HW 3, Math 319, Fall 2016

Nasser M. Abbasi (Discussion section 44272, Th 4:35PM-5:25 PM)

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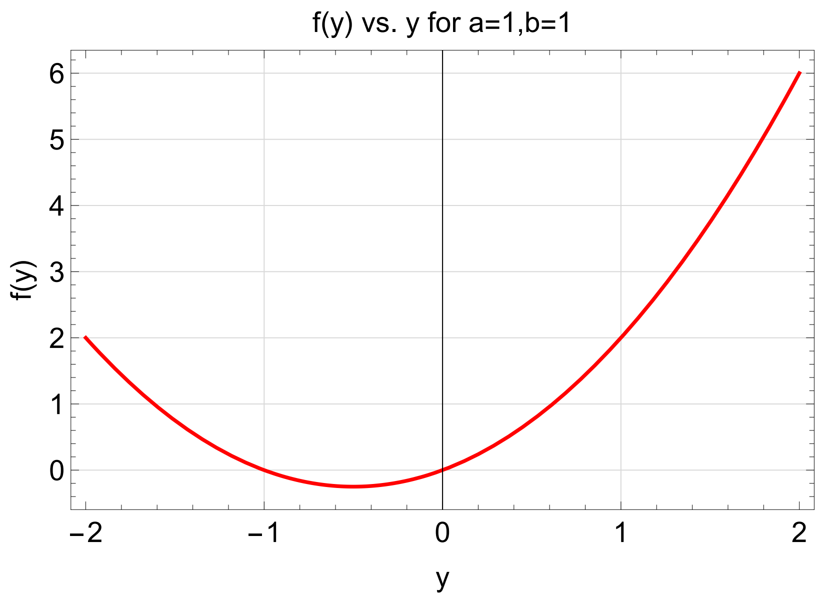
0.1 Section 2.5 problem 1

Sketch the graph of $f(y)$ vs. y and determine critical points and classify each as stable or not stable. $\frac{dy}{dt} = ay + by^2; a > 0, b > 0, y_0 \geq 0$

$$f(y) = ay + by^2$$

The following is sketch of $f(y)$ for $a = 1, b = 1$. Or $f(y) = y + y^2$



The critical points are solution of

$$f(y) = 0$$
$$y(a + by) = 0$$

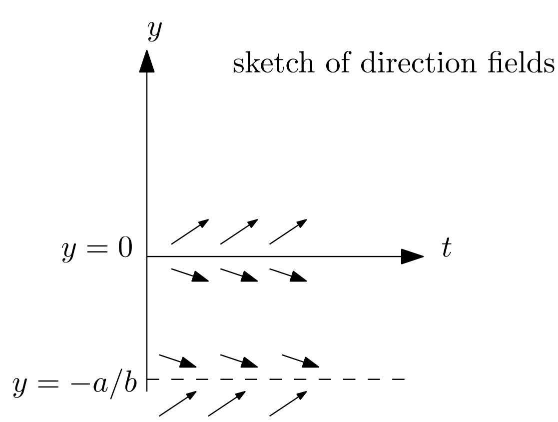
Therefore the critical points are

$$y_1 = 0$$
$$y_2 = \frac{-a}{b} \qquad \text{(not in domain)}$$

Or

$$y_1 = 0$$
$$y_2 = -1$$

Notice that since $a > 0, b > 0$, then $y_2 < 0$. Here is sketch of the direction field.



Therefore $y = 0$ is not stable, and $y = \frac{-a}{b}$ is stable. However, since $y_0 \geq 0$, then $y = \frac{-a}{b}$ will not be reached. Per discussion, only lines above y_0 are to be considered. In the following problem, since $-\infty < y < \infty$, then all lines will be considered. This is the only difference between this problem and the next one.

0.1.1 Appendix

This is extra. The problem is also solved to determined which is the stable and which is the unstable critical points. But using direction field as above, is simpler method. The ODE is $\frac{dy}{dt} = ay + by^2$. This is separable

$$\frac{dy}{y\left(a+by\right)} = dx$$

Integrating

$$\int \frac{dy}{y\left(a+by\right)} = \int dx \tag{1}$$

For the integral $\int \frac{dy}{y\left(a+by\right)}$, partial fractions is used to split it. Let $\frac{A}{y} + \frac{B}{a+by} = \frac{1}{y\left(a+by\right)}$, therefore

$$\begin{aligned} A\left(a+by\right) + By &= 1 \\ Aa + y\left(Ab+B\right) &= 1 \end{aligned}$$

Hence comparing terms, gives

$$\begin{aligned} Ab + B &= 0 \\ Aa &= 1 \end{aligned}$$

Solving for A, B, gives

$$\begin{aligned} A &= \frac{1}{a} \\ B &= -\frac{b}{a} \end{aligned}$$

Hence the integral becomes

$$\begin{aligned} \int \frac{dy}{y\left(a+by\right)} &= \frac{1}{a}\int \frac{1}{y}dy - \frac{b}{a}\int \frac{dy}{a+by} \\ &= \frac{1}{a}\ln\left|y\right| - \frac{b}{a}\int \frac{dy}{a+by} \end{aligned}$$

Let $u = a+by, \rightarrow du = bdy$, hence $\int \frac{dy}{a+by} = \frac{1}{b}\int \frac{du}{u} = \frac{1}{b}\ln\left|u\right| = \frac{1}{b}\ln\left|a+by\right|$ and the above becomes

$$\begin{aligned} \int \frac{dy}{y\left(a+by\right)} &= \frac{1}{a}\ln\left|y\right| - \frac{b}{a}\frac{1}{b}\ln\left|a+by\right| \\ &= \frac{1}{a}\ln\left|y\right| - \frac{1}{a}\ln\left|a+by\right| \\ &= \frac{1}{a}\left(\ln\left|y\right| - \ln\left|a+by\right|\right) \\ &= \frac{1}{a}\ln\left|\frac{y}{a+by}\right| \end{aligned}$$

Hence (1) becomes

$$\frac{1}{a}\ln\left|\frac{y}{a+by}\right| = x + c$$

Where c is constant of integration. Therefore

$$\ln\left|\frac{y}{a+by}\right| = ax + ac$$

Let $ac = c_0$ a new constant. Then

$$\begin{aligned} \ln\left|\frac{y}{a+by}\right| &= ax + c_0 \\ \left|\frac{y}{a+by}\right| &= e^{ax+c_0} \\ \frac{y}{a+by} &= C_0 e^{ax} \end{aligned}$$

Solving for y

$$\begin{aligned} y &= aC_0e^{ax} + byC_0e^{ax} \\ y\left(1 - bC_0e^{ax}\right) &= aC_0e^{ax} \\ y &= \frac{aC_0e^{ax}}{\left(1 - bC_0e^{ax}\right)} \end{aligned}$$

$$\lim_{x\rightarrow\infty} y = \lim_{x\rightarrow\infty} \frac{aC_0}{\frac{1}{e^{ax}} - bC_0}$$

Since $a > 0$ then $e^{ax} \to \infty$ as $x \to \infty$ and the above simplifies to

$$\lim_{x\to\infty} y = \frac{aC_0}{-bC_0} = -\frac{a}{b}$$

Since the limit goes to the point $-\frac{a}{b}$ then this point is stable equilibrium and the point $y = 0$ is not stable.

0.2 Section 2.5 problem 2

Sketch the graph of $f(y)$ vs. y and determine critical points and classify each as stable or not stable.
$\frac{dy}{dt} = ay + by^2; a > 0, b > 0, -\infty < y_0 < \infty$

$$f(y) = ay + by^2$$

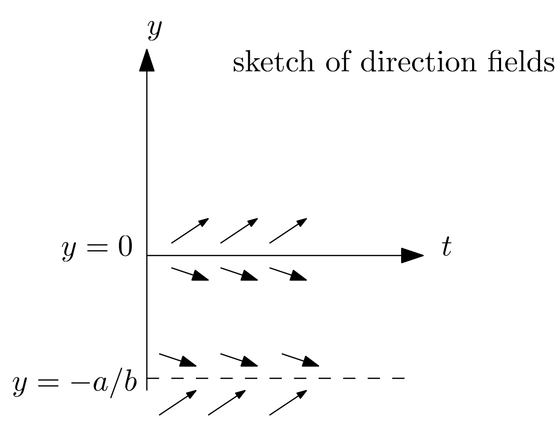
This is the same problem as above, with same direction field. But now the phase line will include both critical points. The critical points are from above

$$y_1 = 0$$
$$y_2 = \frac{-a}{b}$$

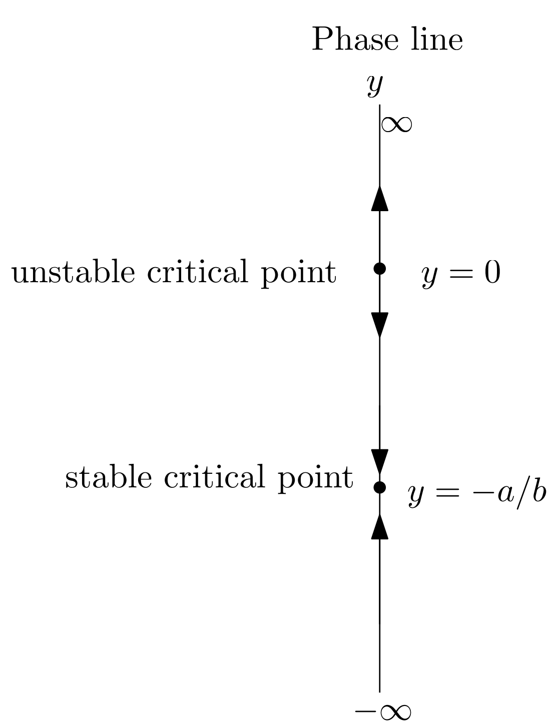
Or

$$y_1 = 0$$
$$y_2 = -1$$

For $a = 1, b = 1$. Here is sketch of the direction field.



Therefore $y = 0$ is not stable, and $y = \frac{-a}{b}$ is stable. The following is the phase line for this problem



0.3 Section 2.6 problem 1

Determine if $(2x+3)+\left(2y-2\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{(2x+3)}^{M(x,y)}+\overbrace{\left(2y-2\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=0$$
$$\frac{\partial N}{\partial x}=0$$

Therefore, it is exact. Before solving, it is always best to apply singular point analysis on $f\left(x,y\right)$ in order to determined if the solution is unique or not. Writing the ODE as $\frac{dy}{dx}=f\left(x,y\right)=\frac{-(2x+3)}{(2y-2)}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $y=1$. Now the ODE is solved. Setting up the two equations

$$\frac{\partial\Psi}{\partial x}=M=2x+3 \tag{1}$$
$$\frac{\partial\Psi}{\partial y}=N=2y-2 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\Psi}{\partial x}dx=\int 2x+3dx$$
$$\Psi=x^2+3x+f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial\Psi}{\partial y}=f'\left(y\right)$$

Comparing the above to (2) shows that $f'\left(y\right)=2y-2$. By integrating $f\left(y\right)$ is found to be

$$f\left(y\right)=y^2-2y+c$$

Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x,y(x)\right)$

$$\Psi\left(x,y(x)\right)=x^2+3x+\left(y^2-2y+c\right)$$

However, since $\frac{d}{dx}\Psi=0$, then $\Psi=c_1$, where c_1 is some constant. Therefore the above can be written as

$$x^2+3x+\left(y^2-2y+c\right)=c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$\boxed{x^2+3x+y^2-2y=c_0 \qquad y\neq 1} \tag{4}$$

0.4 Section 2.6 problem 2

Determine if $\left(2x+4y\right)+\left(2x-2y\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{\left(2x+4y\right)}^{M(x,y)}+\overbrace{\left(2x-2y\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=4$$
$$\frac{\partial N}{\partial x}=2$$

Therefore the ODE is not exact.

0.5 Section 2.6 problem 3

Determine if $\left(3x^2-2xy+2\right)+\left(6y^2-x^2+3\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{\left(3x^2-2xy+2\right)}^{M(x,y)}+\overbrace{\left(6y^2-x^2+3\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=-2x$$
$$\frac{\partial N}{\partial x}=-2x$$

Hence the ODE is exact. Writing the ODE as $\frac{dy}{dx}=f\left(x,y\right)=\frac{-\left(3x^2-2xy+2\right)}{\left(6y^2-x^2+3\right)}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $y=\pm\sqrt{\frac{1}{6}x^2-\frac{1}{2}}$. Now the ODE is solved. Setting up the two equations

$$\frac{\partial\Psi}{\partial x}=M=3x^2-2xy+2 \tag{1}$$
$$\frac{\partial\Psi}{\partial y}=N=6y^2-x^2+3 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\Psi}{\partial x}dx=\int\left(3x^2-2xy+2\right)dx$$
$$\Psi=x^3-x^2y+2x+f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial\Psi}{\partial y}=-x^2+f'\left(y\right)$$

Equating the above to (2) gives

$$-x^2+f'\left(y\right)=6y^2-x^2+3$$
$$f'\left(y\right)=6y^2+3$$

Integrating the above w.r.t. y gives

$$f\left(y\right)=2y^3+3y+c$$

Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x,y(x)\right)$

$$\Psi\left(x,y(x)\right)=x^3-x^2y+2x+2y^3+3y+c$$

However, since $\frac{d}{dx}\Psi=0$, then $\Psi=c_1$, where c_1 is some constant. Therefore the above can be written as

$$x^3-x^2y+2x+2y^3+3y+c=c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$\boxed{x^3-x^2y+2x+2y^3+3y=c_o}$$

The above solution is valid only for $y\neq\pm\sqrt{\frac{1}{6}x^2-\frac{1}{2}}$

0.6 Section 2.6 problem 4

Determine if $\left(2xy^2+2y\right)+\left(2x^2y+2x\right)\frac{dy}{dx}=0$ is exact and solve if so.

$$\overbrace{\left(2xy^2+2y\right)}^{M(x,y)}+\overbrace{\left(2x^2y+2x\right)}^{N(x,y)}\frac{dy}{dx}=0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = 4xy + 2$$
$$\frac{\partial N}{\partial x} = 4xy + 2$$

Hence the <u>ODE is exact</u>. Writing the ODE as $\frac{dy}{dx} = f\left(x, y\right) = \frac{-\left(2xy^2+2y\right)}{\left(2x^2y+2x\right)}$ shows that this is <u>non-linear first order</u> and applying theorem 2, shows that $f(x)$ is <u>not continuous at $y = \frac{-1}{x}$</u> for $x \neq 0$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial \Psi}{\partial x} = M = 2xy^2 + 2y \tag{1}$$
$$\frac{\partial \Psi}{\partial y} = N = 2x^2y + 2x \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int \left(2xy^2 + 2y\right) dx$$
$$\Psi = x^2y^2 + 2yx + f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial \Psi}{\partial y} = 2x^2y + 2x + f'\left(y\right)$$

Equating the above to (2) gives

$$2x^2y + 2x + f'\left(y\right) = 2x^2y + 2x$$
$$f'\left(y\right) = 0$$

Integrating the above w.r.t. y gives

$$f\left(y\right) = c$$

Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x, y(x)\right)$

$$\Psi\left(x, y(x)\right) = x^2y^2 + 2yx + c$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$x^2y^2 + 2yx + c = c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$x^2y^2 + 2yx = c_o \qquad y \neq \frac{-1}{x}, x \neq 0$$

0.7 Section 2.6 problem 5

Determine if $\frac{dy}{dx} = \frac{-\left(ax+by\right)}{bx+cy}$ is exact and solve if so.

$$\overbrace{\left(ax + by\right)}^{M(x,y)} + \overbrace{\left(bx + cy\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = b$$
$$\frac{\partial N}{\partial x} = b$$

Hence the <u>ODE is exact</u>. Writing the ODE as $\frac{dy}{dx} = f\left(x, y\right) = \frac{-\left(ax+by\right)}{bx+cy}$ shows that this is <u>non-linear first order</u> and applying theorem 2, shows that $f(x)$ is <u>not continuous at $y = \frac{-bx}{c}$</u>. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial \Psi}{\partial x} = M = ax + by \tag{1}$$
$$\frac{\partial \Psi}{\partial y} = N = bx + cy \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int \left(ax + by\right) dx$$
$$\Psi = \frac{a}{2}x^2 + byx + f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial \Psi}{\partial y} = bx + f'\left(y\right)$$

Equating the above to (2) gives

$$bx + f'\left(y\right) = bx + cy$$
$$f'\left(y\right) = cy$$

Integrating the above w.r.t. y gives

$$f\left(y\right) = \frac{1}{2}cy^2 + k$$

Where k is constant. Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x, y\left(x\right)\right)$

$$\Psi\left(x, y\left(x\right)\right) = \frac{a}{2}x^2 + byx + \frac{1}{2}cy^2 + k$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = k_1$, where k_1 is some constant. Therefore the above can be written as

$$\frac{a}{2}x^2 + byx + \frac{1}{2}cy^2 + k = k_1$$

Combining constants and simplifying gives the implicit solution for $y\left(x\right)$ as

$$\frac{a}{2}x^2 + byx + \frac{1}{2}cy^2 = k_o$$
$$ax^2 + 2byx + cy^2 = 2k_o = k_2$$

Summary of solution

$$ax^2 + 2byx + cy^2 = k_2 \qquad y \neq \frac{-bx}{c}$$

0.8 Section 2.6 problem 6

Determine if $\frac{dy}{dx} = \frac{-\left(ax-by\right)}{bx-cy}$ is exact and solve if so.

$$\overbrace{\left(ax - by\right)}^{M(x,y)} + \overbrace{\left(bx - cy\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = -b$$
$$\frac{\partial N}{\partial x} = b$$

Hence the ODE is not exact.

0.9 Section 2.6 problem 7

Determine if $\left(e^x \sin y - 2y \sin x\right) + \left(e^x \cos y + 2\cos x\right)\frac{dy}{dx} = 0$ is exact and solve if so.

$$\overbrace{\left(e^x \sin y - 2y \sin x\right)}^{M(x,y)} + \overbrace{\left(e^x \cos y + 2\cos x\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = e^x \cos y - 2\sin x$$
$$\frac{\partial N}{\partial x} = e^x \cos y - 2\sin x$$

Hence the ODE is exact. Writing the ODE as $\frac{dy}{dx} = f\left(x, y\right) = \frac{-\left(e^x \sin y - 2y\sin x\right)}{e^x \cos y + 2\cos x}$ shows that this is non-linear first order and applying theorem 2, shows that $f\left(x\right)$ is not continuous at $y = \arccos\left(\frac{-2\cos x}{e^x}\right)$.

Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial \Psi}{\partial x} = M = e^x \sin y - 2y \sin x \tag{1}$$

$$\frac{\partial \Psi}{\partial y} = N = e^x \cos y + 2\cos x \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x} dx = \int \left(e^x \sin y - 2y \sin x\right) dx$$

$$\Psi = e^x \sin y + 2y \cos x + f\left(y\right) \tag{3}$$

Therefore

$$\frac{\partial \Psi}{\partial y} = e^x \cos y + 2\cos x + f'\left(y\right)$$

Equating the above to (2) gives

$$e^x \cos y + 2\cos x + f'\left(y\right) = e^x \cos y + 2\cos x$$

$$f'\left(y\right) = 0$$

Hence

$$f\left(y\right) = c$$

Where c is constant. Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x, y\left(x\right)\right)$

$$\Psi\left(x, y\left(x\right)\right) = e^x \sin y + 2y \cos x + c$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$e^x \sin y + 2y \cos x + c = c_1$$

Combining constants and simplifying gives the implicit solution for $y\left(x\right)$ as

$$e^x \sin y + 2y \cos x = c_o \qquad y \neq \arccos\left(\frac{-2\cos x}{e^x}\right)$$

Since c_0 is constant, then $c_0 = 0$ is allowed value. This implies $e^x \sin y + 2y \cos x = 0$ is allowed, which means $y\left(x\right) = 0$ is solution also, since when $y = 0$ then $e^x \sin\left(0\right) + 2\left(0\right)\cos x$ gives zero. Hence a second solution is

$$y\left(x\right) = 0$$

Summary

$$\begin{cases} e^x \sin y + 2y \cos x = c_o & y \neq \arccos\left(\frac{-2\cos x}{e^x}\right) \\ y\left(x\right) = 0 & c_0 = 0 \end{cases}$$

0.10 Section 2.6 problem 8

Determine if $\left(e^x \sin y + 3y\right) - \left(3x - e^x \sin y\right)\frac{dy}{dx} = 0$ is exact and solve if so.

$$\overbrace{\left(e^x \sin y + 3y\right)}^{M(x,y)} + \overbrace{\left(-3x + e^x \sin y\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = e^x \sin y + 3$$

$$\frac{\partial N}{\partial x} = -3 + e^x \sin y$$

Hence the ODE is not exact

0.11 Section 2.6 problem 9

Determine if $\left(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x\right) + \left(xe^{xy}\cos 2x - 3\right)\frac{dy}{dx} = 0$ is exact and solve if so.

$$\overbrace{\left(ye^{xy}\cos 2x - 2e^{xy}\sin 2x + 2x\right)}^{M(x,y)} + \overbrace{\left(xe^{xy}\cos 2x - 3\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=e^{xy}\cos 2x+yxe^{xy}\cos 2x-2xe^{xy}\sin 2x$$
$$\frac{\partial N}{\partial x}=e^{xy}\cos 2x+xye^{xy}\cos 2x-2xe^{xy}\sin 2x$$

Hence the ODE is exact. Now the ODE is solved. Setting up the two equations

$$\frac{\partial \Psi}{\partial x}=M=ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x \tag{1}$$
$$\frac{\partial \Psi}{\partial y}=N=xe^{xy}\cos 2x-3 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \Psi}{\partial x}dx=\int \left(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x\right)dx$$
$$\Psi=y\int e^{xy}\cos 2xdx-2\int e^{xy}\sin 2xdx+2\int xdx+f\left(y\right) \tag{3}$$

Let

$$I=\int e^{xy}\cos 2xdx$$

Using integration by parts. Let $u=\cos 2x, dv=e^{xy}\rightarrow du=-2\sin\left(2x\right), v=\frac{e^{xy}}{y}$, hence

$$I=uv-\int vdu$$
$$=\frac{e^{xy}}{y}\cos 2x+\frac{2}{y}\int e^{xy}\sin 2xdx$$

Applying integration by parts again to $\int e^{xy}\sin 2xdx$, where now $u=\sin 2x, dv=e^{xy}\rightarrow du=2\cos\left(2x\right), v=\frac{e^{xy}}{y}$ Therefore the above becomes

$$I=\frac{e^{xy}}{y}\cos 2x+\frac{2}{y}\left(\frac{e^{xy}}{y}\sin 2x-\int \frac{e^{xy}}{y}2\cos 2xdx\right)$$
$$=\frac{e^{xy}}{y}\cos 2x+\frac{2}{y}\left(\frac{e^{xy}}{y}\sin 2x-\frac{2}{y}\int e^{xy}\cos 2xdx\right)$$

But $I=\int e^{xy}\cos 2xdx$ and the above becomes

$$I=\frac{e^{xy}}{y}\cos 2x+\frac{2}{y}\left(\frac{e^{xy}}{y}\sin 2x-\frac{2}{y}I\right)$$

Solving for I

$$I=\frac{e^{xy}}{y}\cos 2x+\frac{2e^{xy}}{y^2}\sin 2x-\frac{4}{y^2}I$$
$$I+\frac{4}{y^2}I=\frac{e^{xy}}{y}\cos 2x+\frac{2e^{xy}}{y^2}\sin 2x$$
$$I\left(\frac{y^2+4}{y^2}\right)=\frac{e^{xy}}{y}\cos 2x+\frac{2e^{xy}}{y^2}\sin 2x$$
$$I=\frac{y^2}{y^2+4}\frac{e^{xy}}{y}\cos 2x+\frac{y^2}{y^2+4}\frac{2e^{xy}}{y^2}\sin 2x$$

Therefore

$$\int e^{xy}\cos 2xdx=\frac{ye^{xy}}{y^2+4}\cos 2x+\frac{2e^{xy}}{y^2+4}\sin 2x \tag{4}$$

Similarly $I=\int e^{xy}\sin 2xdx$ is solve by integration by parts. Let $ev=e^{xy}, u=\sin 2x\rightarrow du=2\cos 2x, v=\frac{e^{xy}}{y}$, hence

$$I=\frac{e^{xy}}{y}\sin 2x-\frac{2}{y}\int e^{xy}\cos 2xdx$$

For $\int e^{xy}\cos 2xdx$, let $u=\cos 2x, dv=e^{xy}\rightarrow du=-2\sin 2x, v=\frac{e^{xy}}{y}$ and the above becomes

$$I=\frac{e^{xy}}{y}\sin 2x-\frac{2}{y}\left(\frac{e^{xy}}{y}\cos 2x+\frac{2}{y}\int e^{xy}\sin 2xdx\right)$$

But $\int e^{xy}\sin 2xdx=I$ and the above becomes

$$I=\frac{e^{xy}}{y}\sin 2x-\frac{2}{y}\left(\frac{e^{xy}}{y}\cos 2x+\frac{2}{y}I\right)$$

Solving for I

$$
\begin{aligned}
I &= \frac{e^{xy}}{y}\sin 2x - \left(\frac{2e^{xy}}{y^2}\cos 2x + \frac{4}{y^2}I\right)\\
I + \frac{4}{y^2}I &= \frac{e^{xy}}{y}\sin 2x - \frac{2e^{xy}}{y^2}\cos 2x\\
I\left(\frac{y^2+4}{y^2}\right) &= \frac{e^{xy}}{y}\sin 2x - \frac{2e^{xy}}{y^2}\cos 2x\\
I &= \frac{y^2}{y^2+4}\frac{e^{xy}}{y}\sin 2x - \frac{y^2}{y^2+4}\frac{2e^{xy}}{y^2}\cos 2x
\end{aligned}
$$

Hence

$$
\int e^{xy}\sin 2x dx = \frac{ye^{xy}}{y^2+4}\sin 2x - \frac{2e^{xy}}{y^2+4}\cos 2x \tag{5}
$$

Substituting (4,5) into (3) gives

$$
\Psi = y\left(\frac{ye^{xy}}{y^2+4}\cos 2x + \frac{2e^{xy}}{y^2+4}\sin 2x\right) - 2\left(\frac{ye^{xy}}{y^2+4}\sin 2x - \frac{2e^{xy}}{y^2+4}\cos 2x\right) + x^2 + f\left(y\right)
$$

Simplifying

$$
\begin{aligned}
\Psi &= \frac{y^2e^{xy}}{y^2+4}\cos 2x + \frac{2ye^{xy}}{y^2+4}\sin 2x - \frac{2ye^{xy}}{y^2+4}\sin 2x + \frac{4e^{xy}}{y^2+4}\cos 2x + x^2 + f\left(y\right)\\
&= \frac{y^2e^{xy}}{y^2+4}\cos 2x + \frac{4e^{xy}}{y^2+4}\cos 2x + x^2 + f\left(y\right)\\
&= e^{xy}\cos(2x)\left(\frac{y^2}{y^2+4} + \frac{4}{y^2+4}\right) + x^2 + f\left(y\right)\\
&= e^{xy}\cos(2x)\left(\frac{4+y^2}{y^2+4}\right) + x^2 + f\left(y\right)
\end{aligned}
$$

Therefore

$$
\Psi = e^{xy}\cos(2x) + x^2 + f\left(y\right) \tag{6}
$$

Therefore

$$
\frac{\partial \Psi}{\partial y} = xe^{xy}\cos(2x) + f'\left(y\right)
$$

Equating the above to (2) gives

$$
\begin{aligned}
xe^{xy}\cos(2x) + f'\left(y\right) &= xe^{xy}\cos 2x - 3\\
f'\left(y\right) &= -3
\end{aligned}
$$

Hence

$$
f\left(y\right) = -3y + c
$$

Where c is constant. Substituting $f\left(y\right)$ back into (6) gives

$$
\Psi\left(x, y(x)\right) = e^{xy}\cos(2x) + x^2 - 3y + c
$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$
e^{xy}\cos(2x) + x^2 - 3y + c = c_1
$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$
e^{xy}\cos(2x) + x^2 - 3y = c_o
$$

0.12 Section 2.6 problem 10

Determine if $\left(\frac{y}{x} + 6x\right) + (\ln x - 2)\frac{dy}{dx} = 0; x > 0$ is exact and solve if so.

$$
\overbrace{\left(\frac{y}{x} + 6x\right)}^{M(x,y)} + \overbrace{(\ln x - 2)}^{N(x,y)}\frac{dy}{dx} = 0
$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{1}{x}\\
\frac{\partial N}{\partial x} &= \frac{1}{x}
\end{aligned}
$$

Hence the ODE is exact. Writing the ODE as $\frac{dy}{dx} = f\left(x,y\right) = \frac{-\left(\frac{y}{x}+6x\right)}{\ln x-2}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $x = e^2$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial\Psi}{\partial x} = M = \frac{y}{x} + 6x \tag{1}$$

$$\frac{\partial\Psi}{\partial y} = N = \ln x - 2 \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\Psi}{\partial x}dx = \int \left(\frac{y}{x} + 6x\right)dx$$

$$\Psi = y\ln(x) + 3x^2 + f\left(y\right) \tag{3}$$

No need to use $\ln|x|$ since the problem said that $x > 0$. Therefore

$$\frac{\partial\Psi}{\partial y} = \ln(x) + f'\left(y\right)$$

Equating the above to (2) gives

$$\ln(x) + f'\left(y\right) = \ln(x) - 2$$

$$f'\left(y\right) = -2$$

Hence

$$f\left(y\right) = -2y + c$$

Where c is constant. Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x, y(x)\right)$

$$\Psi\left(x, y(x)\right) = y\ln(x) + 3x^2 - 2y + c$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$y\ln(x) + 3x^2 - 2y + c = c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$y\ln(x) + 3x^2 - 2y = c_o \qquad x > 0; x \neq e^2$$

0.13 Section 2.6 problem 11

Determine if $\left(x\ln\left(y\right) + xy\right) + \left(y\ln(x) + xy\right)\frac{dy}{dx} = 0; x > 0; y > 0$ is exact and solve if so.

$$\overbrace{\left(x\ln\left(y\right) + xy\right)}^{M(x,y)} + \overbrace{\left(y\ln(x) + xy\right)}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y} = \frac{x}{y} + x = \frac{x\left(1+y\right)}{y}$$

$$\frac{\partial N}{\partial x} = \frac{y}{x} + y = \frac{y(1+x)}{x}$$

Hence this ODE is not exact.

0.14 Section 2.6 problem 12

Determine if $\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}} + \frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}\frac{dy}{dx} = 0$ is exact and solve if so.

$$\overbrace{\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}}^{M(x,y)} + \overbrace{\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}}^{N(x,y)}\frac{dy}{dx} = 0$$

ODE is exact if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Applying this to the above gives

$$\frac{\partial M}{\partial y}=\frac{-3}{2}\frac{x}{\left(x^2+y^2\right)^{\frac{5}{2}}}(2y)=\frac{-3xy}{\left(x^2+y^2\right)^{\frac{5}{2}}}$$
$$\frac{\partial N}{\partial x}=\frac{-3}{2}\frac{y}{\left(x^2+y^2\right)^{\frac{5}{2}}}(2x)=\frac{-3xy}{\left(x^2+y^2\right)^{\frac{5}{2}}}$$

Hence ODE is exact. Writing the ODE as $\frac{dy}{dx}=f\left(x,y\right)=\frac{-\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}}{\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}}=\frac{-x}{y}$ shows that this is non-linear first order and applying theorem 2, shows that $f(x)$ is not continuous at $y=0$. Now the ODE is solved under these assumptions. Setting up the two equations

$$\frac{\partial \Psi}{\partial x}=M=\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}} \qquad (1)$$
$$\frac{\partial \Psi}{\partial y}=N=\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}} \qquad (2)$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial \Psi}{\partial x}dx=\int\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}dx$$

Let $u=x^2+y^2$, then $\frac{du}{dx}=2x$. Substituting this into $\int\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}dx$ gives

$$\begin{aligned}\int\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}dx&=\int\frac{x}{u^{\frac{3}{2}}}\frac{du}{2x}\\&=\frac{1}{2}\int u^{\frac{-3}{2}}du\\&=\frac{1}{2}\frac{u^{\frac{-1}{2}}}{\frac{-1}{2}}+f\left(y\right)\\&=-\frac{1}{u^{\frac{1}{2}}}+f\left(y\right)\\&=-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}+f\left(y\right)\end{aligned}$$

Hence

$$\Psi=-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}+f\left(y\right) \qquad (3)$$

Therefore

$$\begin{aligned}\frac{\partial \Psi}{\partial y}&=\frac{1}{2}\left(x^2+y^2\right)^{-\frac{3}{2}}\left(2y\right)+f'\left(y\right)\\&=\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}+f'\left(y\right)\end{aligned}$$

Equating the above to (2) gives

$$\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}+f'\left(y\right)=\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$
$$f'\left(y\right)=0$$

Hence

$$f\left(y\right)=c$$

Where c is constant. Substituting $f\left(y\right)$ back into (3) gives $\Psi\left(x,y\left(x\right)\right)$

$$\Psi\left(x,y\left(x\right)\right)=-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}}+c$$

However, since $\frac{d}{dx}\Psi = 0$, then $\Psi = c_1$, where c_1 is some constant. Therefore the above can be written as

$$-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}} + c = c_1$$

Combining constants and simplifying gives the implicit solution for $y(x)$ as

$$-\frac{1}{\left(x^2+y^2\right)^{\frac{1}{2}}} = c_o$$

$$\left(x^2+y^2\right)^{\frac{1}{2}} = -\frac{1}{c_0} = c_2$$