HW 9, Math 319, Fall 2016

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Contents

1 HW 9

1.1 Section 6.6 problem 1

Question: Establish

1. $f \circledast g = g \circledast f$
2. $f \circledast \left(g_1 + g_2\right) = f \circledast g_1 + f \circledast g_2$
3. $f \circledast \left(g \circledast h\right) = \left(f \circledast g\right) \circledast h$

1.1.1 Part (a)

From definition

$$f(t) \circledast g(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) \, d\tau$$

Let $u = t - \tau$, hence $\frac{du}{d\tau} = -1$. When $\tau = -\infty \rightarrow u = +\infty$ and when $\tau = +\infty \rightarrow u = -\infty$, hence the above becomes

$$f(t) \circledast g(t) = \int_{+\infty}^{-\infty} f(u) g(t-u) (-du)$$

Pulling the minus sign outside and changing the integration limits

$$f(t) \circledast g(t) = \int_{-\infty}^{\infty} g(t-u) f(u) \, du$$

But since u is arbitrary, we can relabel u as τ in the above. Hence the above RHS can be written as

$$f(t) \circledast g(t) = \int_{-\infty}^{\infty} g(t-\tau) f(\tau) \, d\tau$$

But $\int_{-\infty}^{\infty} g(t-\tau) f(\tau) \, d\tau = g(t) \circledast f(t)$, hence

$$f(t) \circledast g(t) = g(t) \circledast f(t)$$

QED.

1.1.2 Part (b)

From definition

$$f(t) \circledast \left(g_1(t) + g_2(t)\right) = \int_{-\infty}^{\infty} f(t-\tau) \left(g_1(\tau) + g_1(\tau)\right) d\tau$$

By linearity of the integral operation, we can break the integral above

$$\int_{-\infty}^{\infty} f(t-\tau) \left(g_1(\tau) + g_1(\tau)\right) d\tau = \int_{-\infty}^{\infty} f(t-\tau) g_1(\tau) \, d\tau + \int_{-\infty}^{\infty} f(t-\tau) g_1(\tau) \, d\tau$$

But $\int_{-\infty}^{\infty} f(t-\tau) g_1(\tau) \, d\tau = f(t) \circledast g_1(t)$ and $\int_{-\infty}^{\infty} f(t-\tau) g_1(\tau) \, d\tau = f(t) \circledast g_2(t)$, hence the above becomes

$$\int_{-\infty}^{\infty} f(t-\tau) \left(g_1(\tau) + g_1(\tau)\right) d\tau = \left(f(t) \circledast g_1(t)\right) + \left(f(t) \circledast g_2(t)\right)$$

Therefore

$$f(t) \circledast \left(g_1(t) + g_2(t)\right) = \left(f(t) \circledast g_1(t)\right) + \left(f(t) \circledast g_2(t)\right)$$

QED.

1.1.3 Part (c)

From definition

$$\begin{aligned}
\left(\left(f \circledast g\right) \circledast h\right)(t) &= \int_{\Re} \left(f \circledast g\right)(\tau) h(t-\tau) \, d\tau \\
&= \int_{\Re} \left[\int_{\Re} f(\tau_1) g(\tau - \tau_1) \, d\tau_1\right] h(t-\tau) \, d\tau \\
&= \int_{\Re} \int_{\Re} f(\tau_1) g(\tau - \tau_1) h(t-\tau) \, d\tau_1 d\tau
\end{aligned}$$

By Fubini, we can change order of integration

$$\begin{aligned}
\left(\left(f \circledast g\right) \circledast h\right)(t) &= \int_{\Re} \int_{\Re} f(\tau_1) g(\tau - \tau_1) h(t-\tau) \, d\tau d\tau_1 \\
&= \int_{\Re} f(\tau_1) \left[\int_{\Re} g(\tau - \tau_1) h(t-\tau) \, d\tau\right] d\tau_1
\end{aligned}$$

By translation, if we add τ_1 to τ for both functions in the inner integral above, we obtain

$$\begin{aligned}\left(\left(f\circledast g\right)\circledast h\right)(t) &= \int_{\Re} f(\tau_1)\left[\int_{\Re} g((\tau+\tau_1)-\tau_1)\,h(t-(\tau+\tau_1))\,d\tau\right]d\tau_1\\ &= \int_{\Re} f(\tau_1)\left[\int_{\Re} g(\tau)\,h((t-\tau_1)-\tau)\,d\tau\right]d\tau_1\end{aligned}$$

But now we see that inner integral is $\int_{\Re} g(\tau)\,h((t-\tau_1)-\tau)\,d\tau = \left(g\circledast h\right)(t-\tau_1)$, hence the above becomes

$$\begin{aligned}\left(\left(f\circledast g\right)\circledast h\right)(t) &= \int_{\Re} f(\tau_1)\left(g\circledast h\right)(t-\tau_1)\,d\tau_1\\ &= \left(f\circledast\left(g\circledast h\right)\right)(t)\end{aligned}$$

QED

1.2 Section 6.6 problem 2

Find an example showing $\left(f\circledast 1\right)(t)$ need not be equal to $f(t)$

<u>Solution</u> Let $f(t)=e^t$, hence

$$\begin{aligned}\left(f\circledast 1\right)(t) &= \int_0^t f(t-\tau)\times 1 d\tau\\ &= \int_0^t e^{(t-\tau)}d\tau\\ &= \left[\frac{e^{(t-\tau)}}{-1}\right]_{\tau=0}^{\tau=t}\\ &= -\left[e^{(t-t)}-e^{(t-0)}\right]\\ &= -\left[e^0-e^t\right]\\ &= -\left(1-e^t\right)\\ &= e^t-1\end{aligned}$$

Which is not the same as e^t. QED

1.3 Section 6.6 problem 3

Show that $\left(f\circledast f\right)(t)$ is not necessarily non-negative, using $f(t)=\sin(t)$

<u>Solution</u> From definition

$$\left(f\circledast f\right)(t) = \int_0^t \sin(\tau)\sin(t-\tau)\,d\tau$$

Using $\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))$ on the integrand gives

$$\begin{aligned}\left(f\circledast f\right)(t) &= \int_0^t \frac{1}{2}(\cos(\tau-(t-\tau))-\cos(\tau+(t-\tau)))\,d\tau\\ &= \frac{1}{2}\int_0^t \cos(\tau-(t-\tau))\,d\tau-\frac{1}{2}\int_0^t\cos(t)\,d\tau\\ &= \frac{1}{2}\int_0^t \cos(2\tau-t)\,d\tau-\frac{1}{2}\int_0^t\cos(t)\,d\tau\end{aligned}$$

For the second integral above, since it is w.r.t τ, then we can pull $\cos(t)$ outside, which gives

$$\begin{aligned}\left(f\circledast f\right)(t) &= \frac{1}{2}\left(\frac{\sin(2\tau-t)}{2}\right)_{\tau=0}^{\tau=t}-\frac{1}{2}\cos(t)\int_0^t d\tau\\ &= \frac{1}{4}(\sin(2t-t)-\sin(-t))-\frac{1}{2}t\cos t\\ &= \frac{1}{4}(\sin(t)+\sin(t))-\frac{1}{2}t\cos t\\ &= \frac{1}{2}\sin t-\frac{1}{2}t\cos t\end{aligned}$$

Let $t=2\pi$ then

$$\begin{aligned}\left(f\circledast f\right)(t) &= 0-\frac{1}{2}(2\pi)\\ &= -\pi\end{aligned}$$

Which is negative. Hence we showed that $\left(f\circledast f\right)(t)$ can be negative at some t. QED.

1.4 Section 6.6 problem 4

Find Laplace transform of $f(t) = \int_0^t (t-\tau)^2 \cos(2\tau)\, d\tau$

Solution We see that

$$f(t) = t^2 \circledast \cos(2t)$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{t^2\right\}\mathscr{L}\{\cos(2t)\}$$

But $\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$ and $\mathscr{L}\{\cos(2t)\} = \frac{s}{s^2+4}$, hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{f(t)\right\} &= \left(\frac{2}{s^3}\right)\left(\frac{s}{s^2+4}\right)\\ &= \frac{2}{s^2}\frac{1}{s^2+4}\end{aligned}$$

1.5 Section 6.6 problem 5

Find Laplace transform of $f(t) = \int_0^t e^{-(t-\tau)} \sin(\tau)\, d\tau$

Solution We see that

$$f(t) = e^{-t} \circledast \sin(t)$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{e^{-t}\right\}\mathscr{L}\{\sin(t)\}$$

But $\mathscr{L}\left\{e^{-t}\right\} = \frac{1}{s+1}$ and $\mathscr{L}\{\sin(t)\} = \frac{1}{s^2+1}$, hence the above becomes

$$\mathscr{L}\left\{f(t)\right\} = \frac{1}{(s+1)\left(s^2+1\right)}$$

1.6 Section 6.6 problem 6

Find Laplace transform of $f(t) = \int_0^t (t-\tau)\, e^{\tau} d\tau$

Solution We see that

$$f(t) = t \circledast e^t$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\{t\}\,\mathscr{L}\left\{e^t\right\}$$

But $\mathscr{L}\{t\} = \frac{1}{s^2}$ and $\mathscr{L}\left\{e^t\right\} = \frac{1}{s-1}$, hence the above becomes

$$\mathscr{L}\left\{f(t)\right\} = \left(\frac{1}{s^2}\right)\left(\frac{1}{s-1}\right)$$

1.7 Section 6.6 problem 7

Find Laplace transform of $f(t) = \int_0^t \sin(t-\tau)\cos\tau d\tau$

Solution We see that

$$f(t) = \sin(t) \circledast \cos(t)$$

Therefore, using convolution theorem

$$\mathscr{L}\left\{f(t)\right\} = \mathscr{L}\{\sin t\}\,\mathscr{L}\{\cos t\}$$

But $\mathscr{L}\{\sin t\} = \frac{1}{s^2+1}$ and $\mathscr{L}\{\cos t\} = \frac{s}{s^2+1}$, hence the above becomes

$$\mathscr{L}\left\{f(t)\right\} = \left(\frac{1}{s^2+1}\right)\left(\frac{s}{s^2+1}\right)$$

1.8 Section 6.6 problem 8

Find the inverse Laplace transform of $F(s) = \frac{1}{s^4\left(s^2+1\right)}$ using convolution theorem.

Solution We see that

$$\begin{aligned} F(s) &= \frac{1}{s^4}\frac{1}{s^2+1} \\ &= \mathscr{L}\left(\frac{t^3}{6}\right)\mathscr{L}(\sin t) \end{aligned}$$

Hence, using convolution theorem

$$\begin{aligned} f(t) &= \frac{t^3}{6} \circledast \sin t \\ &= \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, d\tau \end{aligned}$$

Integrate by parts. $\int udv = uv - \int vdu$. Let $u = (t-\tau)^3, dv = \sin\tau \rightarrow du = -3(t-\tau)^2, v = -\cos\tau$, hence

$$\begin{aligned} \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, d\tau &= \frac{1}{6}\left(-\left[(t-\tau)^3\cos\tau\right]_0^t - \int_0^t -3(t-\tau)^2(-\cos\tau) \, d\tau\right) \\ &= \frac{1}{6}\left(-\left[(t-t)^3\cos t - (t-0)^3\cos 0\right] - 3\int_0^t (t-\tau)^2(\cos\tau) \, d\tau\right) \\ &= \frac{1}{6}\left(-\left[0-t^3\right] - 3\int_0^t (t-\tau)^2(\cos\tau) \, d\tau\right) \\ &= \frac{1}{6}\left(t^3 - 3\int_0^t (t-\tau)^2(\cos\tau) \, d\tau\right) \end{aligned}$$

Integrate by parts. Let $u = (t-\tau)^2, dv = \cos\tau \rightarrow du = -2(t-\tau), v = \sin\tau$, hence

$$\begin{aligned} \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, d\tau &= \frac{1}{6}\left(t^3 - 3\left[\left((t-\tau)^2\sin\tau\right)_0^t - \int_0^t -2(t-\tau)\sin\tau d\tau\right]\right) \\ &= \frac{1}{6}\left(t^3 - 3\left[\left((t-t)^2\sin t - (t-0)^2\sin 0\right)_0^t + 2\int_0^t (t-\tau)\sin\tau d\tau\right]\right) \\ &= \frac{1}{6}\left(t^3 - 3\left[0 + 2\int_0^t (t-\tau)\sin\tau d\tau\right]\right) \\ &= \frac{1}{6}\left(t^3 - 6\int_0^t (t-\tau)\sin\tau d\tau\right) \end{aligned}$$

Integrate by parts. Let $u = (t-\tau), dv = \sin\tau \rightarrow du = -1, v = -\cos\tau$, hence above becomes

$$\begin{aligned} \frac{1}{6}\int_0^t (t-\tau)^3 \sin\tau \, d\tau &= \frac{1}{6}\left(t^3 - 6\left[\left(-(t-\tau)\cos\tau\right)_0^t - \int_0^t \cos\tau d\tau\right]\right) \\ &= \frac{1}{6}\left(t^3 - 6\left[-((t-t)\cos t - (t-0)\cos 0) - (\sin\tau)_0^t\right]\right) \\ &= \frac{1}{6}\left(t^3 - 6\left[-(0-t) - \sin t\right]\right) \\ &= \frac{1}{6}\left(t^3 - 6(t - \sin t)\right) \\ &= \frac{1}{6}\left(t^3 - 6t + 6\sin t\right) \end{aligned}$$

Hence

$$f(t) = \frac{1}{6}\left(t^3 - 6t + 6\sin t\right)$$

1.9 Section 6.6 problem 9

Find the inverse Laplace transform of $F(s) = \frac{s}{(s+1)(s^2+4)}$ using convolution theorem.

Solution We see that

$$\begin{aligned} F(s) &= \frac{1}{s+1}\frac{s}{s^2+4} \\ &= \mathscr{L}\left(e^{-t}\right)\mathscr{L}(\cos 2t) \end{aligned}$$

Hence, using convolution theorem

$$\begin{aligned} f(t) &= e^{-t} \circledast \cos 2t \\ &= \int_0^t e^{-(t-\tau)} \cos 2\tau \, d\tau \end{aligned}$$

Integrate by parts. $\int udv = uv - \int vdu$. Let $u = \cos 2\tau, dv = e^{-(t-\tau)} \to du = -2\sin 2\tau, v = e^{-(t-\tau)}$, hence

$$\begin{aligned}\int_0^t e^{-(t-\tau)} \cos 2\tau \, d\tau &= \left(\cos 2\tau e^{-(t-\tau)}\right)_0^t - \int_0^t e^{-(t-\tau)} (-2\sin 2\tau)\, d\tau \\ &= \left(\cos 2t e^{-(t-t)} - \cos 0 e^{-(t-0)}\right) + 2\int_0^t e^{-(t-\tau)} \sin 2\tau d\tau \\ &= \left(\cos 2t - e^{-t}\right) + 2\int_0^t e^{-(t-\tau)} \sin 2\tau d\tau\end{aligned}$$

Integrate by parts. Let $u = \sin 2\tau, dv = e^{-(t-\tau)} \to du = 2\cos 2\tau, v = e^{-(t-\tau)}$, hence

$$\begin{aligned}\int_0^t e^{-(t-\tau)} \cos 2\tau \, d\tau &= \left(\cos 2t - e^{-t}\right) + 2\left[\left(\sin 2\tau e^{-(t-\tau)}\right)_0^t - \int_0^t e^{-(t-\tau)} 2\cos 2\tau d\tau\right] \\ &= \left(\cos 2t - e^{-t}\right) + 2\left[\left(\sin 2t e^{-(t-t)} - 0\right) - 2\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau\right] \\ &= \left(\cos 2t - e^{-t}\right) + 2\left[\sin 2t - 2\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau\right] \\ &= \left(\cos 2t - e^{-t}\right) + 2\sin 2t - 4\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau\end{aligned}$$

Hence

$$\begin{aligned}\int_0^t e^{-(t-\tau)} \cos 2\tau \, d\tau + 4\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau &= \cos 2t - e^{-t} + 2\sin 2t \\ 5\int_0^t e^{-(t-\tau)} \cos 2\tau d\tau &= \cos 2t - e^{-t} + 2\sin 2t \\ \int_0^t e^{-(t-\tau)} \cos 2\tau d\tau &= \frac{1}{5}\left(\cos 2t - e^{-t} + 2\sin 2t\right)\end{aligned}$$

Therefore

$$f(t) = \frac{1}{5}\left(\cos 2t - e^{-t} + 2\sin 2t\right)$$

1.10 Section 6.6 problem 10

Find the inverse Laplace transform of $F(s) = \frac{1}{(s+1)^2(s^2+4)}$ using convolution theorem.

Solution We see that

$$\begin{aligned}F(s) &= \frac{1}{(s+1)^2} \frac{1}{s^2+4} \\ &= \mathscr{L}\left(te^{-t}\right) \mathscr{L}\left(\frac{1}{2}\sin 2t\right)\end{aligned}$$

Hence, using convolution theorem

$$\begin{aligned}f(t) &= te^{-t} \circledast \frac{1}{2}\sin 2t \\ &= \frac{1}{2}\int_0^t (t-\tau)\, e^{-(t-\tau)} \sin 2\tau \\ &= \frac{1}{2}\int_0^t te^{-(t-\tau)} \sin 2\tau \, d\tau - \frac{1}{2}\int_0^t \tau e^{-(t-\tau)} \sin 2\tau \, d\tau\end{aligned}$$

The first integral is

$$\int_0^t te^{-(t-\tau)} \sin 2\tau \, d\tau = t\int_0^t e^{-(t-\tau)} \sin 2\tau \, d\tau$$

This is similar to one we did in problem 10 but now we have $\sin 2\tau$. Using integration by parts again as before gives

$$\begin{aligned}t\int_0^t e^{-(t-\tau)} \sin 2\tau \, d\tau &= t\left(\frac{1}{5}\left(2e^{-t} - 2\cos 2t + \sin 2t\right)\right) \\ &= \frac{t}{5}\left(2e^{-t} - 2\cos 2t + \sin 2t\right)\end{aligned}$$

Now we need to evaluate the second integral $\int_0^t \tau e^{-(t-\tau)} \sin 2\tau \, d\tau$. This can also be done using integration by part. But I used CAS here, the result is

$$\int_0^t \tau e^{-(t-\tau)} \sin 2\tau \, d\tau = \frac{1}{25}\left(-4e^{-t} + (4-10t)\cos 2t + (3+5t)\sin 2t\right)$$

Therefore

$$\begin{aligned}f(t) &= \frac{1}{2}\frac{t}{5}\left(2e^{-t} - 2\cos(2t) + \sin(2t)\right) - \frac{1}{2}\frac{1}{25}\left(-4e^{-t} + (4-10t)\cos(2t) + (3+5t)\sin(2t)\right)\\ &= \frac{2}{25}e^{-t} - \frac{2}{25}\cos 2t - \frac{3}{50}\sin 2t + \frac{1}{5}te^{-t}\end{aligned}$$

1.11 Section 6.6 problem 11

Find the inverse Laplace transform of $F(s) = \frac{G(s)}{s^2+1}$ using convolution theorem.

Solution We see that

$$F(s) = G(s)\frac{1}{s^2+1} = G(s)\mathscr{L}(\sin t)$$

Hence, using convolution theorem

$$f(t) = g(t) \circledast \sin t = \int_0^t \sin(t-\tau)\,g(\tau)\,d\tau$$

Or

$$f(t) = \int_0^t g(t-\tau)\sin(\tau)\,d\tau$$