HW 10, Math 322, Fall 2016

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Contents

1 HW 10

Math 322 (Smith): Problem Set 10

Due Wednesday Dec. 7, 2016

1-4) For the following problems, determine a representation of the solution in terms of a symmetric Green's function. Use appropriate homogeneous boundary conditions for the Green's function. Show that the boundary terms can also be understood using homogeneous solutions of the differential equation.

$$\frac{d^2u}{dx^2} = f(x), \quad 0 < x < 1, \quad u(0) = A, \quad \frac{du}{dx}(1) = B \tag{1}$$

$$\frac{d^2u}{dx^2} + u = f(x), \quad 0 < x < L, \quad u(0) = A, \quad u(L) = B, \quad L \neq n\pi \tag{2}$$

$$\frac{d^2u}{dx^2} = f(x), \quad 0 < x < L, \quad u(0) = A, \quad \frac{du}{dx}(L) + hu(L) = 0 \tag{3}$$

$$\frac{d^2u}{dx^2} + 2\frac{du}{dx} + u = f(x), \quad 0 < x < 1, \quad u(0) = 0, \quad u(1) = 1 \tag{4}$$

1.1 Problem 1

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$$\frac{d^2u}{dx^2} = f(x); 0 < x < L; u(0) = A; \frac{du}{dx}(1) = B$$

Note: I used L for the length instead of one. Will replace L by one at the very end. This makes it more clear. Compare the above to the standard form (Sturm-Liouville)

$$-\frac{d}{dx}\left(p\frac{du}{dx}\right) + qu = f(x)$$

Therefore it becomes

$$-\frac{d}{dx}\left(p\frac{du}{dx}\right) = -f(x)$$

$$p(x) = 1$$

Green function is $G(x, x_0)$ (will use x_0 which is what the book uses, instead of a, as x_0 is more clear). Green function is the solution to

$$\frac{d^2G(x,x_0)}{dx^2} = \delta(x - x_0)$$
$$G(0, x_0) = 0$$
$$\frac{dG(L, x_0)}{dx} = 0$$

Where x_0 is the location of the impulse. Since $\frac{d^2G(x,x_0)}{dx^2} = 0$ for $x \neq x_0$, then the solution to $\frac{d^2G(x,x_0)}{dx^2} = 0$, which is a linear function in this case, is broken into two regions

$$G(x, x_0) = \begin{cases} A_1x + A_2 & 0 < x < x_0 \\ B_1x + B_2 & x_0 < x < L \end{cases}$$

The first solution, using $G(0, x_0) = 0$ gives $A_2 = 0$ and the second solution using $\frac{dG(1,x_0)}{dx} = 0$ gives $B_1 = 0$, hence the above reduces to

$$G(x, x_0) = \begin{cases} A_1x & x < x_0 \\ B_2 & x_0 < x \end{cases} \tag{1}$$

We are left with constants to A_1, B_2 to find. The continuity condition at $x = x_0$ gives

$$A_1x_0 = B_2 \tag{2}$$

The jump discontinuity of the derivative of $G(x,x_0)$ at $x=x_0$, gives the final equation

$$\left(\frac{d}{dx}G(x,x_0)\right)_{x_0<x}-\left(\frac{d}{dx}G(x,x_0)\right)_{x<x_0}=\frac{-1}{p(x_0)}=-1 \tag{2A}$$

Since $p(x)=1$ in this problem. But

$$\frac{dG(x,x_0)}{dx}=\begin{cases}A_1 & x<x_0\\ 0 & x_0<x\end{cases} \tag{3}$$

Hence (2A) becomes

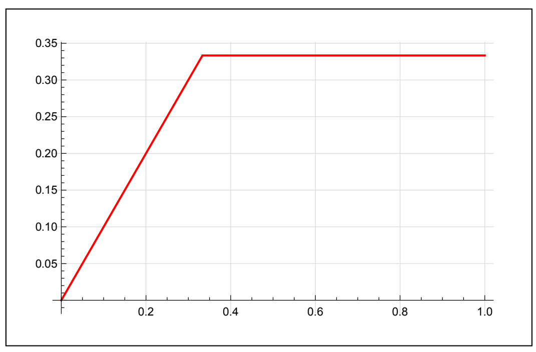
$$0-(A_1)=-1$$

Therefore

$$A_1=1 \tag{4}$$

Solving (2,4) gives $B_2=x_0$. Hence the Green function is, from (1)

$$G(x,x_0)=\begin{cases}x & x<x_0\\ x_0 & x_0<x\end{cases}$$



And $\frac{dG(x,x_0)}{dx_0}$ where now derivative is w.r.t. x_0, is

$$\frac{dG(x,x_0)}{dx_0}=\begin{cases}0 & x<x_0\\ 1 & x_0<x\end{cases}$$

We now have all the information needed to evaluate the solution to the original ODE.

$$u(x)=\overbrace{\int_0^x G(x,x_0)f(x_0)\,dx_0}^{\text{particular solution}}+\overbrace{\left[p(x_0)G(x,x_0)\frac{du(x_0)}{dx_0}-p(x_0)u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}}^{\text{boundary terms}}$$

Since $p(x_0)=1$ then

$$y(x)=\int_0^x G(x,x_0)f(x_0)\,dx_0+\left[G(x,x_0)\frac{du(x_0)}{dx_0}-u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}$$

Let $u_h=\left[G(x,x_0)\frac{du(x_0)}{dx_0}-u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}$, hence

$$u_h=G(x,L)\frac{du(x_0)}{dx_0}(L)-u(L)\frac{dG(x,x_0)}{dx_0}(L)-G(x,0)\frac{du(x_0)}{dx_0}(0)+u(0)\frac{dG(x,x_0)}{dx_0}(0)$$

But

$$\begin{aligned}G(x,L)&=x\\ \frac{du(x_0)}{dx_0}(L)&=B\\ \frac{dG(x,x_0)}{dx_0}(L)&=0\\ G(x,0)&=0\\ \frac{dG(x,x_0)}{dx_0}(0)&=1\\ u(0)&=A\end{aligned}$$

Then

$$u_h=xB+A$$

We see that the boundary terms are linear in x, which is expected as the fundamental solutions for the homogenous solution as linear. The complete solution is

$$
\begin{aligned}
y(x) &= \int_0^L G(x,x_0) f(x_0)\,dx_0 + (xB + A) \\
&= \int_0^L G(x,x_0) f(x_0)\,dx_0 + xB + A \\
&= \int_0^x x_0 f(x_0)\,dx_0 + \int_x^L x f(x_0)\,dx_0 + xB + A
\end{aligned}
$$

For example, if $f(x) = x$, or $f(x_0) = x_0$ then (but remeber, we have to use $-f(x)$ since we are using S-L form)

$$
\begin{aligned}
y(x) &== \int_0^x G(x,x_0)(-f(x_0))dx_0 + \int_x^1 G(x,x_0)(-f(x_0))dx_0 + (xB + A) \\
&= -\int_0^x x_0 x_0 dx_0 - \int_x^1 x x_0 dx_0 + xB + A \\
&= -\left(\frac{x_0^3}{3}\right)_0^x - x\left(\frac{x_0^2}{2}\right)_x^1 + xB + A \\
&= -\left(\frac{x^3}{3}\right) - x\left(\frac{1}{2} - \frac{x^2}{2}\right) + xB + A \\
&= A - \frac{1}{2}x + Bx + \frac{1}{6}x^3
\end{aligned}
$$

To verify the result, this was solved directly, with $f(x) = x$, giving same answer as above.

```
In[39]:= DSolve[{u''[x] == x, u[0] == A0, u'[1] == B0}, u[x], x]
Out[39]= {{u[x] -> 1/6 (6 A0 - 3 x + 6 B0 x + x^3)}}

In[40]:= Expand[%]
Out[40]= {{u[x] -> A0 - x/2 + B0 x + x^3/6}}
```

And if $f(x) = x^2$, or $f(x_0) = x_0^2$, then

$$
\begin{aligned}
y(x) &= \int_0^x x_0\left(-f(x_0)\right)dx_0 + \int_x^L x\left(-f(x_0)\right)dx_0 + xB + A \\
&= -\int_0^x x_0 x_0^2 dx_0 - x\int_x^1 x_0^2 dx_0 + xB + A \\
&= -\left(\frac{x_0^4}{4}\right)_0^x - x\left(\frac{x_0^3}{3}\right)_x^1 + xB + A \\
&= -\left(\frac{x^4}{4}\right) - x\left(\frac{1}{3} - \frac{x^3}{3}\right) + xB + A \\
&= A - \frac{1}{3}x + Bx + \frac{1}{12}x^4
\end{aligned}
$$

To verify the result, this was solved directly, with $f(x) = x^2$, giving same answer as above.

```
In[41]:= DSolve[{u''[x] == x^2, u[0] == A0, u'[1] == B0}, u[x], x]
Out[41]= {{u[x] -> 1/12 (12 A0 - 4 x + 12 B0 x + x^4)}}

In[42]:= Expand[%]
Out[42]= {{u[x] -> A0 - x/3 + B0 x + x^4/12}}
```

This shows the benefit of Green function. Once we know $G(x,x_0)$, then changing the source term, requires only convolution to find the new solution, instead of solving the ODE again as normally done.

1.2 Problem 2

$$
\frac{d^2u}{dx^2} + u = f(x); 0 < x < L; u(0) = A; u(L) = B; L \neq n\pi
$$

Solution

Compare the above to the standard form

$$-\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu=f(x)$$
$$-pu''+qu=f(x)$$

Therefore

$$p(x)=-1$$

Green function is the solution to

$$\frac{d^2G(x,x_0)}{dx^2}+G(x,x_0)=\delta(x-x_0)$$
$$G(0,x_0)=0$$
$$G(L,x_0)=0$$

Where x_0 is the location of the impulse. Since $\frac{d^2G(x,x_0)}{dx^2}=0$ for $x\neq x_0$, then the solution to $\frac{d^2G(x,x_0)}{dx^2}+G(x,x0)=0$, is broken into two regions

$$G(x,x_0)=\begin{cases}A_1\cos x+A_2\sin x & x<x_0\\ B_1\cos x+B_2\sin x & x_0<x\end{cases}$$

The first boundary condition on the left gives $A_1=0$. Second boundary conditions on the right gives

$$B_1\cos L+B_2\sin L=0$$
$$B_1=-B_2\frac{\sin L}{\cos L}$$

Hence the solution now looks like

$$G(x,x_0)=\begin{cases}A_2\sin x & x<x_0\\ -B_2\frac{\sin L}{\cos L}\cos x+B_2\sin x & x_0<x\end{cases}$$

But

$$-B_2\frac{\sin L}{\cos L}\cos x+B_2\sin x=\frac{B_2}{\cos L}(\sin x\cos L-\cos x\sin L)$$

Using trig identity $\sin(a-b)=\sin a\cos b-\cos a\sin b$, the above can be written as $\frac{B_2}{\cos L}\sin(x-L)$, hence the solution becomes

$$G(x,x_0)=\begin{cases}A_2\sin x & x<x_0\\ \frac{B_2}{\cos L}\sin(x-L) & x_0<x\end{cases}\tag{1}$$

Continuity at x_0 gives

$$A_2\sin x_0=\frac{B_2}{\cos L}\sin(x_0-L)\tag{2}$$

And jump discontinuity on derivative of G gives

$$\frac{B_2}{\cos L}\cos(x_0-L)-A_2\cos x_0=-\frac{1}{p(x)}=1$$
$$\frac{B_2}{\cos L}\cos(x_0-L)-A_2\cos x_0=1\tag{3}$$

Now we need to solve (2,3) for A_2,B_2 to obtain the final solution for $G(x,x_0)$. From (2),

$$A_2=\frac{B_2}{\cos L\sin x_0}\sin(x_0-L)\tag{4}$$

Plug into (3)

$$\frac{B_2}{\cos L}\cos(x_0-L)-\frac{B_2}{\cos L\sin x_0}\sin(x_0-L)\cos x_0=1$$
$$\frac{B_2}{\cos L}\cos(x_0-L)-\frac{B_2}{\cos L}\sin(x_0-L)\frac{\cos x_0}{\sin x_0}=1$$
$$B_2\cos(x_0-L)-B_2\sin(x_0-L)\frac{\cos x_0}{\sin x_0}=\cos L$$
$$B_2\left(\cos(x_0-L)-\sin(x_0-L)\frac{\cos x_0}{\sin x_0}\right)=\cos L$$
$$B_2(\sin x_0\cos(x_0-L)-\cos x_0\sin(x_0-L))=\cos L\sin x_0$$

But using trig identity $\sin(a-b)=\sin a\cos b-\cos a\sin b$ we can write above as

$$
\begin{aligned}
B_2\left(\sin\left(x_0-(x_0-L)\right)\right) &= \cos L\sin x_0\\
B_2\sin L &= \cos L\sin x_0\\
B_2 &= \frac{\cos L\sin x_0}{\sin L}
\end{aligned}
$$

Now that we found B_2, we go back and find A_2 from (4)

$$
\begin{aligned}
A_2 &= \frac{\cos L\sin x_0}{\sin L}\frac{1}{\cos L\sin x_0}\sin(x_0-L)\\
&= \frac{\sin(x_0-L)}{\sin L}
\end{aligned}
$$

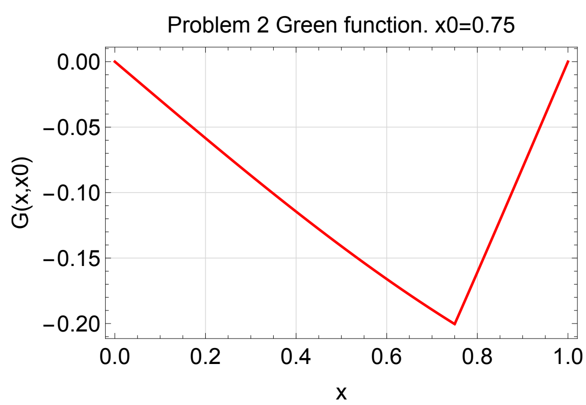
Therefore Green function is, from (1)

$$
G(x,x_0)=\begin{cases}\frac{\sin(x_0-L)}{\sin L}\sin x & x<x_0\\ \frac{\cos L\sin x_0}{\sin L}\frac{1}{\cos L}\sin(x-L) & x_0<x\end{cases}
$$

Or

$$
G(x,x_0)=\begin{cases}\frac{\sin(x_0-L)}{\sin L}\sin x & x<x_0\\ \frac{\sin x_0}{\sin L}\sin(x-L) & x_0<x\end{cases} \tag{5}
$$

It is symmetrical. Here is a plot of $G(x,x_0)$ for some arbitrary x_0 located at $x=0.75$ for $L=1$.



Now comes the hard part. We need to find the solution using

$$
y(x)=\overbrace{\int_0^x G(x,x_0)f(x_0)\,dx_0}^{\text{particular solution}}+\overbrace{\left[p(x_0)G(x,x_0)\frac{du(x_0)}{dx_0}-p(x_0)u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}}^{\text{boundary terms}} \tag{6}
$$

The first step is to find $\frac{dG(x,x_0)}{dx_0}$. From (5), we find

$$
\frac{dG(x,x_0)}{dx_0}=\begin{cases}\frac{\cos(x_0-L)}{\sin L}\sin x & x<x_0\\ \frac{\cos x_0}{\sin L}\sin(x-L) & x_0<x\end{cases} \tag{7}
$$

Now we plug everything in (6). But remember that $G(0,x_0)=0, G(L,x_0)=0, u(0)=A, u(L)=B$. Hence

$$
\begin{aligned}
\Delta &= \left[p(x_0)G(x,x_0)\frac{du(x_0)}{dx_0}-p(x_0)u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}\\
&= G(x,L)\frac{du(L)}{dx_0}-u(L)\frac{dG(x,L)}{dx_0}-G(x,0)\frac{du(0)}{dx_0}+u(0)\frac{dG(x,0)}{dx_0}\\
&= 0-(B)\left(\frac{\cos(x_0-L)}{\sin L}\sin x\right)_{x_0=L}-0+(A)\left(\frac{\cos x_0}{\sin L}\sin(x-L)\right)_{x_0=0}\\
&= -(B)\left(\frac{1}{\sin L}\sin x\right)+(A)\left(\frac{\sin(x-L)}{\sin L}\right)\\
&= -B\frac{\sin x}{\sin L}+A\frac{\sin(x-L)}{\sin L}
\end{aligned}
$$

But $p=-1$, hence the above becomes

$$\begin{aligned}
\Delta &= -G(x,L)\frac{du(L)}{dx_0}+u(L)\frac{dG(x,L)}{dx_0}+G(x,0)\frac{du(0)}{dx_0}-u(0)\frac{dG(x,0)}{dx_0}\\
&= 0+(B)\left(\frac{\cos(x_0-L)}{\sin L}\sin x\right)_{x_0=L}+0-(A)\left(\frac{\cos x_0}{\sin L}\sin(x-L)\right)_{x_0=0}\\
&= +(B)\left(\frac{1}{\sin L}\sin x\right)-(A)\left(\frac{\sin(x-L)}{\sin L}\right)\\
&= B\frac{\sin x}{\sin L}-A\frac{\sin(x-L)}{\sin L}
\end{aligned}$$

We see that the boundary terms are linear combination of sin and cosine in x, which is expected as the fundamental solutions for the homogenous solution as linear combination of sin and cosine in x as was found initially above. Equation (6) becomes

$$y(x)=\int_0^x G(x,x_0)f(x_0)\,dx_0+B\frac{\sin x}{\sin L}-A\frac{\sin(x-L)}{\sin L} \tag{8}$$

Now we can do the integration part. Therefore

$$\int_0^x G(x,x_0)f(x_0)\,dx_0=\int_0^x\left(\frac{\sin(x-L)}{\sin L}\sin x_0\right)f(x_0)\,dx_0+\int_x^L\left(\frac{\sin x}{\sin L}\sin(x_0-L)\right)f(x_0)\,dx_0$$

We can test the solution to see if it correct. Let $f(x)=x$ or $f(x_0)=x_0$, hence

$$\begin{aligned}
\int_0^x G(x,x_0)f(x_0)\,dx_0 &= \int_0^x x_0\left(\frac{\sin(x-L)}{\sin L}\sin x_0\right)dx_0+\int_x^L x_0\left(\frac{\sin x}{\sin L}\sin(x_0-L)\right)dx_0\\
&= \frac{\sin(x-L)}{\sin L}\int_0^x x_0\sin x_0 dx_0+\frac{\sin x}{\sin L}\int_x^L x_0\sin(x_0-L)\,dx_0\\
&= \frac{\sin(x-L)}{\sin L}(-x\cos x+\sin x)+\frac{\sin x}{\sin L}(-L+x\cos(x-L)-\sin(x-L))\\
&= \frac{-x\cos x\sin(x-L)}{\sin L}+\frac{\sin x\sin(x-L)}{\sin L}-L\frac{\sin x}{\sin L}+\frac{x\cos(x-L)\sin x}{\sin L}-\frac{\sin x\sin(x-L)}{\sin L}\\
&= \frac{-x\cos x\sin(x-L)}{\sin L}-L\frac{\sin x}{\sin L}+\frac{x\cos(x-L)\sin x}{\sin L}\\
&= \frac{1}{\sin L}(-L\sin x+x\cos(x-L)\sin x-x\cos x\sin(x-L))
\end{aligned}$$

Hence the solution is

$$\begin{aligned}
u(x) &= \frac{1}{\sin L}(-L\sin x+x\cos(x-L)\sin x-x\cos x\sin(x-L))+\left(B\frac{\sin x}{\sin L}-A\frac{\sin(x-L)}{\sin L}\right)\\
&= \frac{1}{\sin L}(-L\sin x+x\cos(x-L)\sin x-x\cos x\sin(x-L)+B\sin x-A\sin(x-L))
\end{aligned}$$

To verify, the problem is solved directly using CAS, and solution above using Green function was compared, same answer confirmed.

```
In[81]:= f = x;
L0 = 1;
A0 = 1;
B0 = 2;
computerSolution = u[x] /. First@DSolve[{u''[x] + u[x] == f, u[0] == A0, u[L0] == B0}, u[x], x];
mySolUsingGreenFunction = 1 / Sin[L0] (-L0 Sin[x] + x Cos[x - L0] Sin[x] - x Cos[x] Sin[x - L0] + B0 Sin[x] - A0 Sin[x - L0]);
Simplify[computerSolution - mySolUsingGreenFunction]
Out[87]= 0
```

1.3 Problem 3

$$\frac{d^2u}{dx^2}=f(x);0<x<L;u(0)=A;\frac{du}{dx}(L)+hu(L)=0$$

<u>Solution</u>

Compare the above to the standard form

$$\begin{aligned}
-\frac{d}{dx}\left(p\frac{du}{dx}\right)&=f(x)\\
-pu''&=f(x)
\end{aligned}$$

Therefore

$$p(x)=-1$$

Green function is the solution to

$$\frac{d^2 G(x,x_0)}{dx^2} = \delta(x-x_0)$$
$$G(0,x_0) = 0$$
$$\frac{d}{dx}G(L,x_0) + hG(L,x_0) = 0$$

Where x_0 is the location of the impulse. Since $\frac{d^2G(x,x_0)}{dx^2} = 0$ for $x \neq x_0$, then the solution to $\frac{d^2G(x,x_0)}{dx^2} = 0$, is broken into two regions

$$G(x,x_0) = \begin{cases} A_1 x + A_2 & x < x_0 \\ B_1 x + B_2 & x_0 < x \end{cases}$$

The first boundary condition on the left gives $A_2 = 0$. Second boundary conditions on the right gives

$$\begin{aligned} B_1 + h(B_1 L + B_2) &= 0 \\ B_1(1+hL) &= -hB_2 \\ B_1 &= \frac{-hB_2}{1+hL} \end{aligned}$$

Hence the solution now looks like

$$G(x,x_0) = \begin{cases} A_1 x & x < x_0 \\ \left(\frac{-hB_2}{1+hL}\right)x + B_2 & x_0 < x \end{cases}$$

But

$$\begin{aligned} \left(\frac{-hB_2}{1+hL}\right)x + B_2 &= \left(\frac{-hB_2}{1+hL}\right)x + \frac{B_2(1+hL)}{1+hL} \\ &= \frac{B_2(1+hL-hx)}{1+hL} \end{aligned}$$

Hence

$$G(x,x_0) = \begin{cases} A_1 x & x < x_0 \\ \frac{B_2(1+hL-hx)}{1+hL} & x_0 < x \end{cases} \tag{1}$$

Continuity at x_0 gives

$$A_1 x_0 = \frac{B_2(1+hL-hx_0)}{1+hL} \tag{2}$$

And jump discontinuity on derivative of G gives

$$\frac{B_2}{\cos L}\cos(x_0-L) - A_2\cos x_0 = -\frac{1}{p(x)} = 1$$
$$\frac{-hB_2}{1+hL} - A_1 = -\frac{1}{p(x)} = 1 \tag{3}$$

We solve (2,3) for A_1, B_2. From (3)

$$\begin{aligned} \frac{-hB_2}{1+hL} - A_1 &= 1 \\ A_1 &= \frac{-hB_2}{1+hL} - 1 \\ &= \frac{-hB_2}{1+hL} - \frac{(1+hL)}{1+hL} \\ &= \frac{-hB_2 - 1 - hL}{1+hL} \end{aligned}$$

Substituting in (2)

$$\begin{aligned} \frac{-hB_2 - 1 - hL}{1+hL}x_0 &= \frac{B_2(1+hL-hx_0)}{1+hL} \\ (-hB_2 - 1 - hL)x_0 &= B_2(1+hL-hx_0) \\ -hB_2x_0 - x_0 - hLx_0 &= B_2 + hLB_2 - hx_0B_2 \\ B_2(-hx_0 - 1 - hL + hx_0) &= x_0 + hLx_0 \\ B_2 &= \frac{(1+hL)x_0}{-1-hL} \\ &= \frac{-(1+hL)}{1+hL}x_0 \\ &= -x_0 \end{aligned}$$

Hence

$$\begin{aligned} A_1 &= \frac{-hB_2 - 1 - hL}{1 + hL} \\ &= \frac{-h(-x_0) - 1 - hL}{1 + hL} \\ &= \frac{hx_0 - 1 - hL}{1 + hL} \\ &= \frac{hx_0}{1 + hL} - \frac{1 + hL}{1 + hL} \\ &= \frac{hx_0}{1 + hL} - 1 \end{aligned}$$

Therefore (1) becomes

$$G(x, x_0) = \begin{cases} \left(\frac{hx_0}{1+hL} - 1\right) x & x < x_0 \\ \frac{-x_0(1+hL-hx)}{1+hL} & x_0 < x \end{cases} \tag{1}$$

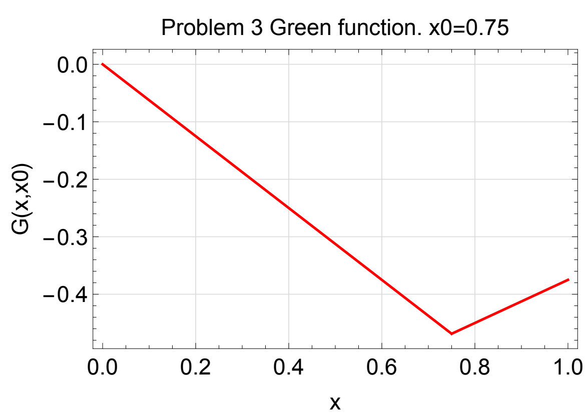
But

$$\begin{aligned} \frac{-x_0(1 + hL - hx)}{1 + hL} &= x_0 \frac{(-1 - hL + hx)}{1 + hL} \\ &= x_0 \left(\frac{hx}{1 + hL} - \frac{1 + hL}{1 + hL}\right) \\ &= x_0 \left(\frac{hx}{1 + hL} - 1\right) \end{aligned}$$

Hence (1) becomes

$$G(x, x_0) = \begin{cases} \left(\frac{hx_0}{1+hL} - 1\right) x & x < x_0 \\ \left(\frac{hx}{1+hL} - 1\right) x_0 & x_0 < x \end{cases} \tag{2}$$

We see they are symmetrical in x, x_0. Here is a plot of $G(x, x_0)$ for some arbitrary x_0 located at $x = 0.75, h = 1$, for $L = 1$.



We need to find the solution using

$$y(x) = \int_0^x G(x, x_0) f(x_0)\, dx_0 + p(x_0) \left[G(x, x_0) \frac{du(x_0)}{dx_0} - u(x_0) \frac{dG(x, x_0)}{dx_0}\right]_{x_0=0}^{x_0=L} \tag{3}$$

The first step is to find $\frac{dG(x,x_0)}{dx_0}$. From (2), we find

$$\frac{dG(x, x_0)}{dx_0} = \begin{cases} \frac{hx}{1+hL} & x < x_0 \\ \frac{hx}{1+hL} - 1 & x_0 < x \end{cases} \tag{4}$$

Now we plug everything in (3). But remember that $G(0, x_0) = 0$, $\frac{dG(L,x_0)}{dx} = -hG(L, x_0)$, $u(0) = A$, $\frac{du(L)}{dx} =$

$-hu(L)$. Hence

$$
\begin{aligned}
\Delta &= \left[G(x,x_0)\frac{du(x_0)}{dx_0} - u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L} \\
&= G(x,L)\frac{du(L)}{dx_0} - u(L)\frac{dG(x,L)}{dx_0} - G(x,0)\frac{du(0)}{dx_0} + u(0)\frac{dG(x,0)}{dx_0} \\
&= G(x,L)(-hu(L)) - u(L)(-hG(x_0,L)) - 0 + (A)\left(\frac{dG(x,0)}{dx_0}\right) \\
&= \overbrace{-G(x,L)hu(L) + u(L)hG(x,L)}^{0} + A\left(\frac{hx}{1+hL}-1\right)_{x_0=0} \\
&= A\left(\frac{hx}{1+hL}-1\right)
\end{aligned}
$$

Now we do the integration, From (3), and since $p=-1$ then we obtain

$$
\begin{aligned}
y(x) &= \int_0^x G(x,x_0)f(x_0)\,dx_0 - A\left(\frac{hx}{1+hL}-1\right) \\
&= \overbrace{\int_0^x G(x,x_0)f(x_0)\,dx_0 + \int_x^L G(x,x_0)f(x_0)\,dx_0}^{\text{particular solution}} - \overbrace{A\left(\frac{hx}{1+hL}-1\right)}^{\text{boundary terms}}
\end{aligned}
$$

We see that the boundary terms are linear combination x, which is expected as the fundamental solutions for the homogenous solution as linear in x as was found initially above. Plugging values From (3) for $G(x,x_0)$ for each region into the above gives

$$
y(x) = \int_0^x \left(\frac{hx}{1+hL}-1\right)x_0 f(x_0)\,dx_0 + \int_x^L \left(\frac{hx_0}{1+hL}-1\right)x f(x_0)\,dx_0 - A\left(\frac{hx}{1+hL}-1\right)
$$

This completes the solution. Now we should test it. Let $f(x)=x$ or $f(x_0)=x_0$ and compare to direction solution. The above becomes

$$
\begin{aligned}
y(x) &= \left(\frac{hx}{1+hL}-1\right)\int_0^x x_0^2 dx_0 + x\int_x^L \left(\frac{hx_0}{1+hL}-1\right)x_0 dx_0 - A\left(\frac{hx}{1+hL}-1\right) \\
&= \left(\frac{hx}{1+hL}-1\right)\left(\frac{x_0^3}{3}\right)_0^x + x\int_x^L \left(\frac{hx_0^2}{1+hL}-x_0\right)dx_0 - A\left(\frac{hx}{1+hL}-1\right) \\
&= \left(\frac{hx}{1+hL}-1\right)\left(\frac{x^3}{3}\right) + x\left(\frac{h}{1+hL}\frac{x_0^3}{3}-\frac{x_0^2}{2}\right)_x^L - A\left(\frac{hx}{1+hL}-1\right) \\
&= \left(\frac{hx}{1+hL}-1\right)\left(\frac{x^3}{3}\right) + x\left(\frac{h}{1+hL}\frac{L^3}{3}-\frac{L^2}{2}-\frac{h}{1+hL}\frac{x^3}{3}+\frac{x^2}{2}\right) - A\left(\frac{hx}{1+hL}-1\right) \\
&= \frac{1}{6(1+Lh)}\left(-hL^3x - 3L^2x + hLx^3 + x^3\right) - A\left(\frac{hx}{1+hL}-1\right) \\
&= \frac{1}{6(1+Lh)}\left(x^3(1+hL) - x\left(hL^3+3L^2\right)\right) + A\left(1-\frac{hx}{1+hL}\right) \\
&= \frac{1}{6(1+Lh)}\left(x^3(1+hL) - xL^2(hL+3)\right) + A\left(\frac{1+h(L-x)}{1+hL}\right)
\end{aligned}
$$

To verify, the problem is solved directly, and solution above using Green function was compared, same answer confirmed.

```
In[152]:= ClearAll[x, L0, y, A0, h]
          f = x;
          computerSolution = u[x] /. First@DSolve[{u''[x] == f, u[0] == A0, u'[L0] + h u[L0] == 0}, u[x], x];
          mySolUsingGreenFunction = 1/(6 (1 + L0 h)) (x^3 (1 + h L0) - x L0^2 (h L0 + 3)) + A0 (1 + h (L0 - x))/(1 + h L0);
          Simplify[computerSolution - mySolUsingGreenFunction]
Out[156]= 0
```

1.4 Problem 4

$$
u'' + 2u' + u = f(x); 0 < x < L; u(0) = 0; u(L) = 1
$$

Solution

Since the coefficient on u' is 2, then the Integrating factor is $\mu(x) = e^{\int 2dx} = e^{2x}$. Multiplying the ODE by $\mu(x)$ gives

$$\begin{aligned} e^{2x}u'' + 2e^{2x}u' + e^{2x}u &= e^{2x}f(x) \\ \frac{d}{dx}\left(e^{2x}u'\right) + e^{2x}u &= e^{2x}f(x) \end{aligned}$$

To keep the solution consistent with the class notes, we now multiply both sides by -1 in order to obtain the same form as used in class notes. Hence our ODE is

$$-\frac{d}{dx}\left(e^{2x}u'\right) - e^{2x}u = -e^{2x}f(x)$$

We now see from above that

$$p(x) = e^{2x}$$

Once we found $p(x)$, we now find the Green function. The Green function is the solution to

$$\begin{aligned} \frac{d^2G(x,x_0)}{dx^2} + 2\frac{dG(x,x_0)}{dx} + G(x,x_0) &= \delta(x-x_0) \\ G(0,x_0) &= 0 \\ G(L,x_0) &= 0 \end{aligned}$$

Where x_0 is the location of the impulse. We first need to find fundamental solutions to the homogeneous ODE. The solution to $u'' + 2u' + u = 0$ is found by characteristic method. $r^2 + 2r + 1 = 0$, hence $(r+1)^2 = 0$. Therefore the roots are $r = -1$, double root. Hence the fundamental solutions are

$$\begin{aligned} u_1 &= e^{-x} \\ u_2 &= xe^{-x} \end{aligned}$$

Therefore

$$\begin{aligned} G(x,x_0) &= \begin{cases} A_1u_1 + A_2u_2 & x < x_0 \\ B_1u_1 + B_2u_2 & x > x_0 \end{cases} \\ &= \begin{cases} A_1e^{-x} + A_2xe^{-x} & x < x_0 \\ B_1e^{-x} + B_2xe^{-x} & x > x_0 \end{cases} \end{aligned}$$

The first boundary condition on the left end gives $A_1 = 0$ from the first region. The second B.C. on the right end, gives

$$\begin{aligned} B_1e^{-L} + B_2Le^{-L} &= 0 \\ B_1 &= -\frac{B_2Le^{-L}}{e^{-L}} = -B_2L \end{aligned}$$

Hence the above solution now reduces to

$$G(x,x_0) = \begin{cases} A_2xe^{-x} & x < x_0 \\ -B_2Le^{-x} + B_2xe^{-x} & x > x_0 \end{cases}$$

Simplifying $-B_2Le^{-x} + B_2xe^{-x} = B_2(x-L)e^{-x}$, the above can be written as

$$G(x,x_0) = \begin{cases} A_2xe^{-x} & x < x_0 \\ B_2(x-L)e^{-x} & x > x_0 \end{cases} \tag{1}$$

Continuity at x_0 gives

$$\begin{aligned} A_2x_0e^{-x_0} &= B_2(x_0-L)e^{-x_0} \\ A_2x_0 &= B_2(x_0-L) \end{aligned} \tag{2}$$

And jump discontinuity on derivative of G gives

$$\frac{d}{dx}G(x,x_0) = \begin{cases} A_2\left(e^{-x} - xe^{-x}\right) & x < x_0 \\ B_2(1-x+L)e^{-x} & x > x_0 \end{cases}$$

Hence (important note: we use $\frac{-1}{p(x_0)}$ below and not $\frac{1}{p(x_0)}$ because we started with $\frac{-d}{dx}\left(p\frac{dy}{dx}\right) + \cdots$ instead of $+\frac{d}{dx}\left(p\frac{dy}{dx}\right) + \cdots$)

$$B_2(1-x_0+L)e^{-x_0} - A_2\left(e^{-x_0} - x_0e^{-x_0}\right) = \frac{-1}{p(x_0)} = \frac{-1}{e^{2x_0}} = -e^{-2x_0}$$

Dividing by e^{-x_0} to simplify gives

$$B_2(1-x_0+L) - A_2(1-x_0) = -e^{-x_0} \tag{3}$$

We solve (2,3) for A_1, B_2. From (3)

$$B_2 = \frac{-e^{-x_0} + A_2(1-x_0)}{1-x_0+L} \tag{4}$$

Substituting in (2)

$$\begin{aligned}
A_2 x_0 &= \frac{-e^{-x_0} + A_2(1-x_0)}{1-x_0+L}(x_0-L) \\
A_2 x_0(1-x_0+L) &= -e^{-x_0}(x_0-L) + A_2(1-x_0)(x_0-L) \\
A_2(x_0(1-x_0+L) - (1-x_0)(x_0-L)) &= -e^{-x_0}(x_0-L) \\
A_2 &= \frac{-e^{-x_0}(x_0-L)}{x_0(1-x_0+L)-(1-x_0)(x_0-L)} \\
&= \frac{1}{L}e^{-x_0}(L-x_0)
\end{aligned}$$

Hence, from (4)

$$\begin{aligned}
B_2 &= \frac{-e^{-x_0} + \left(\frac{1}{L}e^{-x_0}(L-x_0)\right)(1-x_0)}{1-x_0+L} \\
&= -\frac{1}{L}x_0 e^{-x_0}
\end{aligned}$$

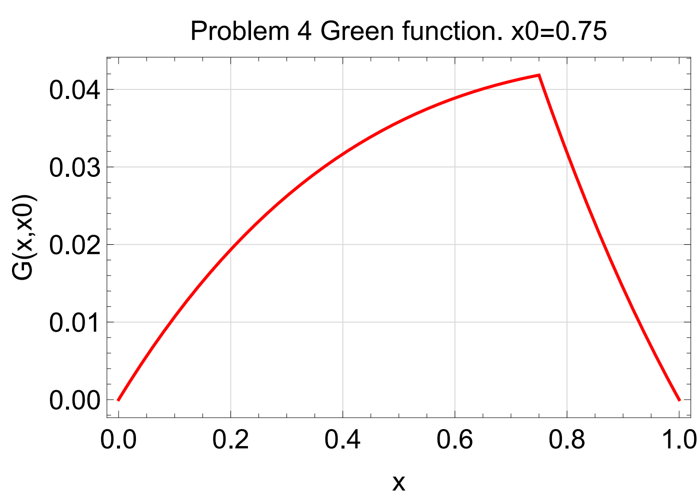
Therefore the solution (1) becomes

$$G(x,x_0) = \begin{cases} \frac{1}{L}e^{-x_0}(L-x_0)\,xe^{-x} & x<x_0 \\ \frac{1}{L}x_0 e^{-x_0}(L-x)\,e^{-x} & x>x_0 \end{cases} \tag{5}$$

Or

$$G(x,x_0) = \begin{cases} \frac{(L-x_0)}{L}xe^{-x_0-x} & x<x_0 \\ \frac{(L-x)}{L}x_0 e^{-x_0-x} & x>x_0 \end{cases} \tag{5}$$

We see they are symmetrical in x, x_0. Here is a plot of $G(x,x_0)$ for some arbitrary x_0 located at $x=0.75$ for $L=1$.



We now need to find the solution using

$$y(x) = \overbrace{\int_0^x G(x,x_0)f(x_0)\,dx_0}^{\text{particular solution}} + \overbrace{\left[p(x_0)G(x,x_0)\frac{du(x_0)}{dx_0} - p(x_0)u(x_0)\frac{dG(x,x_0)}{dx_0}\right]_{x_0=0}^{x_0=L}}^{\text{homogeneous solution/boundary terms}} \tag{6}$$

The first step is to find $\frac{dG(x,x_0)}{dx_0}$. From (5), we find

$$\frac{dG(x,x_0)}{dx_0} = \begin{cases} -\frac{xe^{-x-x_0}}{L} - \frac{xe^{-x-x_0}(L-x_0)}{L} & x<x_0 \\ \frac{e^{-x-x_0}(L-x)}{L} - \frac{(L-x)x_0e^{-x-x_0}}{L} & x>x_0 \end{cases} \tag{7}$$

Now we plug everything in (3). But remember that $G(x,0)=0, G(x,L)=0, u(0)=0, u(L)=1, p(x)=$

e^{2x}. The following is the result of the homogeneous part

$$\Delta = \left[p(x_0) G(x,x_0) \frac{du(x_0)}{dx_0} - p(x_0) u(x_0) \frac{dG(x,x_0)}{dx_0} \right]_{x_0=0}^{x_0=L}$$

$$= \left(e^{2L}\right) G(x,L) \frac{du(L)}{dx_0} - \left(e^{2L}\right) u(L) \frac{dG(x,L)}{dx_0} - \left(e^{2(0)}\right) G(x,0) \frac{du(0)}{dx_0} + \left(e^{2(0)}\right) u(0) \frac{dG(x,0)}{dx_0}$$

$$= 0 - e^{2L}(1) \overbrace{\left(-\frac{xe^{-x-x_0}}{L} - \frac{xe^{-x-x_0}(L-x_0)}{L} \right)_{x_0=L}}^{x<x_0 \text{ branch from (7)}} - 0 + (0)$$

$$= -e^{2L} \left(-\frac{xe^{-x-L}}{L} - \frac{xe^{-x-L}(L-L)}{L} \right)$$

$$= e^{2L} \left(\frac{xe^{-x-L}}{L} \right)$$

$$= \frac{xe^{L-x}}{L}$$

Now we complete the integration, From (3)

$$u(x) = \overbrace{\int_0^x G(x,x_0) f(x_0)\, dx_0 +}^{\text{particular}} \overbrace{\left(\frac{xe^{L-x}}{L} \right)}^{\text{homogeneous}}$$

$$= \int_0^x G(x,x_0) f(x_0)\, dx_0 + \int_x^L G(x,x_0) f(x_0)\, dx_0 + \frac{xe^{L-x}}{L}$$

Plug-in in values From (5) $G(x,x_0)$ for each region,

$$u(x) = \int_0^x \overbrace{\left(\frac{(L-x)}{L} x_0 e^{-x_0-x} \right)}^{\text{from } x>x_0 \text{ branch in (5)}} g(x_0)\, dx_0 + \int_x^L \overbrace{\left(\frac{(L-x_0)}{L} x e^{-x_0-x} \right)}^{\text{from } x<x_0 \text{ branch in (5)}} g(x_0)\, dx_0 + \frac{1}{L} x e^{-x+L}$$

This completes the solution. Now we should test it. Let $f(x) = x$ or $f(x_0) = x_0$. But since we multiplied by $-e^{2x}$ (integrating factor) at start, we should now use $g(x_0) = -e^{2x_0} x_0$ as $f(x_0)$ below.The above becomes

$$u(x) = \int_0^x \left(\frac{(L-x)}{L} x_0 e^{-x_0-x} \right) \left(-e^{2x_0} x_0 \right) dx_0 + \int_x^L \left(\frac{(L-x_0)}{L} x e^{-x_0-x} \right) \left(-e^{2x_0} x_0 \right) dx_0 + \frac{xe^{L-x}}{L}$$

$$= -\frac{(L-x) e^{-x}}{L} \int_0^x x_0^2 e^{x_0} dx_0 - \frac{xe^{-x}}{L} \int_x^L (L-x_0) x_0 e^{x_0} dx_0 + \frac{xe^{L-x}}{L} \tag{8}$$

But

$$\int_0^x x_0^2 e^{x_0} dx_0 = -2 + e^x \left(2 + x^2 - 2x \right)$$

$$\int_x^L e^{x_0} (L-x_0) x_0 dx_0 = e^L (L-2) + e^x \left(2 + L - 2x - Lx + x^2 \right)$$

Hence (8) becomes

$$u(x) = -\frac{(L-x) e^{-x}}{L} \left(-2 + e^x \left(2 + x^2 - 2x \right) \right) - \frac{xe^{-x}}{L} \left(e^L (L-2) + e^x \left(2 + L - 2x - Lx + x^2 \right) \right) + \frac{xe^{L-x}}{L}$$

Which can be simplified to

$$u(x) = \frac{1}{L} e^{-x} \left(\left(3e^L - 2 \right) x + L \left(2 + e^x (x-2) - xe^L \right) \right)$$

For $L = 1$, the above becomes

$$u(x) = x - 2\frac{x}{e^x} + \frac{2}{e^x} + 2x\frac{e}{e^x} - 2$$

<u>Verification</u>

To verify the above, a plot of the solution was compare to Mathematica result. Here is plot of the result. My solution gives exact plot as Mathematica.

```
In[773]:=
        ClearAll[L, x, f]
        f = x;
        computerSolution = u[x] /. First@DSolve[{u''[x] + 2 u'[x] + u[x] == f, u[0] == 0, u[L] == 1}, u[x], x]
```

Out[775]= $\frac{e^{-x}\left(2L - 2e^{x}L - 2x + 3e^{L}x - e^{L}Lx + e^{x}Lx\right)}{L}$

```
In[778]:= mysolution = - (L - x) Exp[-x] / L (-2 + Exp[x] (2 + x^2 - 2 x)) - x Exp[-x] / L (Exp[L] (L - 2) + Exp[x] (2 + L - 2 x - L x + x^2)) + x / L Exp[L - x];
          Simplify[mysolution]
```

Out[779]= $\frac{e^{-x}\left(\left(-2 + 3e^{L}\right)x + L\left(2 + e^{x}(-2 + x) - e^{L}x\right)\right)}{L}$

```
In[780]:= (mysolution - computerSolution) // Simplify
```

Out[780]= 0

```
In[781]:= L = 1;
          p1 = Plot[computerSolution, {x, 0, 1}, PlotRange → All, ImageSize → 300, PlotLabel → "Mathematica answer", PlotStyle → Blue, GridLines → Automatic,
            GridLinesStyle → LightGray];

In[783]:= p2 = Plot[mysolution, {x, 0, 1}, PlotRange → All, ImageSize → 300, PlotLabel → "Manual solution, Green function method", PlotStyle → {Red},
            GridLines → Automatic, GridLinesStyle → LightGray];
```

Out[575]=