_{-\pi}^{\pi} \sin(m\theta)\cos(n\theta)\,d\theta = 0$$

And

$$\int_{-\pi}^{\pi} D_0 \sin(m\theta)\,d\theta = 0$$

Then (9) becomes

$$\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta = \pi B_n \left(b^n + \frac{a^{2n}}{b^n}\right)$$
$$B_n = \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta}{b^n + \frac{a^{2n}}{b^n}}$$

This complete the solution. Summary

$$u(r,\theta) = D_0 + \sum_{n=1}^{\infty} A_n \left(r^n + \frac{a^{2n}}{r^n}\right)\cos(n\theta) + \sum_{n=1}^{\infty} B_n \left(r^n + \frac{a^{2n}}{r^n}\right)\sin(n\theta)$$
$$D_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta$$
$$A_n = \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\cos(n\theta)\,d\theta}{b^n + \frac{a^{2n}}{b^n}}$$
$$B_n = \frac{1}{\pi}\frac{\int_{-\pi}^{\pi} g(\theta)\sin(n\theta)\,d\theta}{b^n + \frac{a^{2n}}{b^n}}$$

The following are some plots of the solution.

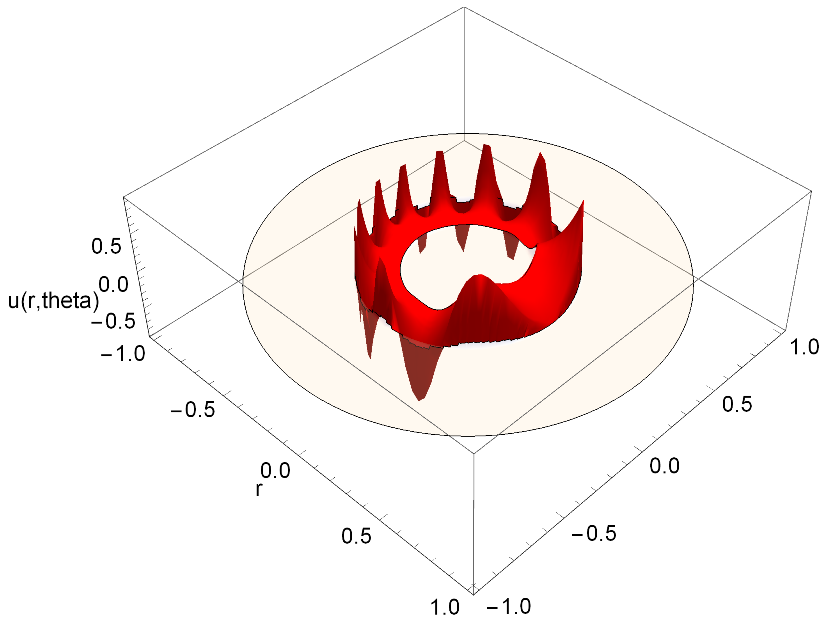


Figure 10: Solution using $f(\theta) = \sin(3\theta^2), a = 0.3, b = 0.5$

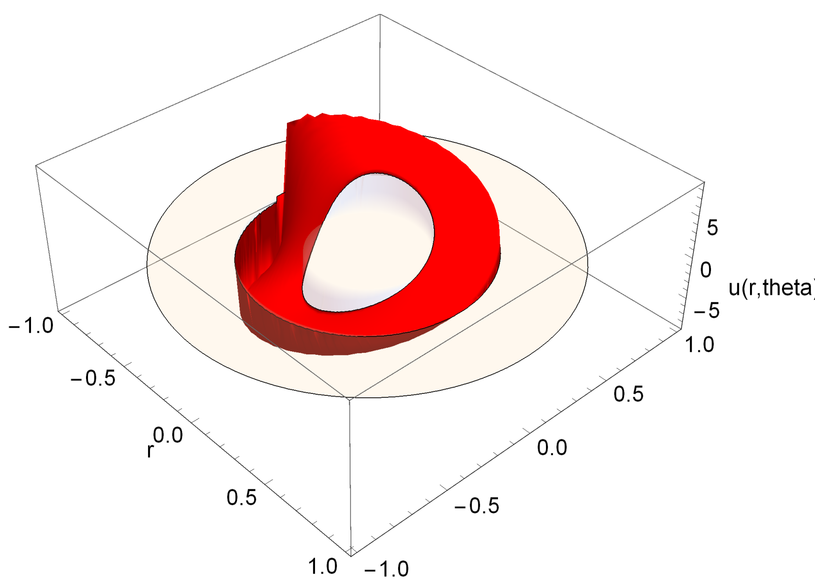


Figure 11: Solution using $f(\theta) = 3\theta, a = 0.3, b = 0.6$

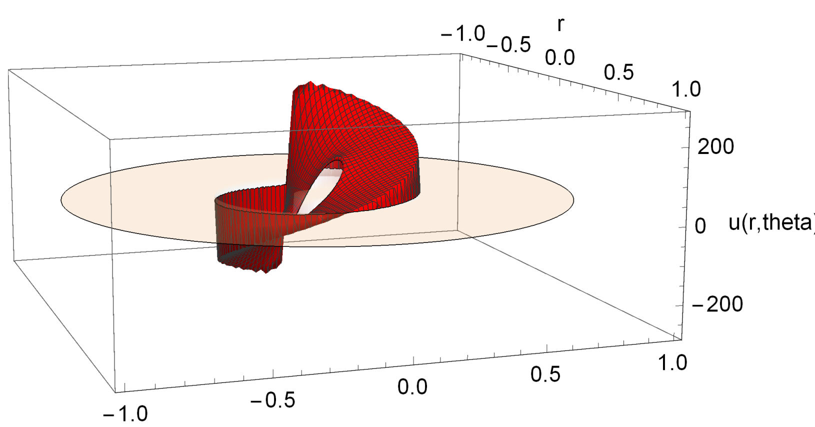


Figure 12: Solution using $f(\theta) = 100\theta, a = 0.1, b = 0.4$

0.5 Problem 2.5.14 (problem 5)

2.5.14. Show that the "backward" heat equation

$$\frac{\partial u}{\partial t} = -k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0,t) = u(L,t) = 0$ and $u(x,0) = f(x)$, is *not* well posed. [*Hint:* Show that if the data are changed an arbitrarily small amount, for example,

$$f(x) \to f(x) + \frac{1}{n}\sin\frac{n\pi x}{L}$$

for large n, then the solution $u(x,t)$ changes by a large amount.]

$$\begin{aligned}\frac{-1}{k}\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}\\ u(0,t) &= 0\\ u(L,t) &= 0\\ u(x,0) &= f(x)\end{aligned}$$

Assume $u(x,t) = XT$. Hence the PDE becomes

$$\begin{aligned}-\frac{1}{k}T'X &= X''T\\ -\frac{1}{k}\frac{T'}{T} &= \frac{X''}{X}\end{aligned}$$

Hence, forλ real

$$-\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The space ODE was solved before. Only positive eigenvalues exist. The solution is

$$\begin{aligned}X(x) &= \sum_{n=1}^{\infty} B_n \sin\left(\sqrt{\lambda_n}x\right)\\ \lambda_n &= \left(\frac{n\pi}{L}\right)^2 \qquad n = 1,2,3,\cdots\end{aligned}$$

The time ODE becomes

$$\begin{aligned}T_n' &= \lambda_n T_n\\ T_n' - \lambda_n T_n &= 0\end{aligned}$$

With solution

$$\begin{aligned}T_n(t) &= A_n e^{\lambda_n t}\\ T(t) &= \sum_{n=1}^{\infty} A_n e^{\lambda_n t}\end{aligned}$$

For the same eigenvalues. Therefore the full solution is

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{\left(\frac{n\pi}{L}\right)^2 t} \tag{1}$$

Where $C_n = A_n B_n$. Applying initial conditions gives

$$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)$$

Multiplying by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating results in

$$\begin{aligned}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx &= \int_0^L \sin\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right)dx\\ &= \sum_{n=1}^{\infty} C_n \int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx\\ &= C_m\frac{L}{2}\end{aligned}$$

Therefore

$$C_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

The solution (1) becomes

$$u(x,t)=\frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right)\left(\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}\right) \tag{2}$$

Assuming initial data is changed to $f(x)+\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)$ then

$$f(x)+\frac{1}{m}\sin\left(\frac{m\pi}{L}x\right)=\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi}{L}x\right)$$

Multiplying both sides by $\sin\left(\frac{m\pi}{L}x\right)$ and integrating

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx+\frac{1}{m}\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx=\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sum_{n=1}^{\infty}C_n\sin\left(\frac{n\pi}{L}x\right)$$

$$\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx+\frac{1}{m}\frac{L}{2}=C_m\frac{L}{2}$$

$$C_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{1}{n}$$

Therefore, the new solution is

$$\begin{aligned}\tilde{u}(x,t)&=\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx+\frac{1}{n}\right)\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}\\&=\sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}+\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}\\&=\sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}+\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}\end{aligned}$$

But $\sum_{n=1}^{\infty}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{m\pi}{L}x\right)dx\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}=u(x,t)$, therefore the above can be written as

$$\tilde{u}(x,t)=u(x,t)+\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{\left(\frac{n\pi}{L}\right)^2 t}$$

For large n, the difference between initial data $f(x)$ and $f(x)+\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)$ is very small, since $\frac{1}{n}\to 0$. However, the effect in the solution above, due to the presence of $e^{\left(\frac{n\pi}{L}\right)^2 t}$ is that $\frac{1}{n}e^{\left(\frac{n\pi}{L}\right)^2 t}$ increases now for large n, since the exponential is to the positive power, and it grows at a faster rate than $\frac{1}{n}$ grows small as n increases, with the net effect that the produce blow up for large n. This is because the power of the exponential is positive and not negative is normally would be the case. Also by looking at the series of $e^{\left(\frac{n\pi}{L}\right)^2 t}$ which is $1+\left(\frac{n\pi}{L}\right)^4\frac{t^2}{2}+\left(\frac{n\pi}{L}\right)^6\frac{t^3}{3!}+\cdots$, then $\frac{1}{n}e^{\left(\frac{n\pi}{L}\right)^2 t}$ expands to $\frac{1}{n}+\frac{1}{n}\left(\frac{n\pi}{L}\right)^4\frac{t^2}{2}+\frac{1}{n}\left(\frac{n\pi}{L}\right)^6\frac{t^3}{3!}+\cdots$ which becomes very large for large n.

In the normal PDE case, the above solution would have instead been the following

$$\tilde{u}(x,t)=u(x,t)+\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{n\pi}{L}\right)^2 t}$$

And now as $n\to\infty$ then $\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{L}x\right)e^{-\left(\frac{n\pi}{L}\right)^2 t}\to 0$ as well. Notice that $\sin\left(\frac{n\pi}{L}x\right)$ term is not important for this analysis, as its value oscillates between −1 and +1.

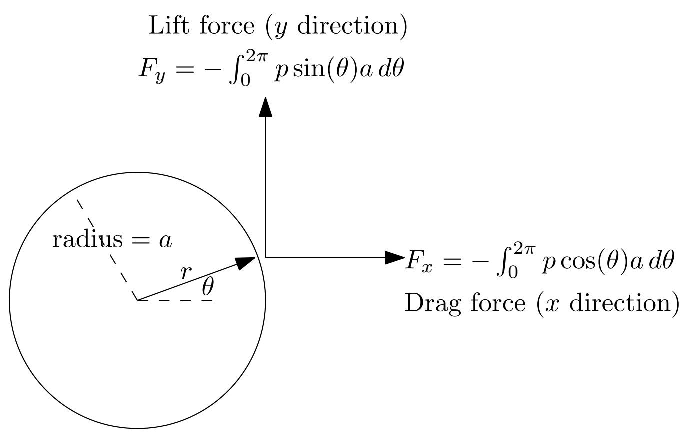
0.6 Problem 2.5.22 (problem 6)

2.5.22. Show the drag force is zero for a uniform flow past a cylinder including circulation.

The force exerted by the fluid on the cylinder is given by equation 2.5.56, page 77 of the text as

$$\bar{F}=-\int_0^{2\pi}p\langle\cos\theta,\sin\theta\rangle\,ad\theta$$

Where a is the cylinder radius, p is the fluid pressure. This vector has 2 components. The x component is the drag force and the y component is the left force as illustrated by this diagram.



Therefore the drag force (per unit length) is

$$F_x = -\int_0^{2\pi} p\cos\theta a d\theta \tag{1}$$

Now the pressure p needs to be determined in order to compute the above. The fluid pressure p is related to fluid flow velocity by the Bernoulli condition

$$p + \frac{1}{2}\rho |\bar{u}|^2 = C \tag{2}$$

Where C is some constant and ρ is fluid density and $\bar{u}$ is the flow velocity vector. Hence in order to find p, the fluid velocity is needed. But the fluid velocity is given by

$$\begin{aligned}\bar{u} &= u_r\hat{r} + u_\theta\hat{\theta}\\ &= \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{r} - \frac{\partial\Psi}{\partial r}\hat{\theta}\end{aligned}$$

Since the radial component of the fluid velocity is zero at the surface if the cylinder (This is one of the boundary conditions used to derive the solution), then only the tangential component comes into play. Hence $|\bar{u}| = \left|-\frac{\partial\Psi}{\partial r}\right|$ but

$$\Psi(r,\theta) = c_1 \ln\left(\frac{r}{a}\right) + u_0\left(r - \frac{a^2}{r}\right)\sin\theta$$

Therefore

$$\frac{\partial\Psi}{\partial r} = \frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

And hence

$$\begin{aligned}|\bar{u}| &= \left|-\frac{\partial\Psi}{\partial r}\right|\\ &= \left|-\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right|\end{aligned}$$

At the surface $r = a$, hence

$$|\bar{u}| = \left|-\frac{c_1}{a} + 2u_0\sin\theta\right|$$

Substituting this into (2) in order to solve for pressure p gives

$$\begin{aligned}p + \frac{1}{2}\rho\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2 &= C\\ p &= C - \frac{1}{2}\rho\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2\end{aligned}$$

Substituting the above into (1) in order to solve for the drag gives

$$F_x = -\int_0^{2\pi}\left[C - \frac{1}{2}\rho\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2\right]\cos\theta a d\theta$$

The above is the quantity that needs to be shown to be zero.

$$F_x = -aC\int_0^{2\pi}\cos\theta d\theta - \frac{a}{2}\rho\int_0^{2\pi}\left(-\frac{c_1}{a} + 2u_0\sin\theta\right)^2\cos\theta d\theta$$

But $\int_0^{2\pi} \cos\theta d\theta = 0$ hence the above simplifies to

$$\begin{aligned}
F_x &= -\frac{a}{2}\rho \int_0^{2\pi} \left(-\frac{c_1}{a} + 2u_0 \sin\theta\right)^2 \cos\theta d\theta \\
&= -\frac{a}{2}\rho \int_0^{2\pi} \frac{c_1^2}{a^2}\cos\theta + 4u_0^2 \sin^2\theta\cos\theta - 4\frac{c_1}{a}u_0 \sin\theta\cos\theta d\theta \\
&= -\frac{a}{2}\rho \left[\frac{c_1^2}{a^2}\int_0^{2\pi} \cos\theta d\theta + 4u_0^2 \int_0^{2\pi} \sin^2\theta\cos\theta d\theta - 4\frac{c_1}{a}u_0 \int_0^{2\pi} \sin\theta\cos\theta d\theta\right]
\end{aligned}$$

But $\int_0^{2\pi} \cos\theta d\theta = 0$ and $\int_0^{2\pi} \sin\theta\cos\theta d\theta = 0$ hence the above reduces to

$$F_x = -4a\rho u_0^2 \int_0^{2\pi} \sin^2\theta\cos\theta d\theta$$

But $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ and the above becomes

$$\begin{aligned}
F_x &= -4a\rho u_0^2 \int_0^{2\pi} \left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\cos\theta d\theta \\
&= -4a\rho u_0^2 \left(\frac{1}{2}\int_0^{2\pi} \cos\theta d\theta - \frac{1}{2}\int_0^{2\pi} \cos(2\theta)\cos\theta d\theta\right)
\end{aligned}$$

But $\int_0^{2\pi} \cos\theta d\theta = 0$ and by orthogonality of cos function $\int_0^{2\pi} \cos(2\theta)\cos(\theta)\, d\theta = 0$ as well. Therefore the above reduces to

$$F_x = 0$$

The drag force (x component of the force exerted by fluid on the cylinder) is zero just outside the surface of the surface of the cylinder. Which is what the question asks to show.

0.7 Problem 2.5.24 (problem 7)

2.5.24. Consider the velocity u_θ at the cylinder. If the circulation is negative, show that the velocity will be larger above the cylinder than below.

Introduction. The stream velocity $\bar{u}$ in Cartesian coordinates is

$$\begin{aligned}
\bar{u} &= u\hat{\imath} + v\hat{\jmath} \\
&= \frac{\partial\Psi}{\partial y}\hat{\imath} - \frac{\partial\Psi}{\partial x}\hat{\jmath} \qquad (1)
\end{aligned}$$

Where Ψ is the stream function which satisfies Laplace PDE in 2D $\nabla^2\Psi = 0$. In Polar coordinates the above becomes

$$\begin{aligned}
\bar{u} &= u_r\hat{r} + u_\theta\hat{\theta} \\
&= \frac{1}{r}\frac{\partial\Psi}{\partial\theta}\hat{r} - \frac{\partial\Psi}{\partial r}\hat{\theta} \qquad (2)
\end{aligned}$$

The solution to $\nabla^2\Psi = 0$ was found under the following conditions

1. When r very large, or in other words, when too far away from the cylinder or the wing, the flow lines are horizontal only. This means at $r = \infty$ the y component of $\bar{u}$ in (1) is zero. This means $\frac{\partial\Psi(x,y)}{\partial x} = 0$. Therefore $\Psi(x,y) = u_0 y$ where u_0 is some constant. In polar coordinates this implies $\Psi(r,\theta) = u_0 r\sin\theta$, since $y = r\sin\theta$.

2. The second condition is that radial component of $\bar{u}$ is zero. In other words, $\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = 0$ when $r = a$, where a is the radius of the cylinder.

3. In addition to the above two main condition, there is a condition that $\Psi = 0$ at $r = 0$

Using the above three conditions, the solution to $\nabla^2\Psi = 0$ was derived in lecture Sept. 30, 2016, to be

$$\Psi(r,\theta) = c_1 \ln\left(\frac{r}{a}\right) + u_0\left(r - \frac{a^2}{r}\right)\sin\theta$$

Using the above solution, the velocity $\bar{u}$ can now be found using the definition in (2) as follows

$$\frac{1}{r}\frac{\partial\Psi}{\partial\theta} = \frac{1}{r}u_0\left(r - \frac{a^2}{r}\right)\cos\theta$$
$$\frac{\partial\Psi}{\partial r} = \frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

Hence, in polar coordinates

$$\boxed{\bar{u} = \left(\frac{1}{r}u_0\left(r - \frac{a^2}{r}\right)\cos\theta\right)\hat{r} - \left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)\hat{\theta}} \tag{3}$$

Now the question posed can be answered. The circulation is given by

$$\Gamma = \int_0^{2\pi} u_\theta r d\theta$$

But from (3) $u_\theta = -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right)$, therefore the above becomes

$$\Gamma = \int_0^{2\pi} -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta\right) r d\theta$$

At $r = a$ the above simplifies to

$$\begin{aligned}\Gamma &= \int_0^{2\pi} -\left(\frac{c_1}{a} + 2u_0\sin\theta\right) a d\theta \\ &= \int_0^{2\pi} -c_1 - 2au_0\sin\theta d\theta \\ &= -\int_0^{2\pi} c_1 d\theta - 2au_0\int_0^{2\pi}\sin\theta d\theta\end{aligned}$$

But $\int_0^{2\pi}\sin\theta d\theta = 0$, hence

$$\begin{aligned}\Gamma &= -c_1\int_0^{2\pi} d\theta \\ &= -2c_1\pi\end{aligned}$$

Since $\Gamma < 0$, then $\underline{c_1 > 0}$. Now that c_1 is known to be positive, then the velocity is calculated at $\theta = \frac{-\pi}{2}$ and then at $\theta = \frac{+\pi}{2}$ to see which is larger. Since this is calculated at $r = a$, then the radial velocity is zero and only u_θ needs to be evaluated in (3).

At $\theta = \frac{-\pi}{2}$

$$\begin{aligned}u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{-\pi}{2}\right)\right) \\ &= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\ &= -\left(\frac{c_1}{r} - u_0\left(1 + \frac{a^2}{r^2}\right)\right)\end{aligned}$$

At $r = a$

$$\begin{aligned}u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} - 2u_0\right) \\ &= -\frac{c_1}{a} + 2u_0\end{aligned} \tag{4}$$

At $\theta = \frac{+\pi}{2}$

$$\begin{aligned}u_{\left(\frac{+\pi}{2}\right)} &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\sin\left(\frac{\pi}{2}\right)\right) \\ &= -\left(\frac{c_1}{r} + u_0\left(1 + \frac{a^2}{r^2}\right)\right)\end{aligned}$$

At $r = a$

$$\begin{aligned}u_{\left(\frac{-\pi}{2}\right)} &= -\left(\frac{c_1}{a} + 2u_0\right) \\ &= -\frac{c_1}{a} - 2u_0\end{aligned} \tag{5}$$

Comparing (4),(5), and since $c_1 > 0$, then the magnitude of u_θ at $\frac{\pi}{2}$ is larger than the magnitude of u_θ at $\frac{-\pi}{2}$. Which implies the stream flows faster above the cylinder than below it.