HW 7, Math 319, Fall 2016

Nasser M. Abbasi (Discussion section 44272, Th 4:35PM-5:25 PM)

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1 HW 7

1.1 Section 3.6 problem 1

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $y'' - 5y' + 6y = 2e^t$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' - 5y' + 6y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to compare with.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 - 5r + 6 = 0$ or $(r-3)(r-2) = 0$. Therefore the roots are $r_1 = 3, r_2 = 2$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= e^{3t} \\ y_2 &= e^{2t} \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{3t} + c_2 e^{2t} \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = 2e^{5t} - 3e^{5t} = -e^{5t}$$

Letting $g(t) = 2e^t$ therefore the particular solution is

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W} dt = -\int \frac{e^{2t} 2e^t}{-e^{5t}} dt = 2\int \frac{e^{3t}}{e^{5t}} dt = 2\int e^{-2t} dt = 2\left[\frac{e^{-2t}}{-2}\right] = -e^{-2t}$$

And

$$u_2(t) = \int \frac{y_1(t) g(t)}{W} dt = \int \frac{e^{3t} 2e^t}{-e^{5t}} dt = -2\int \frac{e^{4t}}{e^{5t}} dt = -2\int e^{-t} dt = -2\left[\frac{e^{-t}}{-1}\right] = 2e^{-t}$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \left(-e^{-2t}\right) e^{3t} + 2e^{-t} e^{2t} \\ &= -e^t + 2e^t \\ &= e^t \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{3t} + c_2 e^{2t} + e^t \end{aligned}$$

Finding y_p using undetermined coefficients

From the form of $g(t)$ in the problem, particular solution is assumed to be

$$y_p = Ae^t$$

Hence

$$\begin{aligned} y_p' &= Ae^t \\ y_p'' &= Ae^t \end{aligned}$$

Plugging back into the original ODE gives

$$\begin{aligned} y_p'' - 5y_p' + 6y_p &= 2e^t \\ Ae^t - 5Ae^t + 6Ae^t &= 2e^t \end{aligned}$$

Dividing by $e^t \neq 0$ gives

$$\begin{aligned} A - 5A + 6A &= 2 \\ 2A &= 2 \\ A &= 1 \end{aligned}$$

Therefore

$$y_p = e^t$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

1.2 Section 3.6 problem 2

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $y'' - y' - 2y = 2e^{-t}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' - y' - 2y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to compare with.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 - r - 2 = 0$ or $(r+1)(r-2) = 0$. Therefore the roots are $r_1 = -1, r_2 = 2$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= e^{-t} \\ y_2 &= e^{2t} \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-t} + c_2 e^{2t} \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 2e^t + e^t = 3e^t$$

Letting $g(t) = 2e^{-t}$ therefore the particular solution is

$$y_p(t) = u_1(t) y_1(t) + u_2(t) y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t) g(t)}{W} dt = -\int \frac{e^{2t} 2e^{-t}}{3e^t} dt = -\frac{2}{3} \int \frac{e^t}{e^t} dt = -\frac{2}{3} t$$

And

$$u_2(t) = \int \frac{y_1(t) g(t)}{W} dt = \int \frac{e^{-t} 2e^{-t}}{3e^t} dt = \frac{2}{3} \int \frac{e^{-2t}}{e^t} dt = \frac{2}{3} \int e^{-3t} dt = \frac{2}{3} \left[\frac{e^{-3t}}{-3} \right] = -\frac{2}{9} e^{-3t}$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \left(-\frac{2}{3} t \right) e^{-t} - \frac{2}{9} e^{-3t} e^{2t} \\ &= -\frac{2}{3} t e^{-t} - \frac{2}{9} e^{-t} \end{aligned}$$

We notice something here. The extra term $-\frac{2}{9} e^{-t}$ above is constant times one of the fundamental solutions (one of the solutions to the homogenous equation), which is y_1 in this case found earlier. But adding a multiple of a fundamental solution to a particular solution gives another particular solution. So the term $-\frac{2}{9} e^{-t}$ will be merged with the term from the homogenous solution. Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 e^{2t} - \frac{2}{3} t e^{-t} - \frac{2}{9} e^{-t} \end{aligned}$$

We can now combine $\frac{2}{9}e^{-t}$ that shows up from the particular solution with the $c_1 e^{-t}$ term from the homogenous solution, since c_1 is arbitrary constant, which simplifies the above to

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-t} + c_2 e^{2t} - \frac{2}{3} t e^{-t} \end{aligned}$$

Finding y_p using undetermined coefficients

From the form of $g(t)$ in the problem, and since e^{-t} is already one of the fundamental solutions, then particular solution is assumed to be

$$y_p = Ate^{-t}$$

Hence

$$\begin{aligned} y_p' &= A\left(e^{-t} - te^{-t}\right) \\ y_p'' &= A\left(-e^{-t} - e^{-t} + te^{-t}\right) \\ &= A\left(-2e^{-t} + te^{-t}\right) \end{aligned}$$

Plugging back into the original ODE gives

$$\begin{aligned} y_p'' - y_p' - 2y_p &= 2e^{-t} \\ A\left(-2e^{-t} + te^{-t}\right) - A\left(e^{-t} - te^{-t}\right) - 2Ate^{-t} &= 2e^{-t} \end{aligned}$$

Dividing by $e^{-t} \neq 0$ gives

$$\begin{aligned} A(-2+t) - A(1-t) - 2At &= 2 \\ t(A + A - 2A) - 2A - A &= 2 \\ -3A &= 2 \\ A &= \frac{-2}{3} \end{aligned}$$

Therefore

$$y_p = \frac{-2}{3} te^{-t}$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

1.3 Section 3.6 problem 3

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $y'' + 2y' + y = 3e^{-t}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 2y' + y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to compare with.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2+2r+1 = 0$ or $(r+1)(r+1) = 0$, Therefore the roots are duplicate $r_1 = -1$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= e^{-t} \\ y_2 &= te^{-t} \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{-t} + c_2 te^{-t} \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$
\begin{aligned}
W(t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} \\
&= \left(e^{-t}\right)\left(e^{-t} - te^{-t}\right) + \left(te^{-t}\right)\left(e^{-t}\right) \\
&= e^{-2t} - te^{-2t} + te^{-2t} \\
&= e^{-2t}
\end{aligned}
$$

Letting $g(t) = 3e^{-t}$ therefore the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W} dt = -\int \frac{te^{-t}\left(3e^{-t}\right)}{e^{-2t}} dt = -3\int t dt = -\frac{3}{2}t^2$$

And

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W} dt = \int \frac{e^{-t}\left(3e^{-t}\right)}{e^{-2t}} dt = 3\int dt = 3t$$

Hence the particular solution becomes

$$
\begin{aligned}
y_p &= u_1 y_1 + u_2 y_2 \\
&= \left(-\frac{3}{2}t^2\right) e^{-t} + 3t\left(te^{-t}\right) \\
&= -\frac{3}{2}t^2 e^{-t} + 3t^2 e^{-t} \\
&= \frac{3}{2}t^2 e^{-t}
\end{aligned}
$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= c_1 e^{-t} + c_2 te^{-t} + \frac{3}{2}t^2 e^{-t}
\end{aligned}
$$

Finding y_p using undetermined coefficients

From the form of $g(t) = 3e^{-t}$ in the problem, we want to try e^{-t} but since e^{-t} is already one of the fundamental solutions, we then look at te^{-t} but this is also one fundamental solutions, then we look for $t^2 e^{-t}$. Hence

$$y_p = At^2 e^{-t}$$

Hence

$$
\begin{aligned}
y_p' &= A\left(2te^{-t} - t^2 e^{-t}\right) \\
y_p'' &= A\left(2e^{-t} - 2te^{-t} - \left(2te^{-t} - t^2 e^{-t}\right)\right) \\
&= A\left(2e^{-t} - 2te^{-t} - 2te^{-t} + t^2 e^{-t}\right) \\
&= A\left(2e^{-t} - 4te^{-t} + t^2 e^{-t}\right)
\end{aligned}
$$

Plugging back into the original ODE gives

$$
\begin{aligned}
y_p'' + 2y_p' + y_p &= 3e^{-t} \\
A\left(2e^{-t} - 4te^{-t} + t^2 e^{-t}\right) + 2A\left(2te^{-t} - t^2 e^{-t}\right) + At^2 e^{-t} &= 3e^{-t}
\end{aligned}
$$

Dividing by $e^{-t} \neq 0$ gives

$$
\begin{aligned}
A\left(2 - 4t + t^2\right) + 2A\left(2t - t^2\right) + At^2 &= 3 \\
t(-4A + 4A) + t^2(A - 2A + A) + 2A &= 3 \\
A &= \frac{3}{2}
\end{aligned}
$$

Therefore

$$y_p = \frac{3}{2}te^{-t}$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

1.4 Section 3.6 problem 4

Use method of variations of parameters to find particular solution and check your solution using method of undetermined coefficients. $4y'' - 4y' + y = 16e^{\frac{t}{2}}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $4y'' - 4y' + y = 0$ and y_p is a particular solution which is found using variations of parameters and also using undetermined coefficients to compare with.

Finding y_h

The first step is to put the ODE in standard form, with the coefficient of y'' being one. Hence it becomes

$$y'' - y' + \frac{1}{4}y = 4e^{\frac{t}{2}}$$

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 - r + \frac{1}{4} = 0$ or $\left(r - \frac{1}{2}\right)\left(r - \frac{1}{2}\right) = 0$, Therefore the roots are duplicate $r = \frac{1}{2}$. Hence the two fundamental solutions are

$$y_1 = e^{\frac{1}{2}t}$$
$$y_2 = te^{\frac{1}{2}t}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t} \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$\begin{aligned} W(t) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\frac{1}{2}t} & te^{\frac{1}{2}t} \\ \frac{1}{2}e^{\frac{1}{2}t} & e^{\frac{1}{2}t} + \frac{1}{2}te^{\frac{1}{2}t} \end{vmatrix} \\ &= \left(e^{\frac{1}{2}t}\right)\left(e^{\frac{1}{2}t} + \frac{1}{2}te^{\frac{1}{2}t}\right) - \left(te^{\frac{1}{2}t}\right)\left(\frac{1}{2}e^{\frac{1}{2}t}\right) \\ &= e^t + \frac{1}{2}te^t - \frac{1}{2}te^t \\ &= e^t \end{aligned}$$

Letting $g(t) = 4e^{\frac{t}{2}}$ therefore the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W} dt = -\int \frac{te^{\frac{1}{2}t}\left(4e^{\frac{t}{2}}\right)}{e^t} dt = -4\int t dt = -2t^2$$

And

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W} dt = \int \frac{e^{\frac{1}{2}t}\left(4e^{\frac{t}{2}}\right)}{e^t} dt = 4\int dt = 4t$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \left(-2t^2\right)e^{\frac{1}{2}t} + 4t\left(te^{\frac{1}{2}t}\right) \\ &= -2t^2 e^{\frac{1}{2}t} + 4t^2 e^{\frac{1}{2}t} \\ &= 2t^2 e^{\frac{1}{2}t} \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t} + 2t^2 e^{\frac{1}{2}t} \end{aligned}$$

Finding y_p using undetermined coefficients

From the form of $g(t) = 4e^{\frac{t}{2}}$ in the problem, we want to try $e^{\frac{t}{2}}$ but since $e^{\frac{t}{2}}$ is already one of the fundamental solutions, we then look at $te^{\frac{t}{2}}$ but this is also one fundamental solutions, then we look for $t^2 e^{\frac{t}{2}}$. Hence

$$y_p = At^2 e^{\frac{t}{2}}$$

Hence

$$\begin{aligned} y_p' &= A\left(2te^{\frac{t}{2}} + \frac{1}{2}t^2 e^{\frac{t}{2}}\right) \\ y_p'' &= A\left(2e^{\frac{t}{2}} + te^{\frac{t}{2}} + te^{\frac{t}{2}} + \frac{1}{4}t^2 e^{\frac{t}{2}}\right) \\ &= A\left(2e^{\frac{t}{2}} + 2te^{\frac{t}{2}} + \frac{1}{4}t^2 e^{\frac{t}{2}}\right) \end{aligned}$$

Plugging back into the original ODE gives

$$\begin{aligned} y_p'' - y_p' + \frac{1}{4}y_p &= 4e^{\frac{t}{2}} \\ A\left(2e^{\frac{t}{2}} + 2te^{\frac{t}{2}} + \frac{1}{4}t^2 e^{\frac{t}{2}}\right) - A\left(2te^{\frac{t}{2}} + \frac{1}{2}t^2 e^{\frac{t}{2}}\right) + \frac{1}{4}At^2 e^{\frac{t}{2}} &= 4e^{\frac{t}{2}} \end{aligned}$$

Dividing by $e^{\frac{t}{2}} \neq 0$ gives

$$\begin{aligned} A\left(2 + 2t + \frac{1}{4}t^2\right) - A\left(2t + \frac{1}{2}t^2\right) + \frac{1}{4}At^2 &= 4 \\ t(2A - 2A) + t^2\left(\frac{1}{4}A - \frac{1}{2}A + \frac{1}{4}A\right) + 2A &= 4 \\ A &= 2 \end{aligned}$$

Therefore

$$y_p = 2t^2 e^{\frac{t}{2}}$$

Which agrees with variation of parameters particular solution found earlier. Therefore the same general solution is obtained as expected. QED.

1.5 Section 3.6 problem 5

Find the general solution of $y'' + y = \tan t$ for $0 < t < \frac{\pi}{2}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + y = 0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 + 1 = 0$ or $r = \pm i$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= \cos t \\ y_2 &= \sin t \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos t + c_2 \sin t \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

Let $g(t) = \tan t$, therefore the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$\begin{aligned} u_1(t) &= -\int \frac{y_2(t)\, g(t)}{W(t)} dt = -\int \frac{\sin t \tan t}{1} dt = -\int \sin t \frac{\sin t}{\cos t} dt = -\int \frac{\sin^2 t}{\cos t} dt \\ &= -\int \frac{1-\cos^2 t}{\cos t} dt = \int \frac{\cos^2 t - 1}{\cos t} dt = \int \cos t - \frac{1}{\cos t} dt \\ &= \int \cos t dt - \int \frac{1}{\cos t} dt \\ &= \sin t - \int \sec t dt \\ &= \sin t - \ln(\sec(t) + \tan(t)) \end{aligned}$$

And

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(t)} dt = \int \frac{\cos t \tan t}{1} dt = \int \cos t \frac{\sin t}{\cos t}\, dt = \int \sin t\, dt = -\cos t$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= (\sin t - \ln(\sec(t) + \tan(t))) \cos t + (-\cos t) \sin t \\ &= -\cos(t) \ln(\sec(t) + \tan(t)) \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \cos t + c_2 \sin t - \cos(t) \ln(\sec(t) + \tan(t)) \end{aligned}$$

1.6 Section 3.6 problem 6

Find the general solution of $y'' + 9y = 9\sec^2 3t$ for $0 < t < \frac{\pi}{6}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 9y = 0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 + 9 = 0$ or $r = \pm 3i$. Hence the two fundamental solutions are

$$\begin{aligned} y_1 &= \cos 3t \\ y_2 &= \sin 3t \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos 3t + c_2 \sin 3t \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 3t & \sin 3t \\ -3\sin 3t & 3\cos 3t \end{vmatrix} = 3\cos^2 t + 3\sin^2 t = 3$$

Let $g(t) = \frac{9}{\cos^2 3t}$, therefore the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(t)} dt = -\int \frac{9\sin(3t)}{3\cos^2(3t)} dt = -3\int \frac{\sin(3t)}{\cos^2(3t)} dt$$

Let $u=\cos(3t)$, hence $\frac{du}{dt}=-3\sin 3t\rightarrow dt=\frac{du}{-3\sin 3t}$ and the above integral becomes

$$u_1(t)=-3\int\frac{\sin(3t)}{u^2}\frac{du}{-3\sin 3t}=\int\frac{1}{u^2}du=\frac{-1}{u}=\frac{-1}{\cos 3t}=-\sec(3t)$$

And

$$u_2(t)=\int\frac{y_1(t)g(t)}{W(t)}dt=\int\frac{9\cos 3t}{3\cos^2(3t)}dt=3\int\frac{1}{\cos(3t)}\ dt=3\int\sec(3t)\ dt=\ln(\sec(3t)+\tan(3t))$$

Hence the particular solution becomes

$$\begin{aligned}y_p&=u_1y_1+u_2y_2\\&=-\sec(3t)\cos 3t+\ln(\sec(3t)+\tan(3t))\sin 3t\\&=-1+\ln(\sec(t)+\tan(t))\sin 3t\end{aligned}$$

Therefore the general solution is

$$\begin{aligned}y&=y_h+y_p\\&=c_1\cos 3t+c_2\sin 3t-1+\sin 3t\ln(\sec(t)+\tan(t))\end{aligned}$$

1.7 Section 3.6 problem 7

Find the general solution of $y''+4y'+4y=t^{-2}e^{-2t}$ for $t>0$

solution

The general solution is

$$y=y_h+y_p$$

Where y_h is the solution to the homogenous ode $y''+4y'+4y=0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2+4r+4=0$ or $(r+2)(r+2)=0$. Hence double root $r=-2$ and the fundamental solutions are

$$\begin{aligned}y_1&=e^{-2t}\\y_2&=te^{-2t}\end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned}y_h&=c_1y_1+c_2y_2\\&=c_1e^{-2t}+c_2te^{-2t}\end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$\begin{aligned}W(t)&=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}=\begin{vmatrix}e^{-2t}&te^{-2t}\\-2e^{-2t}&e^{-2t}-2te^{-2t}\end{vmatrix}=e^{-2t}\left(e^{-2t}-2te^{-2t}\right)+2e^{-2t}\left(te^{-2t}\right)\\&=e^{-4t}-2te^{-4t}+2te^{-4t}\\&=e^{-4t}\end{aligned}$$

Let $g(t)=t^{-2}e^{-2t}$, therefore the particular solution is

$$y_p(t)=u_1(t)y_1(t)+u_2(t)y_2(t)$$

Where

$$u_1(t)=-\int\frac{y_2(t)g(t)}{W(t)}dt=-\int\frac{te^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt=-\int t^{-1}dt=-\ln|t|$$

And

$$u_2(t)=\int\frac{y_1(t)g(t)}{W(t)}dt=\int\frac{e^{-2t}t^{-2}e^{-2t}}{e^{-4t}}dt=\int t^{-2}dt=-\frac{1}{t}$$

Hence the particular solution becomes

$$\begin{aligned}y_p&=u_1y_1+u_2y_2\\&=-\ln|t|e^{-2t}-\frac{1}{t}te^{-2t}\\&=-e^{-2t}\ln|t|-e^{-2t}\end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t}\ln|t| - e^{-2t} \end{aligned}$$

We can combine e^{-2t} that shows up from the particular solution with the $c_1 e^{-2t}$ term from the homogenous solution, since c_1 is arbitrary constant, which simplifies the above to

$$y = c_1 e^{-2t} + c_2 t e^{-2t} - e^{-2t}\ln|t|$$

1.8 Section 3.6 problem 8

Find the general solution of $y'' + 4y = 3\frac{1}{\sin 2t}$ for $0 < t < \frac{\pi}{2}$

solution

The general solution is

$$y = y_h + y_p$$

Where y_h is the solution to the homogenous ode $y'' + 4y = 0$ and y_p is a particular solution which is found using variations of parameters.

Finding y_h

Since ODE has constant coefficients, then the characteristic equation is used. It is given by $r^2 + 4 = 0$ or $r = \pm 2i$. The fundamental solutions are

$$\begin{aligned} y_1 &= \cos 2t \\ y_2 &= \sin 2t \end{aligned}$$

And the homogenous solution is therefore given by

$$\begin{aligned} y_h &= c_1 y_1 + c_2 y_2 \\ &= c_1 \cos 2t + c_2 \sin 2t \end{aligned}$$

Finding y_p using variation of parameters

First step is to find Wronskian W given by

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2$$

Let $g(t) = \frac{3}{\sin 2t}$, therefore the particular solution is

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

Where

$$u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(t)} dt = -\int \frac{\sin(2t)\, 3}{2\sin 2t} dt = -\frac{3}{2}\int dt = \frac{-3}{2} t$$

And

$$u_2(t) = \int \frac{y_1(t)\, g(t)}{W(t)} dt = \int \frac{\cos(2t)\, 3}{2\sin 2t} dt = \frac{3}{2}\int \frac{\cos(2t)}{\sin(2t)} dt$$

Let $u = \sin 2t \rightarrow du = 2\cos 2t dt$ and the above integral becomes

$$u_2(t) = \frac{3}{2}\int \frac{\cos(2t)}{u}\frac{du}{2\cos 2t} = \frac{3}{4}\int \frac{1}{u} du = \frac{3}{4}\ln|u| = \frac{3}{4}\ln|\sin 2t|$$

Hence the particular solution becomes

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 \\ &= \frac{-3}{2} t\cos 2t + \frac{3}{4}\ln|\sin 2t|\sin 2t \end{aligned}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= c_1 \cos 2t + c_2 \sin 2t - \frac{3}{2} t\cos 2t + \frac{3}{4}\sin(2t)\ln|\sin 2t| \end{aligned}$$