HW1, Math 322, Fall 2016

Nasser M. Abbasi

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Contents

Reference table used in HW

$\vec{\phi}$	flux (class uses $\vec{q}$)	vector field. thermal energy per unit time per unit area. $\boxed{\frac{M}{T^3}}$
$\vec{\phi}\cdot\hat{n}$	flux	Flux component that is outward normal to the surface $\boxed{\frac{M}{T^3}}$
Q	heat source	heat energy generated per unit volume per unit time. $\boxed{\frac{M}{LT^3}}$
e		thermal energy density. Scalar field. $\boxed{\frac{M}{LT^2}}$
ρ	density	mass density of material which heat flows in. $\boxed{\frac{M}{L^3}}$
c	specific heat	energy to raise temp. of unit mass by one degree Kelvin. $\boxed{\frac{L^2}{T^2 k^o}}$
k_0	Thermal conductivity	Used in flux equation $q = -k_0 \nabla u$, where u is temperature. $\boxed{\frac{ML}{T^3 k^0}}$
κ	Thermal diffusivity	Used in heat equation $\frac{\partial u}{\partial t} = \kappa \nabla u + \tilde{Q}$. Where $\kappa = \frac{k_0}{\rho c}$, u is temperature.
	conservation of energy	$\frac{d}{dt}\int_V e(x,t)\,dv = \int_A \bar{q}\cdot(-\hat{n})\,dA + \int_V Qdv$. Each term has units $\boxed{\frac{ML^2}{T^3}}$
	Fourier law	$\vec{\phi} = -k_0 \bar{\nabla} u$. Relates flux to temperature gradient.
∇	Divergence operator	A vector operator. $\bar{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$

0.1 Problem 1 (1.5.2)

***1.5.2. For conduction of thermal energy, the heat flux vector is $\boldsymbol{\phi} = -K_0 \nabla u$. If in addition the molecules move at an average velocity $\boldsymbol{V}$, a process called convection, then briefly explain why $\boldsymbol{\phi} = -K_0 \nabla u + c\rho u \boldsymbol{V}$. Derive the corresponding equation for heat flow, including both conduction and convection of thermal energy (assuming constant thermal properties with no sources).**

Fourier law is used to relate the flux to the temperature u by $\phi = -k_0 \frac{\partial u}{\partial x}$ for 1D or $\vec{\phi} = -k_0 \bar{\nabla} u$ in general.

In addition to conduction, there is convection present. This implies there is physical material mass flowing out of the control volume carrying thermal energy with it in addition to the process of conduction. Hence the flux is adjusted by this extra amount of thermal energy motion. The amount of mass that flows out of the surface per unit time per unit area is $(\bar{v}\rho) \equiv \left[\frac{L}{T}\frac{M}{L^3}\right] = \left[\frac{M}{T}\frac{1}{L^2}\right]$. Where $\rho \equiv \left[\frac{M}{L^3}\right]$ is the mass density of the material and $\bar{v} \equiv \left[\frac{L}{T}\right]$ is velocity vector of material flow at the surface.

Amount of thermal energy that $(\bar{v}\rho)$ contains is given by $(\bar{v}\rho)\,cu$ where c is the specific heat and u is the temperature. Therefore $(\bar{v}\rho)\,cu$ is the additional flux due to convection part.

Total flux becomes

$$\vec{\phi} = -k_0 \bar{\nabla} u + \bar{v}\rho c u \tag{1}$$

Starting from first principles. Using conservation of thermal energy given by

$$\frac{\partial e}{\partial t} = -\left(\bar{\nabla} \cdot \vec{\phi}\right)$$

Where e is thermal energy density in the control volume. In this problem $Q = 0$ (no energy source). The integral form of the above is

$$\frac{d}{dt} \int_V e(\bar{x}, t)\, dV = \int_S \vec{\phi} \cdot (-\hat{n})\, dA$$

The dot product with the unit normal vector $\hat{n}$ was added to indicate the normal component of $\vec{\phi}$ at the surface. Since $e(\bar{x}, t) = \rho c u$ and by using divergence theorem the above is written as

$$\frac{d}{dt} \int_V \rho c u dV = \int_V \bar{\nabla} \cdot \left(-\vec{\phi}\right) dV$$

Using (1) in the above and moving the time derivative inside the integral (which becomes partial derivative) results in

$$\int_V \rho c \frac{\partial u}{\partial t} dV = \int_V \bar{\nabla} \cdot \left(k_0 \bar{\nabla} u - \bar{v}\rho c u\right) dV$$

Moving all terms under one integral sign

$$\int_V \left[\rho c \frac{\partial u}{\partial t} - \bar{\nabla} \cdot \left(k_0 \bar{\nabla} u - \bar{v}\rho c u\right)\right] dV = 0$$

Since this is zero for all control volumes, therefore the integrand is zero

$$\rho c \frac{\partial u}{\partial t} - \bar{\nabla} \cdot \left(k_0 \bar{\nabla} u - \bar{v}\rho c u\right) = 0$$

Assuming $\kappa = \frac{k_0}{\rho c}$, the above simplifies to

$$\boxed{\frac{\partial}{\partial t} u = \kappa \nabla^2 u - \bar{\nabla} \cdot (\bar{v} u)} \tag{2}$$

Applying to (2) the property of divergence of the product of scalar and a vector given by

$$\bar{\nabla} \cdot (\bar{v} u) = u\left(\bar{\nabla} \cdot \bar{v}\right) + \bar{v} \cdot \left(\bar{\nabla} u\right)$$

Equation (2) becomes

$$\boxed{\frac{\partial u}{\partial t} = \kappa \nabla^2 u - \left(u\left(\bar{\nabla} \cdot \bar{v}\right) + \bar{v} \cdot \left(\bar{\nabla} u\right)\right)}$$

0.2 Problem 2 (1.5.3)

1.5.3. Consider the polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

(a) Since $r^2 = x^2 + y^2$, show that $\frac{\partial r}{\partial x} = \cos\theta$, $\frac{\partial r}{\partial y} = \sin\theta$, $\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}$, and $\frac{\partial\theta}{\partial x} = \frac{-\sin\theta}{r}$.

(b) Show that $\hat{\boldsymbol{r}} = \cos\theta\hat{\boldsymbol{i}} + \sin\theta\hat{\boldsymbol{j}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\boldsymbol{i}} + \cos\theta\hat{\boldsymbol{j}}$.

(c) Using the chain rule, show that $\nabla = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}$ and hence $\nabla u = \frac{\partial u}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial u}{\partial\theta}\hat{\boldsymbol{\theta}}$.

(d) If $\boldsymbol{A} = A_r\hat{\boldsymbol{r}} + A_\theta\hat{\boldsymbol{\theta}}$, show that $\nabla\cdot\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(A_\theta)$, since $\partial\hat{\boldsymbol{r}}/\partial\theta = \hat{\boldsymbol{\theta}}$ and $\partial\hat{\boldsymbol{\theta}}/\partial\theta = -\hat{\boldsymbol{r}}$ follows from part (b).

(e) Show that $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$.

$$x = r\cos\theta \tag{1}$$
$$y = r\sin\theta \tag{2}$$

0.2.1 part (a)

since $r^2 = x^2 + y^2$ then taking derivative w.r.t. x

$$\begin{aligned}
2r\frac{\partial r}{\partial x} &= 2x \\
\frac{\partial r}{\partial x} &= \frac{x}{r} \\
&= \frac{r\cos\theta}{r} \\
&= \cos\theta
\end{aligned} \tag{3}$$

And taking derivative w.r.t. y

$$\begin{aligned}2r\frac{\partial r}{\partial y} &= 2y\\ \frac{\partial r}{\partial y} &= \frac{y}{r}\\ &= \frac{r\sin\theta}{r}\\ &= \sin\theta \qquad (4)\end{aligned}$$

Now taking derivative w.r.t. y of (2) gives

$$1 = \frac{\partial r}{\partial y}\sin\theta + r\frac{\partial\sin\theta}{\partial y}$$

From (4) $\frac{\partial r}{\partial y} = \sin\theta$ and $\frac{\partial\sin\theta}{\partial y} = \cos\theta\left(\frac{\partial\theta}{\partial y}\right)$. Therefore the above becomes

$$\begin{aligned}1 &= \sin^2\theta + r\cos\theta\left(\frac{\partial\theta}{\partial y}\right)\\ \frac{\partial\theta}{\partial y} &= \frac{1-\sin^2\theta}{r\cos\theta}\\ &= \frac{\cos^2\theta}{r\cos\theta}\end{aligned}$$

Hence

$$\boxed{\frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}}$$

Similarly, taking derivative w.r.t. x of (1) gives

$$1 = \frac{\partial r}{\partial x}\cos\theta + r\frac{\partial\cos\theta}{\partial x}$$

From (3), $\frac{\partial r}{\partial x} = \cos\theta$ and $\frac{\partial\cos\theta}{\partial x} = -\sin\theta\left(\frac{\partial\theta}{\partial x}\right)$, Therefore the above becomes

$$\begin{aligned}1 &= \cos^2\theta - r\sin\theta\left(\frac{\partial\theta}{\partial x}\right)\\ \frac{\partial\theta}{\partial x} &= \frac{1-\cos^2\theta}{r\sin\theta}\\ &= \frac{\sin^2\theta}{r\sin\theta}\end{aligned}$$

Hence

$$\boxed{\frac{\partial\theta}{\partial x} = \frac{\sin\theta}{r}}$$

0.2.2 Part (b)

By definition of unit vector

$$\begin{aligned}\hat{r} &= \frac{\bar{r}}{|r|} = \frac{(|r|\cos\theta)\,\hat{\imath} + (|r|\sin\theta)\,\hat{\jmath}}{|r|} \\ &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\end{aligned}$$

To find $\hat{\theta}$, two relations are used.$\left\|\hat{\theta}\right\| = 1$ by definite of unit vector. Also $\hat{\theta}\cdot\hat{r} = 0$ since these are orthogonal vectors (basis vectors). Assuming that $\hat{\theta} = c_1\hat{\imath} + c_2\hat{\jmath}$, the two equations generated are

$$\left\|\hat{\theta}\right\| = 1 = c_1^2 + c_2^2 \tag{1}$$

$$\hat{\theta}\cdot\hat{r} = 0 = \left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right)\cdot\left(c_1\hat{\imath} + c_2\hat{\jmath}\right) = c_1\cos\theta + c_2\sin\theta \tag{2}$$

From (2), $c_1 = \frac{-c_2\sin\theta}{\cos\theta}$. Substituting this into (1) gives

$$\begin{aligned}1 &= \left(\frac{-c_2\sin\theta}{\cos\theta}\right)^2 + c_2^2 \\ &= \frac{c_2^2\sin^2\theta}{\cos^2\theta} + c_2^2\end{aligned}$$

Solving for c_2 gives

$$\begin{aligned}\cos^2\theta &= c_2^2\left(\sin^2\theta + \cos^2\theta\right) \\ c_2 &= \cos\theta\end{aligned}$$

Since c_2 is now known, c_1 is found from (2)

$$\begin{aligned}0 &= c_1\cos\theta + c_2\sin\theta \\ 0 &= c_1\cos\theta + (\cos\theta)\sin\theta \\ c_1 &= \frac{-(\cos\theta)\sin\theta}{\cos\theta}\end{aligned}$$

Hence $c_1 = -\sin\theta$. Therefore

$$\boxed{\hat{\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}}$$

0.2.3 Part (c)

$$\nabla = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial x}\hat{\jmath} \tag{1}$$

Since $x \equiv x(r,\theta), y \equiv y(r,\theta)$, then

$$\begin{aligned}\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial x} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y}\end{aligned}$$

Equation (1) becomes

$$\nabla = \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}\right)\hat{\imath} + \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}\right)\hat{\jmath}$$

Using result found in (a), the above becomes

$$\nabla = \left(\frac{\partial}{\partial r}\cos\theta + \frac{\partial}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)\right)\hat{\imath} + \left(\frac{\partial}{\partial r}\sin\theta + \frac{\partial}{\partial \theta}\frac{\cos\theta}{r}\right)\hat{\jmath}$$

Collecting on $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ gives

$$\begin{aligned}\nabla &= \frac{\partial}{\partial r}\left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right) + \frac{\partial}{\partial \theta}\left(-\frac{\sin\theta}{r}\hat{\imath} + \frac{\cos\theta}{r}\hat{\jmath}\right)\\ &= \frac{\partial}{\partial r}\left(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\right)\end{aligned}$$

Using result from (b), the above simplifies to

$$\boxed{\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}}$$

Hence

$$\nabla u = \left(\hat{r}\frac{\partial}{\partial r}u + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}u\right)$$

0.2.4 Part (d)

$$\begin{aligned}\bar{A} &= A_r\hat{r} + A_\theta\hat{\theta}\\ \nabla &= \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\end{aligned}$$

Hence

$$\begin{aligned}\nabla\cdot\bar{A} &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right)\cdot\left(A_r\hat{r} + A_\theta\hat{\theta}\right)\\ &= \left(\hat{r}\frac{\partial}{\partial r}\cdot A_r\hat{r}\right) + \left(\hat{r}\frac{\partial}{\partial r}\cdot A_\theta\hat{\theta}\right) + \left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\cdot A_r\hat{r}\right) + \left(\hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\cdot A_\theta\hat{\theta}\right) \qquad (1)\end{aligned}$$

But

$$\begin{aligned}\hat{r}\frac{\partial}{\partial r}\cdot A_r\hat{r} &= \hat{r}\frac{\partial}{\partial r}\left(A_r\hat{r}\right)\\ &= \hat{r}\cdot\left(\frac{\partial A_r}{\partial r}\hat{r} + A_r\frac{\partial \hat{r}}{\partial r}\right)\\ &= \frac{\partial A_r}{\partial r}\left(\hat{r}\cdot\hat{r}\right) + A_r\left(\hat{r}\cdot\frac{\partial \hat{r}}{\partial r}\right)\\ &= \frac{\partial A_r}{\partial r}(1) + A_r(0)\\ &= \frac{\partial A_r}{\partial r} \qquad (2)\end{aligned}$$

And

$$\begin{aligned}\hat{r}\frac{\partial}{\partial r}\cdot A_\theta\hat{\theta} &= \hat{r}\frac{\partial}{\partial r}\left(A_\theta\hat{\theta}\right)\\ &= \hat{r}\cdot\left(\frac{\partial A_\theta}{\partial r}\hat{\theta}+A_\theta\frac{\partial\hat{\theta}}{\partial r}\right)\\ &= \frac{\partial A_\theta}{\partial r}\left(\hat{r}\cdot\hat{\theta}\right)+A_\theta\left(\hat{r}\cdot\frac{\partial\hat{\theta}}{\partial r}\right)\\ &= \frac{\partial A_\theta}{\partial r}(0)+A_\theta(0)=0 \end{aligned}\tag{3}$$

And

$$\begin{aligned}\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot A_r\hat{r} &= \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}(A_r\hat{r})\\ &= \frac{1}{r}\hat{\theta}\cdot\left(\frac{\partial A_r}{\partial\theta}\hat{r}+A_r\frac{\partial\hat{r}}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial A_r}{\partial\theta}\left(\hat{\theta}\cdot\hat{r}\right)+\frac{1}{r}A_r\left(\hat{\theta}\cdot\frac{\partial\hat{r}}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial A_r}{\partial\theta}(0)+\frac{1}{r}A_r\left(\hat{\theta}\cdot\hat{\theta}\right) \end{aligned}$$

Since $\frac{\partial\hat{r}}{\partial\theta}=\hat{\theta}$. Therefore

$$\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot A_r\hat{r}=\frac{1}{r}A_r \tag{4}$$

And finally

$$\begin{aligned}\hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\cdot A_\theta\hat{\theta} &= \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\left(A_\theta\hat{\theta}\right)\\ &= \frac{1}{r}\hat{\theta}\cdot\left(\frac{\partial A_\theta}{\partial\theta}\hat{\theta}+A_\theta\frac{\partial\hat{\theta}}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta}\left(\hat{\theta}\cdot\hat{\theta}\right)+\frac{1}{r}A_\theta\left(\hat{\theta}\cdot\frac{\partial\hat{\theta}}{\partial\theta}\right)\\ &= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta}(1)+\frac{1}{r}A_\theta\left(\hat{\theta}\cdot(-\hat{r})\right)\\ &= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta}(1)+\frac{1}{r}A_\theta(0)\\ &= \frac{1}{r}\frac{\partial A_\theta}{\partial\theta} \end{aligned}\tag{5}$$

Substituting (2,3,4,5) into (1) gives

$$\begin{aligned}\nabla\cdot\bar{A} &= \frac{\partial A_r}{\partial r}+0+\frac{1}{r}A_r+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta}\\ &= \frac{1}{r}A_r+\frac{\partial A_r}{\partial r}+\frac{1}{r}\frac{\partial A_\theta}{\partial\theta} \end{aligned}$$

Add since $\frac{\partial}{\partial r}(rA_r) = A_r + r\frac{\partial A_r}{\partial r}$, the above can also be written as

$$\begin{aligned}\nabla \cdot \bar{A} &= \frac{1}{r}\left(A_r + r\frac{\partial A_r}{\partial r}\right) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta}\end{aligned}$$

0.2.5 Part (e)

From part (c), it was found that

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}$$

But

$$\begin{aligned}\nabla^2 &= \nabla \cdot \nabla \\ &= \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\right)\end{aligned}$$

Using result of part (d), which says that $\nabla \cdot \bar{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta}$, the above becomes (where now $A_r \equiv \frac{\partial}{\partial r}, A_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta}$)

$$\begin{aligned}\nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\end{aligned}$$

Hence

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

0.3 Problem 3 (1.5.4)

1.5.4. Using Exercise 1.5.3(a) and the chain rule for partial derivatives, derive the special case of Exercise 1.5.3(e) if $u(r)$ only.

Let $u \equiv u\,(r)$. From problem 2 part (a) it was found that

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
\frac{\partial r}{\partial x} &= \cos\theta \\
\frac{\partial r}{\partial y} &= \sin\theta \\
\frac{\partial \theta}{\partial y} &= \frac{\cos\theta}{r} \\
\frac{\partial \theta}{\partial x} &= \frac{-\sin\theta}{r}
\end{aligned}$$

And

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{1}$$

But

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cos\theta\right) \\
&= \left(\frac{\partial}{\partial x}\frac{\partial u}{\partial r}\right)\cos\theta + \frac{\partial u}{\partial r}\frac{\partial \cos\theta}{\partial x} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x}\right)\cos\theta + \frac{\partial u}{\partial r}\left(-\sin\theta\frac{\partial \theta}{\partial x}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\cos\theta\right)\cos\theta + \frac{\partial u}{\partial r}\left(-\sin\theta\left(\frac{-\sin\theta}{r}\right)\right) \\
&= \frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{1}{r}\sin^2\theta\frac{\partial u}{\partial r} \qquad (2)
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\sin\theta\right) \\
&= \left(\frac{\partial}{\partial y}\frac{\partial u}{\partial r}\right)\sin\theta + \frac{\partial u}{\partial r}\frac{\partial \sin\theta}{\partial y} \\
&= \left(\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial y}\right)\sin\theta + \frac{\partial u}{\partial r}\left(\cos\theta\frac{\partial \theta}{\partial y}\right) \\
&= \left(\frac{\partial^2 u}{\partial r^2}\sin\theta\right)\sin\theta + \frac{\partial u}{\partial r}\left(\cos\theta\left(\frac{\cos\theta}{r}\right)\right) \\
&= \frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{1}{r}\cos^2\theta\frac{\partial u}{\partial r} \qquad (3)
\end{aligned}$$

Substituting (2),(3) into (1) gives

$$\begin{aligned}
\nabla^2 u &= \left(\frac{\partial^2 u}{\partial r^2}\cos^2\theta + \frac{1}{r}\sin^2\theta\frac{\partial u}{\partial r}\right) + \left(\frac{\partial^2 u}{\partial r^2}\sin^2\theta + \frac{1}{r}\cos^2\theta\frac{\partial u}{\partial r}\right) \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\left(\sin^2\theta\frac{\partial u}{\partial r} + \cos^2\theta\frac{\partial u}{\partial r}\right) \\
&= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}
\end{aligned}$$

Which can be written as

$$\boxed{\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}$$

Which is the special case of problem 2(e) $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ when $u \equiv u(r)$ only.

0.4 Problem 4 (1.5.5)

1.5.5. Assume that the temperature is circularly symmetric: $u = u(r,t)$, where $r^2 = x^2 + y^2$. We will derive the heat equation for this problem. Consider any circular annulus $a \le r \le b$.

(a) Show that the total heat energy is $2\pi \int_a^b c\rho u r \, dr$.

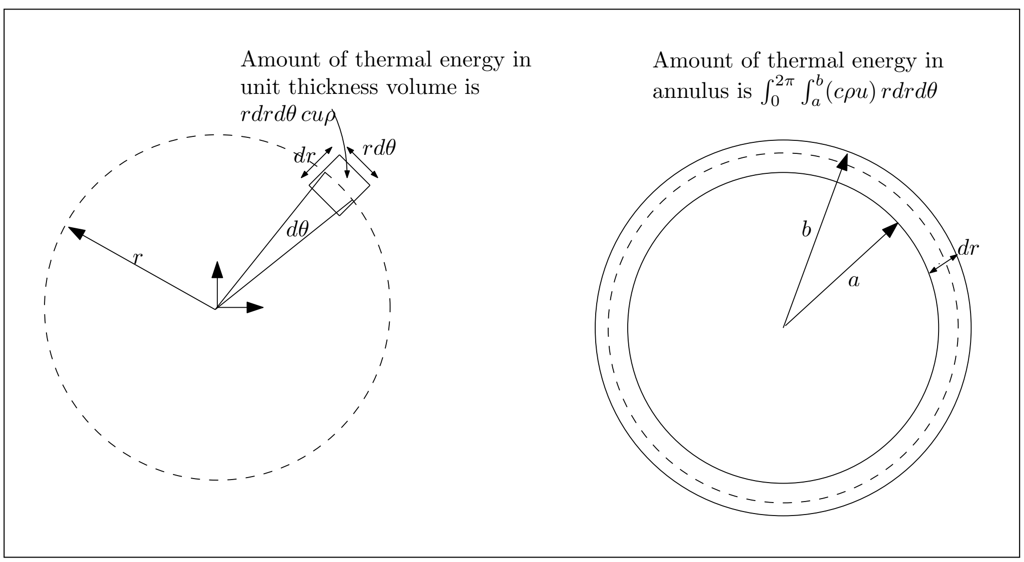
(b) Show that the flow of heat energy per unit time out of the annulus at $r = b$ is $-2\pi b K_0 \partial u/\partial r \,|_{r=b}$. A similar result holds at $r = a$.

(c) Use parts (a) and (b) to derive the circularly symmetric heat equation without sources:

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

0.4.1 Part(a)

Considering the thermal energy in a annulus as shown



Integrating gives total thermal energy

$$\begin{aligned}\int_0^{2\pi}\int_a^b \left(c\rho u\right) rdrd\theta &= \int_0^{2\pi} d\theta \int_a^b \left(c\rho u\right) rdr \\ &= 2\pi \int_a^b \left(c\rho u\right) rdr\end{aligned}$$

0.4.2 Part (b)

Using Fourier law,

$$\begin{aligned}\vec{\phi} &= -k_0 \bar{\nabla} u \\ &= -k_0 \left(\hat{r} \frac{\partial u}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial u}{\partial \theta} \right)\end{aligned}$$

Since symmetric in θ, then $\frac{\partial u}{\partial \theta} = 0$ and the above reduces to

$$\vec{\phi} = -k_0 \hat{r} \frac{\partial u}{\partial r}$$

Hence the heat flow per unit time through surface at $r = b$ is

$$\int_0^{2\pi} \vec{\phi} \cdot (-\hat{n})\, ds$$
$$\int_0^{2\pi} \left(-k_0 \hat{r} \frac{\partial u}{\partial r} \right) \cdot (\hat{n})\, r d\theta$$

But $\hat{n} = \hat{r}$ since radial unit vector. The above becomes

$$\int_0^{2\pi} -k_0 \frac{\partial u}{\partial r} r d\theta = -(2\pi k_0)\, r \frac{\partial u}{\partial r}$$

At $r = b$ the above becomes

$$-(2\pi k_0)\, b \left. \frac{\partial u}{\partial r} \right|_{r=b}$$

Similarly at $r = a$

$$-(2\pi k_0)\, a \left. \frac{\partial u}{\partial r} \right|_{r=a}$$

0.4.3 Part (c)

Applying that the rate of time change of total energy equal to flux through the boundaries gives

$$\begin{aligned}\frac{d}{dt} \left(2\pi \int_a^b (c\rho u)\, r dr \right) &= -(2\pi k_0)\, a \left. \frac{\partial u}{\partial r} \right|_{r=a} + (2\pi k_0)\, b \left. \frac{\partial u}{\partial r} \right|_{r=b} \\ &= 2\pi k_0 \int_a^b \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) dr\end{aligned}$$

Moving $\frac{d}{dt}$ inside the first integral, it become partial

$$2\pi \int_a^b \left(c\rho \frac{\partial u}{\partial t} \right) r dr = 2\pi k_0 \int_a^b \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) dr$$

Moving everything under one integral

$$\int_a^b \left[\left(c\rho \frac{\partial u}{\partial t} \right) r - k_0 \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \right] dr = 0$$

Hence, since this is valid for any annulus, then the integrand is zero

$$\left(c\rho\frac{\partial u}{\partial t}\right)r - k_0\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$

$$\frac{\partial u}{\partial t} = \frac{k_0}{c\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

Hence

$$\boxed{\frac{\partial u}{\partial t} = \frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)}$$

Where $\kappa = \frac{k_0}{c\rho}$.

0.5 Problem 5 (1.5.6)

1.5.6. Modify Exercise 1.5.5 if the thermal properties depend on r.

The earlier problem is now repeated but in this problem $c \equiv c(r), k_0 \equiv k_0(r)$ and $\rho \equiv \rho(r)$. These are the thermal properties in the problem.

0.5.1 Part(a)

$$\int_0^{2\pi}\int_a^b \left(c(r)\rho(r)u\right) r dr d\theta = \int_0^{2\pi} d\theta \int_a^b \left(c(r)\rho(r)u\right) r dr$$

$$= 2\pi\int_a^b \left(c(r)\rho(r)u\right) r dr$$

0.5.2 Part (b)

$$\vec{\phi} = -k_0(r)\hat{r}\frac{\partial u}{\partial r}$$

The heat flow per unit time through surface at r is therefore

$$\int_0^{2\pi}\vec{\phi}\cdot(\hat{n})\,ds = \int_0^{2\pi}\left(-k_0(r)\hat{r}\frac{\partial u}{\partial r}\right)\cdot(\hat{n})\,r d\theta$$

But $\hat{n} = \hat{r}$ since radial therefore

$$\int_0^{2\pi} -k_0(r)\frac{\partial u}{\partial r} r d\theta = -\left(2\pi k_0(r)\right) r\frac{\partial u}{\partial r}$$

At $r = b$ the above becomes

$$-\left(2\pi\, k_0\big|_{r=b}\right) b\left.\frac{\partial u}{\partial r}\right|_{r=b}$$

Similarly at $r = a$

$$-\left(2\pi\, k_0\big|_{r=a}\right) a\, \frac{\partial u}{\partial r}\bigg|_{r=a}$$

0.5.3 Part (c)

Applying that the rate of time change of total energy equal to flux through the boundaries gives

$$\begin{aligned}\frac{d}{dt}\left(2\pi \int_a^b \left(c(r)\,\rho(r)\,u\right) r dr\right) &= -\left(2\pi\, k_0\big|_{r=a}\right) a\, \frac{\partial u}{\partial r}\bigg|_{r=a} + \left(2\pi\, k_0\big|_{r=b}\right) b\, \frac{\partial u}{\partial r}\bigg|_{r=b}\\ &= 2\pi \int_a^b \frac{\partial}{\partial r}\left(k_0(r)\, r \frac{\partial u}{\partial r}\right) dr\end{aligned}$$

Moving $\frac{d}{dt}$ inside the first integral, it become partial

$$2\pi \int_a^b \left(c(r)\,\rho(r)\,\frac{\partial u}{\partial t}\right) r dr = 2\pi \int_a^b \frac{\partial}{\partial r}\left(k_0(r)\, r \frac{\partial u}{\partial r}\right) dr$$

Moving everything under one integral

$$\int_a^b \left[\left(c(r)\,\rho(r)\,\frac{\partial u}{\partial t}\right) r - \frac{\partial}{\partial r}\left(k_0(r)\, r \frac{\partial u}{\partial r}\right)\right] dr = 0$$

Since this is valid for any annulus then the integrand is zero

$$\left(c(r)\,\rho(r)\,\frac{\partial u}{\partial t}\right) r - \frac{\partial}{\partial r}\left(k_0(r)\, r \frac{\partial u}{\partial r}\right) = 0$$

Therefore, the heat equation when the thermal properties depends on r becomes

$$\boxed{\frac{\partial u(r,t)}{\partial t} = \frac{1}{\rho(r)c(r)}\frac{1}{r}\frac{\partial}{\partial r}\left(k_0(r)\, r \frac{\partial u(r,t)}{\partial r}\right)}$$

0.6 Problem 6 (1.5.9)

1.5.9. Determine the equilibrium temperature distribution inside a circular annulus ($r_1 \le r \le r_2$):

*(a) if the outer radius is at temperature T_2 and the inner at T_1

(b) if the outer radius is insulated and the inner radius is at temperature T_1

0.6.1 Part (a)

The heat equation is $\frac{\partial u}{\partial t}=\frac{\kappa}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$. At steady state $\frac{\partial u}{\partial t}=0$. And since circular region, symmetry in θ is assumed and therefore temperature u depends only on r only. This means $u(r_0)$ is the same at any angle θ for that specific r_0. This becomes a second order ODE

$$\begin{aligned}\frac{\kappa}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)&=0\\ \frac{\kappa}{r}\left(\frac{du}{dr}+r\frac{d^2u}{dr^2}\right)&=0\\ \frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}&=0\end{aligned}$$

Since $\frac{\kappa}{r}\neq 0$. Assuming $\frac{du}{dr}=v(r)$, the above becomes

$$\begin{aligned}\frac{dv}{dr}+\frac{1}{r}v&=0\\ \frac{dv}{dr}&=-\frac{1}{r}v\\ \frac{dv}{v}&=-\frac{dr}{r}\end{aligned}$$

Integrating

$$\begin{aligned}\ln v&=-\ln r+c_1\\ v&=e^{-\ln r+c_1}\\ &=c_2e^{-\ln r}\\ &=c_2\frac{1}{r}\end{aligned}$$

Where $c_2=e^{c_1}$. Since $\frac{du}{dr}=v$, then

$$\begin{aligned}\frac{du}{dr}&=c_2\frac{1}{r}\\ du&=c_2\frac{1}{r}dr\end{aligned}$$

Integrating

$$u(r)=c_2\ln r+c_3$$

When $r=r_1,u=T_1$, and when $r=r_2,u=T_2$, therefore

$$\begin{aligned}T_1&=c_2\ln r_1+c_3\\ T_2&=c_2\ln r_2+c_3\end{aligned}$$

From first equation, $c_3=T_1-c_2\ln r_1$. Substituting in second equation gives

$$\begin{aligned}T_2&=c_2\ln r_2+T_1-c_2\ln r_1\\ &=c_2(\ln r_2-\ln r_1)+T_1\end{aligned}$$

Therefore

$$c_2 = \frac{T_2 - T_1}{\ln r_2 - \ln r_1}$$

Hence $c_3 = T_1 - \frac{T_2 - T_1}{\ln r_2 - \ln r_1} \ln r_1$. Therefore the steady state solution becomes

$$\begin{aligned}
u(r) &= c_2 \ln r + c_3 \\
&= \frac{T_2 - T_1}{\ln r_2 - \ln r_1} \ln r + T_1 - \frac{T_2 - T_1}{\ln r_2 - \ln r_1} \ln r_1 \\
&= T_1 + \frac{(T_2 - T_1) \ln r - (T_2 - T_1) \ln r_1}{\ln r_2 - \ln r_1} \\
&= T_1 + \frac{(T_2 - T_1)(\ln r - \ln r_1)}{\ln r_2 - \ln r_1} \\
&= T_1 + (T_2 - T_1) \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}
\end{aligned}$$

Hence

$$\boxed{u(r) = T_1 + (T_2 - T_1) \frac{\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}}$$

0.6.2 Part (b)

Insulated condition implies $\frac{du}{dr} = 0$. So the above is repeated, but this new boundary condition is now used at r_2. Starting from the general solution found in part (a)

$$u(r) = c_2 \ln r + c_3$$

When $r = r_1, u = T_1$ and when $r = r_2, \frac{du}{dr} = 0$. But $\frac{du}{dr} = \frac{c_2}{r}$. Hence $r = r_2$ gives $\frac{c_2}{r_2} = 0$ or $c_2 = 0$. Therefore the solution is

$$u(r) = c_3$$

When $r = r_1$, $u = T_1$, hence $c_3 = T_1$. The solution becomes

$$\boxed{u(r) = T_1}$$

The temperature is T_1 everywhere. This makes sense as this is steady state, and no heat escapes to the outside.

0.7 Problem 7 (1.5.10)

1.5.10. Determine the equilibrium temperature distribution inside a circle ($r \leq r_0$) if the boundary is fixed at temperature T_0.

Last problem found the solution to the heat equation in polar coordinates with symmetry in θ to be

$$u(r) = c_2 \ln r + c_3$$

c_2 must be zero since at $r = 0$ the temperature must be finite. The solution becomes

$$u(r) = c_3$$

Applying the boundary conditions at $r = r_0$

$$T_0 = c_3$$

Therefore,

$$u(r) = T_0$$

The temperature everywhere is the the same as on the edge.

0.8 Problem 8 (1.5.11)

*1.5.11. Consider

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \qquad a < r < b$$

subject to

$$u(r,0) = f(r), \ \frac{\partial u}{\partial r}(a,t) = \beta, \ \text{and} \ \frac{\partial u}{\partial r}(b,t) = 1.$$

Using physical reasoning, for what value(s) of β does an equilibrium temperature distribution exist?

For equilibrium the total rate of heat flow at $r = a$ should be the same as at $r = b$. Circumference at $r = a$ is $2\pi a$ and total rate of flow at $r = a$ is given by β. Hence total heat flow rate at $r = a$ is given by

$$(2\pi a) \left.\frac{\partial u}{\partial r}\right|_{r=a} = 2\pi a\beta$$

Similarly, total heat flow rate at $r=b$ is given by

$$(2\pi b)\left.\frac{\partial u}{\partial r}\right|_{r=b} = 2\pi b$$

Therefore $2\pi a\beta = 2\pi a$ or

$$\beta = \frac{a}{b}$$

0.9 Problem 9 (1.5.12)

1.5.12. Assume that the temperature is spherically symmetric, $u = u(r,t)$, where r is the distance from a fixed point ($r^2 = x^2 + y^2 + z^2$). Consider the heat flow (without sources) between any two concentric spheres of radii a and b.

(a) Show that the total heat energy is $4\pi\int_a^b c\rho u r^2\, dr$.

(b) Show that the flow of heat energy per unit time out of the spherical shell at $r=b$ is $-4\pi b^2 K_0\, \partial u/\partial r\,|_{r=b}$. A similar result holds at $r=a$.

(c) Use parts (a) and (b) to derive the spherically symmetric heat equation

$$\frac{\partial u}{\partial t} = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right).$$

0.9.1 Part (a)

Total heat energy is, by definition

$$E = \int_V c\rho u dv \tag{1}$$

Volume v of sphere of radius r is $v = \frac{4}{3}\pi r^3$. Hence

$$\frac{dv}{dr} = 4\pi r^2$$
$$dv = 4\pi r^2 dr$$

Equation (1) becomes, where now the r limits are from a to b

$$E = \int_a^b c\rho u\left(4\pi r^2 dr\right)$$
$$= 4\pi\int_a^b c\rho u r^2 dr$$

0.9.2 Part (b)

By definition, the flux at $r = b$ is

$$\phi_b = -k_0 \left.\frac{\partial u}{\partial r}\right|_{r=b}$$

The above is per unit area. At $r = b$, the surface area of the sphere is $4\pi b^2$. Therefore, the total energy per unit time is $\phi_b \left(4\pi b^2\right)$ or

$$-4\pi b^2 k_0 \left.\frac{\partial u}{\partial r}\right|_{r=b}$$

Similarly for $r = a$.

0.9.3 Part(c)

By conservation of thermal energy

$$\begin{aligned}
\frac{d}{dt}E &= -4\pi a^2 k_0 \left.\frac{\partial u}{\partial r}\right|_{r=a} + 4\pi b^2 k_0 \left.\frac{\partial u}{\partial r}\right|_{r=b} \\
\frac{d}{dt}\left(4\pi \int_a^b c\rho u r^2 dr\right) &= 4\pi k_0 \int_a^b \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) dr \\
\int_a^b c\rho \frac{\partial u}{\partial t} r^2 dr &= k_0 \int_a^b \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) dr
\end{aligned}$$

Moving everything into one integral

$$\int_a^b \left[c\rho \frac{\partial u}{\partial t} r^2 - k_0 \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)\right] dr = 0$$

Since this is valid for any limits the integrand must be zero

$$\begin{aligned}
c\rho \frac{\partial u}{\partial t} r^2 - k_0 \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) &= 0 \\
\frac{\partial u}{\partial t} &= \frac{k_0}{c\rho} \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)
\end{aligned}$$

Therefore

$$\boxed{\frac{\partial u}{\partial t} = \frac{\kappa}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)}$$

Where $\kappa = \frac{k_0}{c\rho}$

0.10 Problem 10 (1.5.13)

***1.5.13. Determine the *steady-state* temperature distribution between two concentric *spheres* with radii 1 and 4, respectively, if the temperature of the outer sphere is maintained at 80° and the inner sphere at 0° (see Exercise 1.5.12).**

The heat equation is $\frac{\partial u}{\partial t} = \frac{\kappa}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$. For steady state $\frac{\partial u}{\partial t} = 0$ and assuming symmetry in θ, the heat equation becomes an ODE in r

$$\begin{aligned}\frac{\kappa}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) &= 0\\ \frac{d}{dr}\left(r^2\frac{du}{dr}\right) &= 0\\ 2r\frac{du}{dr} + r^2\frac{d^2u}{dr^2} &= 0\end{aligned}$$

For $r \neq 0$

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0$$

Let $\frac{du}{dr} = v(r)$, hence

$$\begin{aligned}r\frac{dv}{dr} + 2v = 0\\ \frac{dv}{dr} &= -\frac{2v}{r}\\ \frac{dv}{v} &= -2\frac{dr}{r}\end{aligned}$$

Integrating

$$\begin{aligned}\ln v &= -2\ln r + c\\ v &= e^{-2\ln r + c}\\ &= c_1 e^{-2\ln r}\\ &= c_1\frac{1}{r^2}\end{aligned}$$

Therefore, since $\frac{du}{dr} = v(r)$ then

$$\begin{aligned}\frac{du}{dr} &= c_1\frac{1}{r^2}\\ du &= c_1\frac{dr}{r^2}\end{aligned}$$

Integrating

$$\boxed{u(r) = \frac{-c_1}{r} + c_2}$$

When $r = 1$, $u = 0$ and when $r = 4, u = 80$, hence

$$0 = -c_1 + c_2$$
$$80 = \frac{-c_1}{4} + c_2$$

From first equation, $c_1 = c_2$, and from second equation $80 = \frac{-c_1}{4} + c_1$, hence $\frac{3}{4}c_1 = 80$ or $c_1 = \frac{(4)(80)}{3} = \frac{320}{3}$. Therefore, the general solution becomes

$$u(r) = -\frac{320}{3}\frac{1}{r} + \frac{320}{3}$$

or

$$\boxed{u(r) = \frac{320}{3}\left(1 - \frac{1}{r}\right)}$$