HW 8, Math 319, Fall 2016

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1 HW 8

1.1 Section 6.1 problem 7

Find Laplace Transform of $f(t) = \cosh(bt)$

solution Since $\cosh(bt) = \frac{e^{bt}+e^{-bt}}{2}$ then

$$\begin{aligned}\mathscr{L}\cosh(bt) &= \frac{1}{2}\mathscr{L}\left(e^{bt}+e^{-bt}\right)\\ &= \frac{1}{2}\left(\mathscr{L}e^{bt}+\mathscr{L}e^{-bt}\right)\end{aligned}$$

But

$$\mathscr{L}e^{bt} = \frac{1}{s-b}$$

For $s > b$ and

$$\mathscr{L}e^{bt} = \frac{1}{s-b}$$

For $s < b$. Hence

$$\begin{aligned}\mathscr{L}\cosh(bt) &= \frac{1}{2}\left(\frac{1}{s-b}+\frac{1}{s+b}\right)\\ &= \frac{s^2}{s^2-b^2}\end{aligned}$$

For $s > |b|$

1.2 Section 6.1 problem 8

Find Laplace Transform of $f(t) = \sinh(bt)$

solution Since $\sinh(bt) = \frac{e^{bt}-e^{-bt}}{2}$ then

$$\begin{aligned}\mathscr{L}\sinh(bt) &= \frac{1}{2}\mathscr{L}\left(e^{bt}-e^{-bt}\right)\\ &= \frac{1}{2}\left(\mathscr{L}e^{bt}-\mathscr{L}e^{-bt}\right)\end{aligned}$$

But, as we found in the last problem

$$\mathscr{L}e^{bt} = \frac{1}{s-b} \qquad s > b$$

And

$$\mathscr{L}e^{-bt} = \frac{1}{s+b} \qquad s < b$$

Therefore

$$\begin{aligned}\mathscr{L}\sinh(bt) &= \frac{1}{2}\left(\frac{1}{s-b}-\frac{1}{s+b}\right) \qquad s>b; s<b \\ &= \frac{b}{s^2-b^2} \qquad s>|b|\end{aligned}$$

1.3 Section 6.1 problem 9

Find Laplace Transform of $f(t) = e^{at}\cosh(bt)$

solution Using the property that

$$e^{at}f(t) \Longleftrightarrow F(s-a)$$

Where $f(t) = \cosh(bt)$ now. We already found above that $\cosh(bt) \Longleftrightarrow \frac{s}{s^2-b^2}$, for $s>|b|$. In other words, $F(s) = \frac{s}{s^2-b^2}$, therefore

$$e^{at}\cosh(bt) \Longleftrightarrow \frac{(s-a)}{(s-a)^2-b^2} \qquad s-a>|b|$$

1.4 Section 6.1 problem 10

Find Laplace Transform of $f(t) = e^{at}\sinh(bt)$

solution Using the property that

$$e^{at}f(t) \Longleftrightarrow F(s-a)$$

Where $f(t) = \sinh(bt)$ now. We already found above that $\sinh(bt) \Longleftrightarrow \frac{b}{s^2-b^2}$, for $s>|b|$. In other words, $F(s) = \frac{b}{s^2-b^2}$, therefore

$$e^{at}\sinh(bt) \Longleftrightarrow \frac{b}{(s-a)^2-b^2} \qquad s-a>|b|$$

1.5 Section 6.2 problem 17

Use Laplace transform to solve $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$ for $y(0)=0, y'(0)=1, y''(0)=0, y'''(0)=1$

Solution Taking Laplace transform of the ODE gives

$$\mathscr{L}\left\{y^{(4)}\right\} - 4\mathscr{L}\left\{y'''\right\} + 6\mathscr{L}\left\{y''\right\} - 4\mathscr{L}\left\{y'\right\} + \mathscr{L}\left\{y\right\} = 0 \tag{1}$$

Let $\mathscr{L}\left\{y\right\} = Y(s)$ then

$$\begin{aligned}\mathscr{L}\left\{y^{(4)}\right\} &= s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) \\ &= s^4Y(s) - s^3(0) - s^2(1) - s(0) - 1 \\ &= s^4Y(s) - s^2 - 1\end{aligned}$$

And

$$\begin{aligned}\mathscr{L}\left\{y'''\right\} &= s^3Y(s) - s^2y(0) - sy'(0) - y''(0) \\ &= s^3Y(s) - s^2(0) - s(1) - 0 \\ &= s^3Y(s) - s\end{aligned}$$

And

$$\begin{aligned}\mathscr{L}\left\{y''\right\} &= s^2Y(s) - sy(0) - y'(0)\\ &= s^2Y(s) - s(0) - 1\\ &= s^2Y(s) - 1\end{aligned}$$

And

$$\begin{aligned}\mathscr{L}\left\{y'\right\} &= sY(s) - y(0)\\ &= sY(s)\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\left(s^4Y(s) - s^2 - 1\right) - 4\left(s^3Y(s) - s\right) + 6\left(s^2Y(s) - 1\right) - 4\left(sY(s)\right) + Y(s) &= 0\\ Y(s)\left(s^4 - 4s^3 + 6s^2 - 4s + 1\right) - s^2 - 1 + 4s - 6 &= 0\end{aligned}$$

Therefore

$$\begin{aligned}Y(s) &= \frac{s^2 - 4s + 7}{s^4 - 4s^3 + 6s^2 - 4s + 1}\\ &= \frac{s^2 - 4s + 7}{(s-1)^4}\\ &= \frac{s^2}{(s-1)^4} - \frac{4s}{(s-1)^4} + \frac{7}{(s-1)^4} \qquad (2)\end{aligned}$$

But

$$\begin{aligned}\frac{s^2}{(s-1)^4} &= \frac{(s-1)^2 - 1 + 2s}{(s-1)^4}\\ &= \frac{(s-1)^2}{(s-1)^4} - \frac{1}{(s-1)^4} + 2\frac{(s-1)+1}{(s-1)^4}\\ &= \frac{1}{(s-1)^2} - \frac{1}{(s-1)^4} + 2\frac{(s-1)}{(s-1)^4} + 2\frac{1}{(s-1)^4}\\ &= \frac{1}{(s-1)^2} - \frac{1}{(s-1)^4} + 2\frac{1}{(s-1)^3} + 2\frac{1}{(s-1)^4}\\ &= \frac{1}{(s-1)^2} + \frac{2}{(s-1)^3} + \frac{1}{(s-1)^4}\end{aligned}$$

And

$$\begin{aligned}\frac{4s}{(s-1)^4} &= 4\frac{(s-1)+1}{(s-1)^4}\\ &= 4\frac{(s-1)}{(s-1)^4} + 4\frac{1}{(s-1)^4}\\ &= \frac{4}{(s-1)^3} + \frac{4}{(s-1)^4}\end{aligned}$$

Therefore (2) becomes

$$\begin{aligned}Y(s) &= \left(\frac{1}{(s-1)^2} + \frac{2}{(s-1)^3} + \frac{1}{(s-1)^4}\right) - \left(\frac{4}{(s-1)^3} + \frac{4}{(s-1)^4}\right) + \frac{7}{(s-1)^4}\\ &= \frac{1}{(s-1)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4} \qquad (3)\end{aligned}$$

Now using property the shift property of $F(s)$ together with

$$\begin{aligned}\frac{1}{s^2} &\iff t\\ \frac{1}{s^3} &\iff \frac{t^2}{2}\\ \frac{1}{s^4} &\iff \frac{t^3}{6}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{(s-1)^2} &\iff e^t t\\ \frac{1}{(s-1)^3} &\iff e^t\frac{t^2}{2}\\ \frac{1}{(s-1)^4} &\iff e^t\frac{t^3}{6}\end{aligned}$$

And (3) becomes

$$\begin{aligned}\frac{1}{(s-1)^2}-\frac{2}{(s-1)^3}+\frac{4}{(s-1)^4} &\iff e^t t-2\left(e^t\frac{t^2}{2}\right)+4\left(e^t\frac{t^3}{6}\right)\\ &= e^t t-e^t t^2+\frac{2}{3}e^t t^3\end{aligned}$$

Hence

$$y(t)=e^t\left(t-t^2+\frac{2}{3}t^3\right)$$

1.6 Section 6.2 problem 18

Use Laplace transform to solve $y^{(4)}-y=0$ for $y(0)=1, y'(0)=0, y''(0)=1, y'''(0)=0$

Solution Taking Laplace transform of the ODE gives

$$\mathscr{L}\left\{y^{(4)}\right\}-\mathscr{L}\left\{y\right\}=0 \tag{1}$$

Let $\mathscr{L}\left\{y\right\}=Y(s)$ then

$$\begin{aligned}\mathscr{L}\left\{y^{(4)}\right\} &= s^4Y(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)\\ &= s^4Y(s)-s^3(1)-s^2(0)-s(1)-0\\ &= s^4Y(s)-s^3-s\end{aligned}$$

Hence (1) becomes

$$s^4Y(s)-s^3-s-Y(s)=0$$

Solving for $Y(s)$ gives

$$\begin{aligned} Y(s) &= \frac{s^3+s}{s^4-1} \\ &= \frac{s\left(s^2+1\right)}{s^4-1} \\ &= \frac{s\left(s^2+1\right)}{\left(s^2-1\right)\left(s^2+1\right)} \\ &= \frac{s}{s^2-1} \end{aligned}$$

But, Hence above becomes, where $a=1$

$$\frac{s}{s^2-1} \iff \cosh(t)$$

Hence

$$y(t) = \cosh(at)$$

1.7 Section 6.2 problem 19

Use Laplace transform to solve $y^{(4)} - 4y = 0$ for $y(0) = 1, y'(0) = 0, y''(0) = -2, y'''(0) = 0$

<u>Solution</u> Taking Laplace transform of the ODE gives

$$\mathscr{L}\left\{y^{(4)}\right\} - 4\mathscr{L}\left\{y\right\} = 0 \tag{1}$$

Let $\mathscr{L}\left\{y\right\} = Y(s)$ then

$$\begin{aligned} \mathscr{L}\left\{y^{(4)}\right\} &= s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) \\ &= s^4 Y(s) - s^3(1) - s^2(0) - s(-2) - 0 \\ &= s^4 Y(s) - s^3 + 2s \end{aligned}$$

Hence (1) becomes

$$s^4 Y(s) - s^3 + 2s - 4Y(s) = 0$$

Solving for $Y(s)$ gives

$$\begin{aligned} Y(s) &= \frac{s^3-2s}{s^4-4} \\ &= \frac{s^3-2s}{\left(s^2-2\right)\left(s^2+2\right)} \\ &= \frac{s\left(s^2-2\right)}{\left(s^2-2\right)\left(s^2+2\right)} \\ &= \frac{s}{\left(s^2+2\right)} \end{aligned}$$

Using $\cos(at) \iff \frac{s}{s^2+a^2}$, the above becomes, where $a=\sqrt{2}$

$$\frac{s}{\left(s^2+2\right)} \iff \cos\left(\sqrt{2}t\right)$$

Hence

$$y(t) = \cos\left(\sqrt{2}t\right)$$

1.8 Section 6.2 problem 20

Use Laplace transform to solve $y'' + \omega^2 y = \cos 2t;\ \omega^2 \neq 4;\ y(0) = 1, y'(0) = 0$

<u>Solution</u> Let $Y(s) = \mathscr{L}\left\{y(t)\right\}$. Taking Laplace transform of the ODE, and using $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}$ gives

$$s^2 Y(s) - sy(0) - y'(0) + \omega^2 Y(s) = \frac{s}{s^2+4} \tag{1}$$

Applying initial conditions

$$s^2 Y(s) - s + \omega^2 Y(s) = \frac{s}{s^2+4}$$

Solving for $Y(s)$

$$\begin{aligned} Y(s)\left(s^2+\omega^2\right) - s &= \frac{s}{s^2+4} \\ Y(s) &= \frac{s}{\left(s^2+4\right)\left(s^2+\omega^2\right)} + \frac{s}{\left(s^2+\omega^2\right)} \end{aligned} \tag{2}$$

But

$$\begin{aligned} \frac{s}{\left(s^2+4\right)\left(s^2+\omega^2\right)} &= \frac{As+B}{\left(s^2+4\right)} + \frac{Cs+D}{\left(s^2+\omega^2\right)} \\ s &= (As+B)\left(s^2+\omega^2\right) + (Cs+D)\left(s^2+4\right) \\ s &= 4D + As^3 + Bs^2 + Cs^3 + B\omega^2 + s^2 D + 4Cs + As\omega^2 \\ s &= \left(4D + B\omega^2\right) + s\left(4C + A\omega^2\right) + s^2(B+D) + s^3(A+C) \end{aligned}$$

Hence

$$\begin{aligned} 4D + B\omega^2 &= 0 \\ 4C + A\omega^2 &= 1 \\ B + D &= 0 \\ A + C &= 0 \end{aligned}$$

Equation (2,4) gives $A = \frac{1}{\omega^2-4}, C = \frac{1}{4-\omega^2}$ and (1,3) gives $B = 0, D = 0$. Hence

$$\frac{s}{\left(s^2+4\right)\left(s^2+\omega^2\right)} = \left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{1}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)}$$

Therefore (2) becomes

$$\begin{aligned} Y(s) &= \left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{1}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)} + \frac{s}{\left(s^2+\omega^2\right)} \\ &= \left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{5-\omega^2}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)} \end{aligned}$$

Using $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}$, the above becomes

$$\left(\frac{1}{\omega^2-4}\right)\frac{s}{\left(s^2+4\right)} + \left(\frac{5-\omega^2}{4-\omega^2}\right)\frac{s}{\left(s^2+\omega^2\right)} \Longleftrightarrow \left(\frac{1}{\omega^2-4}\right)\cos(2t) + \left(\frac{5-\omega^2}{4-\omega^2}\right)\cos(\omega t)$$
$$= \left(\frac{1}{\omega^2-4}\right)\cos(2t) + \left(\frac{\omega^2-5}{\omega^2-4}\right)\cos(\omega t)$$

Hence

$$\begin{aligned} y(t) &= \left(\frac{1}{\omega^2-4}\right)\cos(2t) + \left(\frac{\omega^2-5}{\omega^2-4}\right)\cos(\omega t) \\ &= \frac{\left(\omega^2-5\right)\cos(\omega t) + \cos(2t)}{\omega^2-4} \end{aligned}$$

1.9 Section 6.2 problem 21

Use Laplace transform to solve $y'' - 2y' + 2y = \cos t$; $y(0) = 1, y'(0) = 0$

Solution Let $Y(s) = \mathscr{L}\left\{y(t)\right\}$. Taking Laplace transform of the ODE, and using $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}$ gives

$$\left(s^2Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{s}{s^2+1} \tag{1}$$

Applying initial conditions

$$s^2Y(s) - s - 2(sY(s) - 1) + 2Y(s) = \frac{s}{s^2+1}$$

Solving for $Y(s)$

$$\begin{aligned} s^2Y(s) - s - 2sY(s) + 2 + 2Y(s) &= \frac{s}{s^2+1} \\ Y(s)\left(s^2-2s+2\right) - s + 2 &= \frac{s}{s^2+1} \\ Y(s) &= \frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} + \frac{s}{\left(s^2-2s+2\right)} - \frac{2}{\left(s^2-2s+2\right)} \end{aligned} \tag{2}$$

But

$$\begin{aligned} \frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} &= \frac{As+B}{\left(s^2+1\right)} + \frac{Cs+D}{s^2-2s+2} \\ s &= (As+B)\left(s^2-2s+2\right) + (Cs+D)\left(s^2+1\right) \\ s &= 2B + D - 2As^2 + As^3 + Bs^2 + Cs^3 + s^2D + 2As - 2Bs + Cs \\ s &= (2B+D) + s(2A-2B+C) + s^2(-2A+B+D) + s^3(A+C) \end{aligned}$$

Hence

$$\begin{aligned} 2B + D &= 0 \\ 2A - 2B + C &= 1 \\ -2A + B + D &= 0 \\ A + C &= 0 \end{aligned}$$

Solving gives $A=\frac{1}{5},B=-\frac{2}{5},C=-\frac{1}{5},D=\frac{4}{5}$, hence

$$\begin{aligned}\frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} &= \frac{1}{5}\frac{s-2}{\left(s^2+1\right)}-\frac{1}{5}\frac{s-4}{s^2-2s+2}\\ &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{s}{s^2-2s+2}+\frac{4}{5}\frac{1}{s^2-2s+2} \qquad (3)\end{aligned}$$

Completing the squares for

$$\begin{aligned}s^2-2s+2 &= a\left(s+b\right)^2+d\\ &= a\left(s^2+b^2+2bs\right)+d\\ &= as^2+ab^2+2abs+d\end{aligned}$$

Hence $a=1,2ab=-2,\left(ab^2+d\right)=2$, hence $b=-1,d=1$, hence

$$s^2-2s+2=\left(s-1\right)^2+1$$

Hence (3) becomes

$$\begin{aligned}\frac{s}{\left(s^2+1\right)\left(s^2-2s+2\right)} &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{s}{\left(s-1\right)^2+1}+\frac{4}{5}\frac{1}{\left(s-1\right)^2+1}\\ &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{\left(s-1\right)+1}{\left(s-1\right)^2+1}+\frac{4}{5}\frac{1}{\left(s-1\right)^2+1}\\ &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{\left(s-1\right)}{\left(s-1\right)^2+1}-\frac{1}{5}\frac{1}{\left(s-1\right)^2+1}+\frac{4}{5}\frac{1}{\left(s-1\right)^2+1}\\ &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{\left(s-1\right)}{\left(s-1\right)^2+1}+\frac{3}{5}\frac{1}{\left(s-1\right)^2+1}\end{aligned}$$

Therefore (2) becomes

$$\begin{aligned}Y\left(s\right) &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{\left(s-1\right)}{\left(s-1\right)^2+1}+\frac{3}{5}\frac{1}{\left(s-1\right)^2+1}+\frac{s}{\left(s-1\right)^2+1}-\frac{2}{\left(s-1\right)^2+1}\\ &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{\left(s-1\right)}{\left(s-1\right)^2+1}+\frac{3}{5}\frac{1}{\left(s-1\right)^2+1}+\frac{\left(s-1\right)+1}{\left(s-1\right)^2+1}-\frac{2}{\left(s-1\right)^2+1}\\ &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}-\frac{1}{5}\frac{\left(s-1\right)}{\left(s-1\right)^2+1}+\frac{3}{5}\frac{1}{\left(s-1\right)^2+1}+\frac{\left(s-1\right)}{\left(s-1\right)^2+1}+\frac{1}{\left(s-1\right)^2+1}-\frac{2}{\left(s-1\right)^2+1}\\ &= \frac{1}{5}\frac{s}{s^2+1}-\frac{2}{5}\frac{1}{s^2+1}+\frac{4}{5}\frac{\left(s-1\right)}{\left(s-1\right)^2+1}-\frac{2}{5}\frac{1}{\left(s-1\right)^2+1}\end{aligned}$$

Using $\cos\left(at\right)\Longleftrightarrow\frac{s}{s^2+a^2},\sin\left(at\right)\Longleftrightarrow\frac{a}{s^2+a^2}$ and the shift property of Laplace transform, then

$$\begin{aligned}\frac{1}{5}\frac{s}{s^2+1} &\Longleftrightarrow \frac{1}{5}\cos\left(t\right)\\ \frac{2}{5}\frac{1}{s^2+1} &\Longleftrightarrow \frac{2}{5}\sin\left(t\right)\\ \frac{4}{5}\frac{\left(s-1\right)}{\left(s-1\right)^2+1} &\Longleftrightarrow \frac{4}{5}e^t\cos t\\ \frac{2}{5}\frac{1}{\left(s-1\right)^2+1} &\Longleftrightarrow \frac{8}{5}e^t\sin t\end{aligned}$$

Hence

$$y(t) = \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) + \frac{4}{5}e^t\cos t - \frac{2}{5}e^t\sin t$$
$$\frac{1}{5}\left(\cos t - 2\sin t + 4e^t\cos t - 2e^t\sin t\right)$$

1.10 Section 6.2 problem 22

Use Laplace transform to solve $y'' - 2y' + 2y = e^{-t}; y(0) = 0, y'(0) = 1$

Solution Let $Y(s) = \mathscr{L}\{y(t)\}$. Taking Laplace transform of the ODE, and using $e^{-t} \Longleftrightarrow \frac{1}{s+1}$ gives

$$\left(s^2Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{1}{s+1} \tag{1}$$

Applying initial conditions gives

$$s^2Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

Solving for $Y(s)$

$$Y(s)\left(s^2 - 2s + 2\right) - 1 = \frac{1}{s+1}$$
$$Y(s) = \frac{1}{(s+1)\left(s^2-2s+2\right)} + \frac{1}{s^2-2s+2} \tag{2}$$

But

$$\frac{1}{(s+1)\left(s^2-2s+2\right)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$$
$$1 = A\left(s^2-2s+2\right) + (Bs+C)(s+1)$$
$$1 = 2A + C + As^2 + Bs^2 - 2As + Bs + Cs$$
$$1 = (2A+C) + s(-2A+B+C) + s^2(A+B)$$

Hence

$$1 = 2A + C$$
$$0 = -2A + B + C$$
$$0 = A + B$$

Solving gives $A = \frac{1}{5}, B = -\frac{1}{5}, C = \frac{3}{5}$, therefore

$$\frac{1}{(s+1)\left(s^2-2s+2\right)} = \frac{1}{5}\frac{1}{s+1} + \frac{-\frac{1}{5}s + \frac{3}{5}}{s^2-2s+2}$$
$$= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s}{s^2-2s+2} + +\frac{3}{5}\frac{1}{s^2-2s+2}$$

Completing the square for $s^2 - 2s + 2$ which was done in last problem, gives $(s-1)^2 + 1$, hence the above becomes

$$
\begin{aligned}
\frac{1}{(s+1)\left(s^2-2s+2\right)} &= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s}{(s-1)^2+1} + +\frac{3}{5}\frac{1}{(s-1)^2+1}\\
&= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)+1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1}\\
&= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{1}{5}\frac{1}{(s-1)^2+1} + \frac{3}{5}\frac{1}{(s-1)^2+1}\\
&= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{2}{5}\frac{1}{(s-1)^2+1}
\end{aligned}
$$

Therefore (2) becomes

$$
Y(s) = \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} + \frac{2}{5}\frac{1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}
$$

Using $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}, \sin(at) \Longleftrightarrow \frac{a}{s^2+a^2}$ and the shift property of Laplace transform, then

$$
\begin{aligned}
\frac{1}{5}\frac{1}{s+1} &\Longleftrightarrow \frac{1}{5}e^{-t}\\
\frac{1}{5}\frac{(s-1)}{(s-1)^2+1} &\Longleftrightarrow \frac{1}{5}e^{t}\cos t\\
\frac{2}{5}\frac{1}{(s-1)^2+1} &\Longleftrightarrow \frac{2}{5}e^{t}\sin t\\
\frac{1}{(s-1)^2+1} &\Longleftrightarrow e^{t}\sin t
\end{aligned}
$$

Hence

$$
\begin{aligned}
y(t) &= \frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{2}{5}e^{t}\sin t + e^{t}\sin t\\
&= \frac{1}{5}\left(e^{-t} - e^{t}\cos t + 7e^{t}\sin t\right)
\end{aligned}
$$

1.11 Section 6.2 problem 23

Use Laplace transform to solve $y'' + 2y' + y = 4e^{-t}; y(0) = 2, y'(0) = -1$

Solution Let $Y(s) = \mathscr{L}\left\{y(t)\right\}$. Taking Laplace transform of the ODE, and using $e^{-t} \Longleftrightarrow \frac{1}{s+1}$ gives

$$
\left(s^2Y(s) - sy(0) - y'(0)\right) + 2\left(sY(s) - y(0)\right) + Y(s) = \frac{4}{s+1} \tag{1}
$$

Applying initial conditions gives

$$
\left(s^2Y(s) - 2s + 1\right) + 2\left(sY(s) - 2\right) + Y(s) = \frac{4}{s+1}
$$

Solving for $Y(s)$

$$
\begin{aligned}
Y(s)\left(s^2+2s+1\right) - 2s + 1 - 4 &= \frac{4}{s+1}\\
Y(s)\left(s^2+2s+1\right) &= \frac{4}{s+1} + 2s - 1 + 4\\
Y(s) &= \frac{4}{(s+1)\left(s^2+2s+1\right)} + \frac{2s}{\left(s^2+2s+1\right)} - \frac{1}{\left(s^2+2s+1\right)} + \frac{4}{\left(s^2+2s+1\right)}
\end{aligned}
$$

But $\left(s^2+2s+1\right)=(s+1)^2$, hence

$$Y(s)=\frac{4}{(s+1)^3}+\frac{2s}{(s+1)^2}-\frac{1}{(s+1)^2}+\frac{4}{(s+1)^2} \tag{2}$$

But

$$\begin{aligned}\frac{2s}{(s+1)^2}&=2\frac{s+1-1}{(s+1)^2}\\&=2\frac{(s+1)}{(s+1)^2}-2\frac{1}{(s+1)^2}\\&=2\frac{1}{s+1}-2\frac{1}{(s+1)^2}\end{aligned}$$

Hence (2) becomes

$$Y(s)=\frac{4}{(s+1)^3}+2\frac{1}{s+1}-2\frac{1}{(s+1)^2}-\frac{1}{(s+1)^2}+\frac{4}{(s+1)^2} \tag{3}$$

We now ready to do the inversion. Since $\frac{1}{s^3}\Longleftrightarrow\frac{t^2}{2}$ and $\frac{1}{s^2}\Longleftrightarrow t$ and $\frac{1}{s}\Longleftrightarrow 1$ and using the shift property $e^{at}f(t)\Longleftrightarrow F(s-a)$, then using these into (3) gives

$$\begin{aligned}\frac{4}{(s+1)^3}&\Longleftrightarrow 4e^{-t}\left(\frac{t^2}{2}\right)\\2\frac{1}{s+1}&\Longleftrightarrow 2e^{-t}\\2\frac{1}{(s+1)^2}&\Longleftrightarrow 2e^{-t}t\\\frac{1}{(s+1)^2}&\Longleftrightarrow e^{-t}t\\\frac{4}{(s+1)^2}&\Longleftrightarrow 4e^{-t}t\end{aligned}$$

Now (3) becomes

$$\begin{aligned}Y(s)&\Longleftrightarrow 4e^{-t}\left(\frac{t^2}{2}\right)+2e^{-t}-2e^{-t}t-e^{-t}t+4e^{-t}t\\&=e^{-t}\left(2t^2+2-2t-t+4t\right)\\&=e^{-t}\left(2t^2+t+2\right)\end{aligned}$$

1.12 Section 6.3 problem 25

Suppose that $F(s)=\mathscr{L}\left\{f(t)\right\}$ exists for $s>a\geq 0$.

1. Show that if c is positive constant then $\mathscr{L}\left\{f(ct)\right\}=\frac{1}{c}F\left(\frac{s}{c}\right)$ for $s>ca$
2. Show that if k is positive constant then $\mathscr{L}^{-1}\{F(ks)\}=\frac{1}{k}f\left(\frac{t}{k}\right)$
3. Show that if a,b are constants with $a>0$ then $\mathscr{L}^{-1}\{F(as+b)\}=\frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)$

<u>Solution</u>

1.12.1 Part (a)

From definition,

$$\mathscr{L}\left\{f(ct)\right\} = \int_0^\infty f(ct)\,e^{-st}dt$$

Let $ct = \tau$, then when $t = 0, \tau = 0$ and when $t = \infty, \tau = \infty$, and $c = \frac{d\tau}{dt}$. Hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{f(ct)\right\} &= \int_0^\infty f(\tau)\,e^{-s\left(\frac{\tau}{c}\right)}\frac{d\tau}{c}\\ &= \frac{1}{c}\int_0^\infty f(\tau)\,e^{-\tau\left(\frac{s}{c}\right)}d\tau\end{aligned}$$

We see from above that $\mathscr{L}\left\{f(ct)\right\}$ is $\frac{1}{c}F\left(\frac{s}{c}\right)$.Now we look at the conditions which makes the above integral converges. Let

$$\left|f(\tau)\,e^{-\tau\left(\frac{s}{c}\right)}\right| \leq k\left|e^{at}e^{-\tau\left(\frac{s}{c}\right)}\right|$$

Where k is some constant. Then

$$\begin{aligned}\int_0^\infty f(t)\,e^{-t\left(\frac{s}{c}\right)}dt &\leq k\int_0^\infty e^{at}e^{-t\left(\frac{s}{c}\right)}dt\\ &= k\int_0^\infty e^{-t\left(\frac{s}{c}-a\right)}dt\end{aligned}$$

But $\int_0^\infty e^{-t\left(\frac{s}{c}-a\right)}d\tau$ converges if $\frac{s}{c} - a > 0$ or

$$s > ca$$

Hence this is the condition for $\int_0^\infty f(t)\,e^{-t\left(\frac{s}{c}\right)}dt$ to converge. Which is what we required to show.

1.12.2 Part (b)

From definition

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\} &= \frac{1}{k}\mathscr{L}\left\{f\left(\frac{t}{k}\right)\right\}\\ &= \frac{1}{k}\int_0^\infty f\left(\frac{t}{k}\right)e^{-st}dt\end{aligned}$$

Let $\frac{t}{k} = \tau$. When $t = 0, \tau = 0$ and when $t = \infty, \tau = \infty$. $\frac{dt}{d\tau} = k$, hence the above becomes

$$\begin{aligned}\mathscr{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\} &= \frac{1}{k}\int_0^\infty f(\tau)\,e^{-s(k\tau)}\,(k\,d\tau)\\ &= \int_0^\infty f(\tau)\,e^{-\tau(sk)}d\tau\end{aligned}$$

We see from above that $\mathscr{L}\left\{\frac{1}{k}f\left(\frac{t}{k}\right)\right\}$ is $F(sk)$. In other words, $\mathscr{L}^{-1}\left\{F(ks)\right\} = \frac{1}{k}f\left(\frac{t}{k}\right)$.

1.12.3 Part (c)

From definition

$$\mathscr{L}\left\{\frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\} = \frac{1}{a}\mathscr{L}\left\{e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\}$$
$$= \frac{1}{a}\int_0^\infty e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)e^{-st}dt$$

Let $\frac{t}{a}=\tau$, at $t=0,\tau=0$ and at $t=\infty,\tau=\infty$. And $\frac{dt}{d\tau}=a$, hence the above becomes

$$\mathscr{L}\left\{\frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\} = \frac{1}{a}\int_0^\infty e^{\frac{-b(a\tau)}{a}}f(\tau)\,e^{-s(a\tau)}\,(ad\tau)$$
$$= \int_0^\infty e^{-b\tau}f(\tau)\,e^{-\tau(sa)}d\tau$$
$$= \int_0^\infty f(\tau)\,e^{-\tau(sa+b)}d\tau$$

We see from the above, that $\mathscr{L}\left\{\frac{1}{a}e^{\frac{-bt}{a}}f\left(\frac{t}{a}\right)\right\} = F(sa+b)$. Now we look at the conditions which makes the above integral converges. Let

$$\left|f(\tau)\,e^{-t(sa+b)}\right| \leq k\left|e^{at}e^{-t(sa+b)}\right|$$

Where k is some constant. Then

$$\int_0^\infty f(t)\,e^{-t(sa+b)}dt \leq k\int_0^\infty e^{at}e^{-t(sa+b)}dt$$
$$= k\int_0^\infty e^{-t(sa+b-a)}dt$$

But $\int_0^\infty e^{-t(sa+b-a)}dt$ converges if $sa+b-a>0$ or $sa>a-b$ or $s>1-\frac{b}{a}$

1.13 Section 6.3 problem 26

Find inverse Laplace transform of $F(s)=\frac{2^{n+1}n!}{s^{n+1}}$

Solution

We know from tables that

$$\frac{n!}{s^{n+1}} \Longleftrightarrow t^n$$

Hence

$$2^{n+1}\frac{n!}{s^{n+1}} \Longleftrightarrow 2^{n+1}t^n$$
$$= 2\,(2t)^n$$

1.14 Section 6.3 problem 27

Find inverse Laplace transform of $F(s)=\frac{2s+1}{4s^2+4s+5}$

Solution

$$F(s) = \frac{2s}{4s^2+4s+5} + \frac{1}{4s^2+4s+5}$$

But $4s^2+4s+5=4\left(s+\frac{1}{2}\right)^2+4$, hence

$$\begin{aligned} F(s) &= \frac{2s}{4\left(s+\frac{1}{2}\right)^2+4}+\frac{1}{4\left(s+\frac{1}{2}\right)^2+4} \\ &= \frac{s}{2\left(s+\frac{1}{2}\right)^2+2}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1} \\ &= \frac{1}{2}\frac{s}{\left(s+\frac{1}{2}\right)^2+1}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1} \\ &= \frac{1}{2}\frac{s+\frac{1}{2}-\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1} \\ &= \frac{1}{2}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1}-\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1}+\frac{1}{4}\frac{1}{\left(s+\frac{1}{2}\right)^2+1} \\ &= \frac{1}{2}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} \qquad (1) \end{aligned}$$

Now we ready to do the inversion. Using $e^{-at}f(t) \Longleftrightarrow F(s+a)$ and using $\sin(at) \Longleftrightarrow \frac{a}{s^2+a^2}$, and $\cos(at) \Longleftrightarrow \frac{s}{s^2+a^2}$ then

$$\frac{1}{2}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+1} \Longleftrightarrow \frac{1}{2}e^{-\frac{1}{2}t}\cos(t)$$

Hence

$$f(t)=\frac{1}{2}e^{-\frac{1}{2}t}\cos(t)$$

1.15 Section 6.3 problem 28

Find inverse Laplace transform of $F(s)=\frac{1}{9s^2-12s+3}$

<u>Solution</u>

$$\frac{1}{9s^2-12s+3}=\frac{1}{9}\frac{1}{s^2-\frac{4}{3}s+\frac{1}{3}}=\frac{1}{9}\frac{1}{(s-1)\left(s-\frac{1}{3}\right)}$$

But

$$\frac{1}{(s-1)\left(s-\frac{1}{3}\right)} = \frac{A}{s-1} + \frac{B}{s-\frac{1}{3}}$$

$$A = \left(\frac{1}{\left(s-\frac{1}{3}\right)}\right)_{s=1} = \frac{3}{2}$$

$$B = \left(\frac{1}{(s-1)}\right)_{s=\frac{1}{3}} = -\frac{3}{2}$$

Hence

$$\frac{1}{9s^2-12s+3} = \frac{1}{9}\left(\frac{3}{2}\frac{1}{s-1} - \frac{3}{2}\frac{1}{s-\frac{1}{3}}\right) \tag{1}$$

Using

$$e^{at} \Longleftrightarrow \frac{1}{s-a}$$

Then (1) becomes

$$\begin{aligned}\frac{1}{9s^2-12s+3} &\Longleftrightarrow \frac{1}{9}\left(\frac{3}{2}e^t - \frac{3}{2}e^{\frac{1}{3}t}\right)\\ &= \frac{1}{6}e^t - \frac{1}{6}e^{\frac{1}{3}t}\\ &= \frac{1}{6}\left(e^t - e^{\frac{1}{3}t}\right)\end{aligned}$$

1.16 Section 6.3 problem 29

Find inverse Laplace transform of $F(s) = \frac{e^2 e^{-4s}}{2s-1}$

<u>solution</u>

$$F(s) = \frac{e^2}{2}\frac{e^{-4s}}{s-\frac{1}{2}}$$

Using

$$u_c(t) f(t-c) \Longleftrightarrow e^{-cs}F(s) \tag{1}$$

Since

$$\frac{1}{s-\frac{1}{2}} \Longleftrightarrow e^{\frac{1}{2}t}$$

Then using (1)

$$e^{-4s}\frac{1}{s-\frac{1}{2}} \Longleftrightarrow u_4(t)\, e^{\frac{1}{2}(t-4)}$$

Hence

$$\begin{aligned}\frac{e^2}{2}\frac{e^{-4s}}{s-\frac{1}{2}} &\Longleftrightarrow \frac{e^2}{2}u_4(t)\,e^{\frac{1}{2}(t-4)}\\ &= \frac{1}{2}u_4(t)\,e^{\frac{1}{2}(t-4)+2}\\ &= \frac{1}{2}u_4(t)\,e^{\frac{1}{2}t-2+2}\\ &= \frac{1}{2}u_4(t)\,e^{\frac{t}{2}}\end{aligned}$$

Therefore

$$f(t) = \frac{1}{2}u_4(t)\,e^{\frac{t}{2}}$$

ps. Book answer is wrong. It gives

$$f(t) = \frac{1}{2}u_4\left(\frac{t}{2}\right)e^{\frac{t}{2}}$$

1.17 Section 6.3 problem 30

Find Laplace transform of $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$

solution

Writing $f(t)$ in terms of Heaviside step function gives

$$f(t) = u_0(t) - u_1(t)$$

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

Therefore

$$\begin{aligned}\mathscr{L}\{u_0(t)\} &= e^{-0s}\frac{1}{s} = \frac{1}{s}\\ \mathscr{L}\{u_1(t)\} &= e^{-s}\frac{1}{s}\end{aligned}$$

Hence

$$\begin{aligned}\mathscr{L}\{u_0(t) - u_1(t)\} &= \frac{1}{s} - e^{-s}\frac{1}{s}\\ &= \frac{1}{s}\left(1-e^{-s}\right) \qquad s > 0\end{aligned}$$

1.18 Section 6.3 problem 31

Find Laplace transform of $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & 1 \leq t < 2 \\ 1 & 2 \leq t < 3 \\ 0 & t \geq 3 \end{cases}$

solution

Writing $f(t)$ in terms of Heaviside step function gives

$$f(t) = u_0(t) - u_1(t) + u_2(t) - u_3(t)$$

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

But $f(t) = 1$ in this case. Hence $F(s) = \frac{1}{s}$. Therefore

$$\begin{aligned} f(t) &\Longleftrightarrow \frac{1}{s}e^{0s} - \frac{1}{s}e^{-s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} \\ &= \frac{1}{s}\left(1 - e^{-s} + e^{-2s} - e^{-3s}\right) \qquad s > 0 \end{aligned}$$

1.19 Section 6.3 problem 32

Find Laplace transform of $f(t) = 1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)$

<u>solution</u>

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

Therefore

$$\begin{aligned} \mathscr{L}\left\{1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)\right\} &= \mathscr{L}\{1\} + \mathscr{L}\left\{\sum_{k=1}^{2n+1}(-1)^k u_k(t)\right\} \\ &= \frac{1}{s} + \sum_{k=1}^{2n+1}(-1)^k \frac{1}{s}e^{-ks} \\ &= \sum_{k=0}^{2n+1}(-1)^k \frac{1}{s}e^{-ks} \\ &= \frac{1}{s}\sum_{k=0}^{2n+1}\left(-e^{-s}\right)^k \end{aligned}$$

Since $|e^{-s}| < 1$ the sum converges. Using $\sum_0^N a_n = \left(\frac{1-r^{N+1}}{1-r}\right)$. Where $|r| < 1$. So the answer is

$$\begin{aligned} \mathscr{L}\left\{1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)\right\} &= \frac{1}{s}\left(\frac{1-\left(-e^{-s}\right)^{2n+2}}{1-\left(-e^{-s}\right)}\right) \\ &= \frac{1}{s}\left(\frac{1-(-e)^{-(2n+2)s}}{1+e^{-s}}\right) \end{aligned}$$

Since $2n+2$ is even then

$$\mathscr{L}\left\{1 + \sum_{k=1}^{2n+1}(-1)^k u_k(t)\right\} = \frac{1}{s}\left(\frac{1+e^{-(2n+2)s}}{1+e^{-s}}\right) \qquad s > 0$$

1.20 Section 6.3 problem 33

Find Laplace transform of $f(t) = 1 + \sum_{k=1}^{\infty}(-1)^k u_k(t)$

<u>solution</u>

Using

$$u_c(t) \Longleftrightarrow e^{-cs}\frac{1}{s}$$

Therefore

$$\begin{aligned}
\mathscr{L}\left\{1+\sum_{k=1}^{\infty}(-1)^k u_k(t)\right\} &= \mathscr{L}\{1\}+\mathscr{L}\left\{\sum_{k=1}^{\infty}(-1)^k u_k(t)\right\} \\
&= \frac{1}{s}+\sum_{k=1}^{\infty}(-1)^k \frac{1}{s}e^{-ks} \\
&= \frac{1}{s}+\frac{1}{s}\sum_{k=1}^{\infty}(-1)^k e^{-ks} \\
&= \frac{1}{s}+\frac{1}{s}\sum_{k=1}^{\infty}\left(-e^{-s}\right)^k
\end{aligned}$$

But

$$\sum_{k=1}^{\infty} r^k = \frac{r}{1-r} \qquad |r|<1$$

Since $s>0$ then$|e^{-s}|<1$. So the answer is

$$\begin{aligned}
\frac{1}{s}+\frac{1}{s}\frac{-e^{-s}}{1-\left(-e^{-s}\right)} &= \frac{1}{s}-\frac{1}{s}\frac{e^{-s}}{1+e^{-s}} \\
&= \frac{1+e^{-s}-e^{-s}}{s\left(1+e^{-s}\right)} \\
&= \frac{1}{s\left(1+e^{-s}\right)} \qquad s>0
\end{aligned}$$

1.21 Section 6.4 problem 21

21. Consider the initial value problem

$$y''+y=g(t), \qquad y(0)=0, \quad y'(0)=0,$$

where

$$g(t)=u_0(t)+\sum_{k=1}^{n}(-1)^k u_{k\pi}(t).$$

(a) Draw the graph of $g(t)$ on an interval such as $0 \le t \le 6\pi$. Compare the graph with that of $f(t)$ in Problem 19(a).

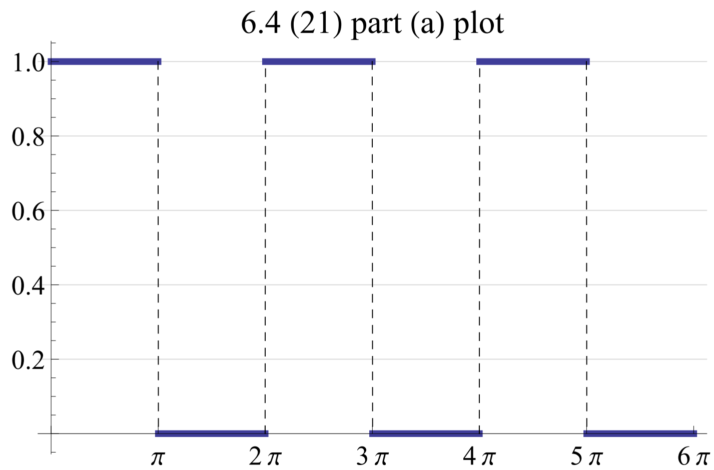
(b) Find the solution of the initial value problem.

(c) Let $n=15$ and plot the graph of the solution for $0 \le t \le 60$. Describe the solution and explain why it behaves as it does. Compare it with the solution of Problem 19.

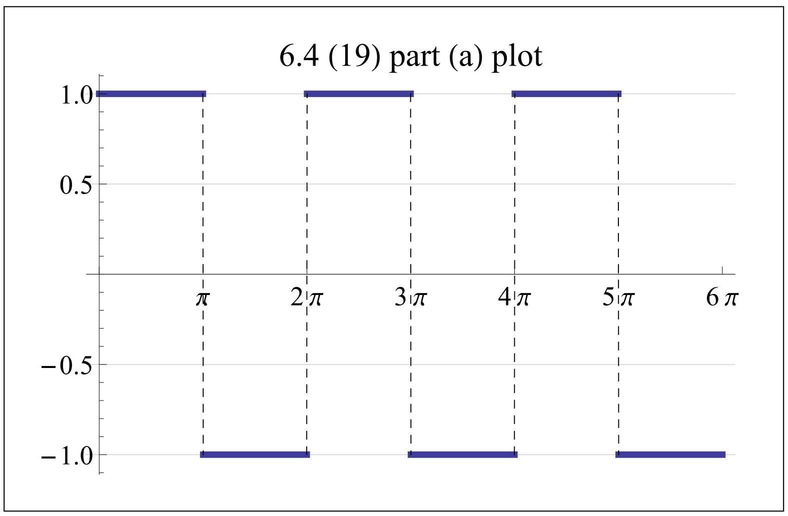
(d) Investigate how the solution changes as n increases. What happens as $n \to \infty$?

1.21.1 Part (a)

A plot of part (a) is the following



And a plot of part(a) for problem 19 is the following



We see the effect of having a 2 inside the sum. It extends the step $u_c(t)$ function to negative side.

1.21.2 Part (b)

The easy way to do this, is to solve for each input term separately, and then add all the solutions, since this is a linear ODE. Once we solve for the first 2-3 terms, we will see the pattern to use for the overall solution. Since the input $g(t)$ is $u_0(t) + \sum_{k=1}^{\infty} (-1)^k u_{k\pi}(t)$, we will first first the response to $u_0(t)$, then for $-u_\pi(t)$ then for $+u_{2\pi}(t)$, and so on, and add them.

When the input is $u_0(t)$, then its Laplace transform is $\frac{1}{s}$, Hence, taking Laplace transform of the ODE gives (where now $Y(s) = \mathscr{L}\left(y(t)\right)$)

$$\left(s^2 Y(s) - sy(0) + y'(0)\right) + Y(s) = \frac{1}{s}$$

Applying initial conditions

$$s^2 Y(s) + Y(s) = \frac{1}{s}$$

Solving for $Y_0(s)$ (called it $Y_0(s)$ since the input is $u_0(t)$)

$$\begin{aligned} Y_0(s) &= \frac{1}{s\left(s^2+1\right)} \\ &= \frac{1}{s} - \frac{s}{s^2+1} \end{aligned}$$

Hence

$$y_0(t) = 1 - \cos t$$

We now do the next input, which is $-u_\pi(t)$, which has Laplace transform of $-\frac{e^{-\pi s}}{s}$, therefore, following what we did above, we obtain now

$$\begin{aligned} Y_\pi(s) &= \frac{-e^{-\pi s}}{s\left(s^2+1\right)} \\ &= -e^{-\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \end{aligned}$$

The effect of $e^{-\pi s}$ is to cause delay in time. Hence the the inverse Laplace transform of the above is the same as $y_0(t)$ but with delay

$$y_\pi(t) = -u_\pi(t)\left(1 - \cos(t-\pi)\right)$$

Similarly, when the input is $+u_{2\pi}(t)$, which which has Laplace transform of $\frac{e^{-2\pi s}}{s}$, therefore, following what we did above, we obtain now

$$\begin{aligned} Y_\pi(s) &= \frac{e^{-2\pi s}}{s\left(s^2+1\right)} \\ &= e^{-2\pi s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) \end{aligned}$$

The effect of $e^{-2\pi s}$ is to cause delay in time. Hence the the inverse Laplace transform of the above is the same as $y_0(t)$ but with now with delay of 2π, therefore

$$y_{2\pi}(t) = +u_{2\pi}(t)\left(1 - \cos(t-2\pi)\right)$$

And so on. We see that if we add all the responses, we obtain

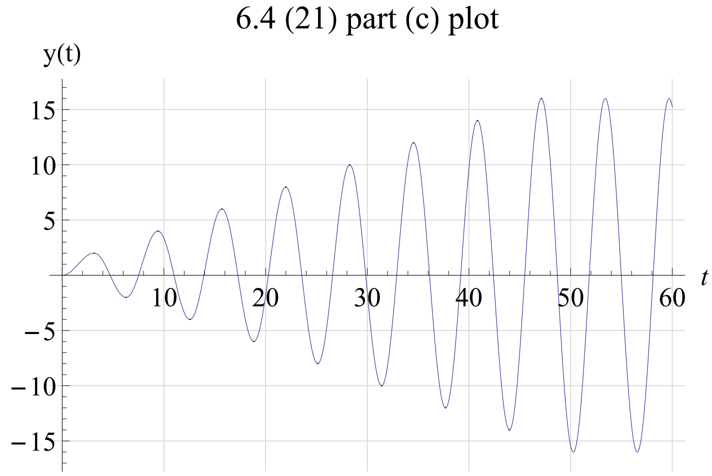
$$\begin{aligned} y(t) &= y_0(t) + y_\pi(t) + y_{2\pi}(t) + \cdots \\ &= (1-\cos t) - u_\pi(t)\left(1-\cos(t-\pi)\right) + u_{2\pi}(t)\left(1-\cos(t-2\pi)\right) - \cdots \end{aligned}$$

Or

$$y(t) = (1-\cos t) + \sum_{k=1}^{n} (-1)^k u_{k\pi}(t)\left(1-\cos(t-k\pi)\right) \tag{1}$$

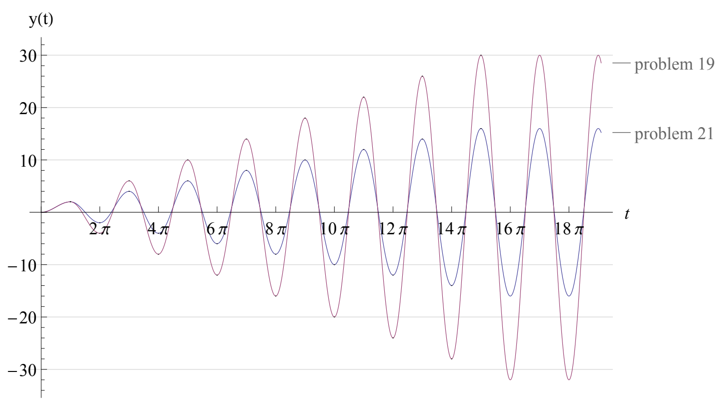
1.21.3 Part (c)

This is a plot of (1) for $n = 15$



We see the solution growing rapidly, they settling down after about $t = 50$ to sinusoidal wave at amplitude of about ±15. This shows the system reached steady state at around $t = 50$.

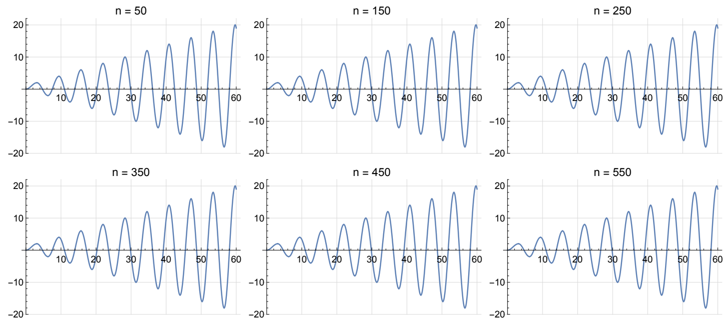
To compare it with problem 19 solution, I used the solution for 19 given in the book, and plotted both solution on top of each others. Also for up to $t = 60$. Here is the result



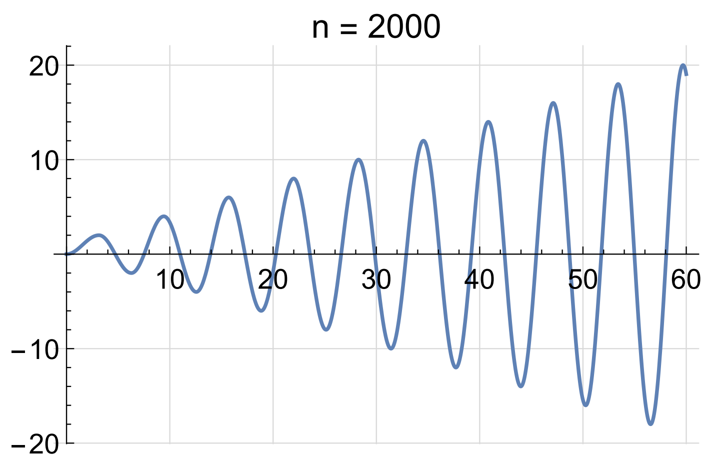
We see that problem 19 output follows the same pattern (since same frequency is used), but with double the amplitude. This is due to the 2 factor used in problem 19 compared to this problem.

1.21.4 Part(d)

At first, I tried it with $n = 50, 150, 250, 350, 450, 550$. I can not see any noticeable change in the plot. Here is the result.



Even at $n = 2000$ there was no change to be noticed.



This shows additional input in the form of shifted unit steps, do not change the steady state solution.