HW2, Math 319, Fall 2016

Nasser M. Abbasi (Discussion section 44272, Th 4:35PM-5:25 PM)

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Contents

0.1 Section 2.2 problem 1

Solve $y' = \frac{x^2}{y}$

This is first order non-linear ODE. In the form $y' = f(x,y)$. The function $f(x,y)$ is continuous everywhere except at the line $y=0$. Now the ODE is solved by separation

$$
\begin{aligned}
y\frac{dy}{dx} &= x^2 \\
ydy &= x^2 dx
\end{aligned}
$$

Integrating

$$
\begin{aligned}
\int ydy &= \int x^2 dx \\
\frac{y^2}{2} &= \frac{x^3}{3} + c
\end{aligned}
$$

Since initial conditions is not gives, the solution is left in implicit form (as mentioned in discussion class, Thursday Sept. 29, 2016)

$$
\boxed{y^2 = \frac{2}{3}x^3 + c_0 \qquad y \neq 0}
$$

0.2 Section 2.2 problem 2

Solve $y' = \frac{x^2}{y(1+x^3)}$

This is first order non-linear ODE. In the form $y' = f(x,y)$. The function $f(x,y)$ is continuous everywhere except at line $y=0$ and at line $x=-1$. Now the ODE is solved by separation

$$
\begin{aligned}
y\frac{dy}{dx} &= \frac{x^2}{(1+x^3)} \\
ydy &= \frac{x^2}{(1+x^3)} dx
\end{aligned}
$$

Integrating

$$
\int ydy = \int \frac{x^2}{(1+x^3)} dx
$$

To integrate $\int \frac{x^2}{(1+x^3)} dx$ let $u = 1+x^3$, hence $\frac{du}{dx} = 3x^2$. Therefore the integral becomes $\int \frac{x^2}{u}\frac{du}{3x^2} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| = \frac{1}{3}\ln\left|1+x^3\right|$. Hence the above becomes

$$
\begin{aligned}
\frac{y^2}{2} &= \frac{1}{3}\ln\left|1+x^3\right| + c \\
y^2 &= \frac{2}{3}\ln\left|1+x^3\right| + c_1
\end{aligned}
$$

Since initial condition is not gives, the solution is left in implicit form

$$\boxed{y^2 = \frac{2}{3}\ln\left|1+x^3\right| + c_1 \qquad y \neq 0, x \neq -1}$$

0.3 Section 2.2 problem 3

Solve $y' = -y^2 \sin x$

This is first order non-linear ODE. In the form $y' = f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous everywhere and $\frac{\partial f}{\partial y} = -2y\sin(x)$ is also continuous everywhere but unbounded at $y = -\infty$. This is separable, assuming $y \neq 0$ and dividing by y^2 the ODE becomes

$$\frac{1}{y^2}\frac{dy}{dx} = -\sin(x)$$
$$\frac{dy}{y^2} = -\sin()\,xdx$$

Integrating

$$\int \frac{dy}{y^2} = -\int \sin(x)\,dx$$
$$-\frac{1}{y} = \cos(x) + c$$
$$\frac{1}{y} = -\cos(x) + c_1$$

Therefore the solution is

$$\boxed{y(x) = \frac{1}{c_1 - \cos(x)} \qquad y \neq 0}$$

The reason for $y \neq 0$ was the assumption to divide by y^2 above. Another solution is

$$\boxed{y(x) = 0}$$

0.4 Section 2.2 problem 4

Solve $y' = \frac{3x^2-1}{3+2y}$

This is first order non-linear ODE. In the form $y' = f\left(x,y\right)$. The function $f\left(x,y\right)$ is continuous everywhere except at $3+2y=0$ or $y = -\frac{3}{2}$. Now the ODE is solved by separation.

$$\left(3+2y\right)\frac{dy}{dx} = 3x^2 - 1$$
$$\left(3+2y\right)dy = \left(3x^2-1\right)dx$$

Integrating

$$\int \left(3+2y\right)dy = \int \left(3x^2-1\right)dx$$
$$y^2 + 3y = x^3 - x + c$$

Complete the square

$$y^2 + 3y + \left(\frac{3}{2}\right)^2 = x^3 - x + c + \left(\frac{3}{2}\right)^2$$

Since initial condition is not gives, the solution is left in implicit form.

$$\boxed{\left(y + \frac{3}{2}\right)^2 = x^3 - x + c_0 \qquad y \neq -\frac{3}{2}}$$

0.5 Section 2.2 problem 5

$y' = \cos^2(x)\cos^2(2y)$

This is first order non-linear ODE. In the form $y' = f(x,y)$. The function $f(x,y)$ is continuous.

$$\begin{aligned}\frac{\partial f}{\partial y} &= \cos^2(x)\,2\cos(2y)\left(-2\sin(2y)\right)\\ &= -4\cos^2(x)\cos(2y)\sin(2y)\end{aligned}$$

Which is continuous everywhere and bounded. Hence a solution exist and is unique. Now the ODE is solved by separation.

Case $\cos^2(2y) \neq 0$

To divide by $\cos^2(2y)$, then for $\cos^2(2y) \neq 0$ or $\cos(2y) \neq 0$ or $2y \neq \left(n + \frac{1}{2}\right)\pi$ or $y \neq \left(n + \frac{1}{2}\right)\frac{\pi}{4}$ for all integers.

$$\begin{aligned}\frac{1}{\cos^2(2y)}\frac{dy}{dx} &= \cos^2(x)\\ \int \frac{dy}{\cos^2(2y)} &= \int \cos^2(x)\,dx \qquad (1)\end{aligned}$$

Now $\int \frac{dy}{\cos^2(2y)} = \frac{1}{2}\tan(2y)$ and

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \frac{1+\cos(2x)}{2}dx\\ &= \int\left(\frac{1}{2} + \frac{\cos(2x)}{2}\right)dx\\ &= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + c_1\\ &= \frac{x}{2} + \frac{\sin(2x)}{4} + c_1\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}\frac{1}{2}\tan(2y) &= \frac{x}{2} + \frac{\sin(2x)}{4} + c_1\\ \tan(2y) &= x + \frac{1}{2}\sin(2x) + c\end{aligned}$$

Since initial condition is not gives, the solution is left in implicit form.

Case $\cos^2(2y) = 0$

This is when $\cos(2y) = 0$ or $2y = \left(n + \frac{1}{2}\right)\pi$ or $y = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$ for all integers. In this case the solution is

$$y = \left(n + \frac{1}{2}\right)\frac{\pi}{2}$$

Summary of solution $y(x)$

$$\begin{cases} \tan(2y) = x + \frac{1}{2}\sin(2x) + c & \cos^2(2y) \neq 0 \\ \left(n + \frac{1}{2}\right)\frac{\pi}{2} & \cos^2(2y) = 0 \end{cases}$$

0.6 Section 2.2 problem 6

Solve $xy' = \left(1 - y^2\right)^{\frac{1}{2}}$

This is nonlinear first order of the form $y' = f(x,y)$ where $f(x,y) = \frac{\left(1-y^2\right)^{\frac{1}{2}}}{x}$. This is continuous everywhere except at $x = 0$. ODE is solved by separation.

Case $1 - y^2 \neq 0$

Or $y^2 \neq 1$ or $y \neq \pm 1$, then dividing by $\left(1 - y^2\right)^{\frac{1}{2}}$ and integrating

$$\begin{aligned} \int \frac{dy}{\left(1 - y^2\right)^{\frac{1}{2}}} &= \int \frac{dx}{x} \\ \arcsin(y) &= \ln|x| + c \\ y(x) &= \sin(\ln|x| + c) \end{aligned}$$

Hence the solution is

$$y(x) = \sin(\ln|x| + c) \qquad y \neq \pm 1, x \neq 0$$

Case $1 - y^2 = 0$

Then

$$y(x) = \pm 1$$

Summary of solutions

$$\begin{cases} y(x) = \sin(\ln|x| + c) & y \neq \pm 1, x \neq 0 \\ y(x) = \pm 1 & x \neq 0 \end{cases}$$

0.7 Section 2.2 problem 7

Solve $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$

This is non-linear first order ODE of the form $y' = f(x,y)$. The function $f(x,y)$ is continuous everywhere except at y which is the solution of $e^y + y = 0$. Using a computer, this is $y_c = -0.567143\cdots$. The

ODE is solved by separation

$$\int \left(y + e^{y}\right) dy = \int \left(x - e^{-x}\right) dx$$
$$\frac{y^2}{2} + e^{y} = \frac{x^2}{2} + e^{-x} + c$$

Hence the solution is given by

$$y^2 + 2e^{y} - x^2 - 2e^{-x} = c_1 \qquad y \neq y_c$$

0.8 Section 2.2 problem 8

Solve $\frac{dy}{dx} = \frac{x^2}{1+y^2}$

This is non-linear first order ODE of the form $y' = f\left(x, y\right)$ where $f\left(x, y\right)$ is continuous everywhere except at $y = \pm 1$. The ODE is solved by separation

$$\left(1 + y^2\right) dy = x^2 dx$$
$$y + \frac{y^3}{3} = \frac{x^3}{3} + c_1$$

Hence the solution is given by

$$y^3 + 3y - x^3 = c \qquad y = \pm 1$$

Since initial condition is not gives, the solution is left in implicit form.

0.9 Section 2.3 problem 1

> 1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

To reduce confusion, let x be the substance which causes the concentration in the die. Let $Q(t)$ be the mass (normally called the amount, but saying mass is more clear than saying amount) of x at time t. Hence $Q(0) = 200g$ since initial concentration was $1[g/L]$ and the volume is $200[L]$.

The goal is to find an ODE that describes how $Q(t)$ changes in time. That is, how the mass of x in the tank changes in time. Knowing the mass of x at any time in the tank, gives the concentration also, since the tank volume is fixed at $200[L]$. So the concentration can always be found using $\frac{Q(t)}{200}$. Using

$$\frac{dQ}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of x moving into the tank, i.e. how many grams of x is being poured in per minute, which is zero, since fresh water is moving in. R_{out} is rate of x moving out, i.e. how many grams of x

is leaving the tank per minute. This is found as follows

$$R_{out} = \frac{Q(t)}{200}\frac{[\text{gram}]}{[\text{L}]} \times 2\left[\frac{\text{L}}{\text{min}}\right]$$
$$= \frac{2}{200}Q(t)\frac{[\text{gram}]}{[\text{min}]}$$

Hence (1) becomes

$$\frac{dQ}{dt} = 0 - 100Q(t)$$
$$= -100Q(t)$$

Solving the ODE, for $Q(t) \neq 0$

$$\frac{dQ}{Q} = -100dt$$
$$\ln|Q| = -100t + c$$

Since Q represent mass, it can not be negative, then there is no need to use $|Q|$.

$$\ln Q = -100t + c$$
$$Q(t) = Ae^{-100t}$$

At $t = 0, Q(0) = 200[g]$, hence $A = 200$ from the above. The solution becomes

$$Q(t) = 200e^{-100t}$$

Since initial Q was $200[g]$ then 1% of that is 2. Solving for time gives

$$2 = 200e^{-100t_0}$$
$$0.01 = e^{-100t_0}$$
$$\ln(0.01) = -100t_0$$

Solving on the computer gives

$$\boxed{t_0 = 460.517[\text{min}]}$$

Hence it takes 460.517 minutes for the mass of x to reach 1% of its original amount of 200 gram. This is also the same amount of time for the concentration of x to reach 1% of its original amount of $1\left[g/L\right]$. It is easier to work with mass in the ODE, and then convert to concentration when needed.

0.10 Section 2.3 problem 2

2. A tank initially contains 120 L of pure water. A mixture containing a concentration of γ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of γ for the amount of salt in the tank at any time t. Also find the limiting amount of salt in the tank as $t \to \infty$.

Let $y(t)$ be the mass of salt at time t in the tank in grams. Hence $y(0) = 0$ since tank initially contains pure water. The goal is to find an ODE that describes how $y(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many grams of salt is being poured in per minute, which is

$$\begin{aligned} R_{in} &= \gamma\left[\frac{\text{gram}}{\text{L}}\right]\times 2\left[\frac{\text{L}}{\text{min}}\right] \\ &= 2\gamma\left[\frac{\text{gram}}{\text{min}}\right] \end{aligned}$$

And R_{out} is rate of salt moving out, i.e. how many grams of salt is leaving the tank per minute. This is found as follows

$$\begin{aligned} R_{out} &= \frac{y(t)}{120}\frac{[\text{gram}]}{[\text{L}]}\times 2\left[\frac{\text{L}}{\text{min}}\right] \\ &= \frac{1}{60}y(t)\left[\frac{\text{gram}}{\text{min}}\right] \end{aligned}$$

Hence (1) becomes

$$\frac{dy(t)}{dt} = 2\gamma - \frac{1}{60}y(t)$$

With $y(0) = 0$. The ODE is linear and first order, of the form $y' + p(t)y = g(t)$ with $p(t) = \frac{1}{60}$ and $g(t) = 2\gamma$. Since both $p(t), g(t)$ are continuous then a solution exist and is unique.

$$y' + \frac{1}{60}y = 2\gamma$$

Integrating factor is $e^{\int \frac{1}{60}dt} = e^{\frac{1}{60}t}$, therefore

$$\frac{d}{dt}\left(e^{\frac{1}{60}t}y\right) = 2\gamma e^{\frac{1}{60}t}$$

Integrating

$$\begin{aligned} e^{\frac{1}{60}t}y &= 2\gamma\int e^{\frac{1}{60}t}dt \\ &= 2\gamma\frac{e^{\frac{1}{60}t}}{\frac{1}{60}} + c \\ &= 120\gamma e^{\frac{1}{60}t} + c \end{aligned}$$

Hence

$$y(t) = 120\gamma + c^{\frac{-t}{60}}$$

In the above, $y(t)$ is the mass of salt in grams in the tank at time t. Hence the concentration of salt in the tank at time t can always be found by dividing $y(t)$ by the volume of the tank. In the limit, as $t \to \infty$ then from above

$$\lim_{t\to\infty} y(t) = 120\gamma$$

0.11 Section 2.3 problem 3

3. A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

This problem is solved in two stages. The first ODE is used to find what the amount of salt in the tank will be after 10 minutes. Then a new ODE is set up, with this value as its initial conditions, in order to find the amount of salt in the tank after an additional 10 minutes.

First 10 minutes

Let $y_1(t)$ be the mass of salt at time t in the tank in lbs. Hence $y_1(0) = 0$ since tank initially contains pure water. The goal is to find an ODE that describes how $y_1(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy_1}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many lbs of salt is being poured in per minute, which is

$$\begin{aligned} R_{in} &= \frac{1}{2}\left[\frac{\text{lb}}{\text{gallon}}\right] \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= 1\left[\frac{\text{lb}}{\text{min}}\right] \end{aligned}$$

And R_{out} is rate of salt moving out, i.e. how many grams of salt is leaving the tank per minute. This is found as follows

$$\begin{aligned} R_{out} &= \frac{y_1(t)}{100} \frac{[\text{lb}]}{[\text{gallon}]} \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= \frac{1}{50} y_1(t) \left[\frac{\text{lb}}{\text{min}}\right] \end{aligned}$$

Hence (1) becomes

$$\frac{dy_1(t)}{dt} = 1 - \frac{1}{50} y_1(t)$$

With $y_1(0) = 0$. The ODE is linear and first order, of the form $y' + p(t)y = g(t)$ with $p(t) = \frac{1}{50}$ and $g(t) = 1$. Since both $p(t), g(t)$ are continuous then a solution exist and is unique.

$$y_1' + \frac{1}{50} y_1 = 1$$

Integrating factor is $e^{\int \frac{1}{50} dt} = e^{\frac{1}{50} t}$, therefore

$$\frac{d}{dt}\left(e^{\frac{1}{50}t} y_1\right) = e^{\frac{1}{50}t}$$

Integrating

$$e^{\frac{1}{50}t}y = \int e^{\frac{1}{50}t}dt$$
$$= 50e^{\frac{1}{50}t} + c$$

Hence

$$y_1(t) = 50 + c^{\frac{-t}{50}}$$

To find c, from initial conditions

$$0 = y_1(0)$$
$$= 50 + c$$
$$c = -50$$

Hence the solution to the first phase is

$$y_1(t) = 50 - 50^{\frac{-t}{50}}$$
$$= 50\left(1 - e^{\frac{-t}{50}}\right)$$

After $t = 10$ minutes

$$y_1(10) = 50\left(1 - e^{\frac{-1}{5}}\right)$$

The above value is now used as initial conditions for new problem. The new problem will use $t = 0$ as initial time for simplicity, but it is understood that 10 minutes has already elapsed in global scale.

Second phase

Let $y_2(t)$ be the mass of salt at time t in the tank in grams. Hence

$$y_2(0) = y_1(10)$$
$$= 50\left(1 - e^{\frac{-1}{5}}\right)$$

From phase one above, this is the amount of salt in lbs in the tank at this moment. The goal is to find an ODE that describes how $y_2(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy_2}{dt} = R_{in} - R_{out} \tag{2}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many lbs of salt is being poured in per minute. But now $R_{in} = 0$ since fresh water is poured in. And R_{out} is rate of salt moving out, i.e. how many lbs of salt is leaving the tank per minute. This is found as follows

$$R_{out} = \frac{y_2(t)}{100}\frac{[\text{lb}]}{[\text{gallon}]} \times 2\left[\frac{\text{gallon}}{\text{min}}\right]$$
$$= \frac{1}{50}y_2(t)\left[\frac{\text{lb}}{\text{min}}\right]$$

Hence (2) becomes

$$\begin{aligned}\frac{dy_2(t)}{dt} &= 0 - \frac{1}{50}y_2(t) \\ &= -\frac{1}{50}y_2(t)\end{aligned}$$

The ODE is linear and first order, of the form $y' + p(t)y = g(t)$ with $p(t) = \frac{1}{50}$ and $g(t) = 0$. Since both $p(t), g(t)$ are continuous then a solution exist and is unique. This is separable.

$$\begin{aligned}\frac{dy_2}{y_2} &= -\frac{1}{50}dt \\ \ln|y_2| &= -\frac{t}{50} + c_1 \\ y_2(t) &= ce^{\frac{-t}{50}}\end{aligned} \tag{3}$$

To find c, from initial conditions $y_2(0) = 50\left(1 - e^{\frac{-1}{5}}\right)$, hence

$$50\left(1 - e^{\frac{-1}{5}}\right) = c$$

Hence the solution (3) to the second phase is

$$y_2(t) = 50\left(1 - e^{\frac{-1}{5}}\right)e^{\frac{-t}{50}}$$

After $t = 10$ minutes (which will be 20 in global scale)

$$\begin{aligned}y_2(10) &= 50\left(1 - e^{\frac{-1}{5}}\right)e^{\frac{-1}{5}} \\ &= 7.4205 \text{ lbs}\end{aligned}$$

Therefore after 20 minutes from the global initial time (or 10 minutes from the start of the second phase), the mass of salt in tank is 7.4205 lbs. Therefore the concentration at the same moment, if needed, will be $\frac{7.4205}{100}\left[\frac{\text{lbs}}{\text{gallon}}\right] = 0.074\left[\frac{\text{lbs}}{\text{gallon}}\right]$.

0.12 Section 2.3 problem 4

4. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow. Find the concentration (in pounds per gallon) of salt in the tank when it is on the point of overflowing. Compare this concentration with the theoretical limiting concentration if the tank had infinite capacity.

Let $y(t)$ be the mass of salt at time t in the tank in lbs. Hence $y(0) = 100$ since tank initially contains that much salt. The goal is to find an ODE that describes how $y(t)$ changes in time. That is, how the mass of salt in the tank changes in time. Using

$$\frac{dy}{dt} = R_{in} - R_{out} \tag{1}$$

Where R_{in} is rate of mass of salt moving into the tank, i.e. how many lbs of salt is being poured in per minute, which is

$$\begin{aligned} R_{in} &= 1\left[\frac{\text{lb}}{\text{gallon}}\right] \times 3\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= 3\left[\frac{\text{lb}}{\text{min}}\right] \end{aligned}$$

And R_{out} is rate of salt moving out, i.e. how many lbs of salt is leaving the tank per minute. This is found as follows

$$R_{out} = \frac{y(t)}{V(t)}\frac{[\text{lb}]}{[\text{gallon}]} \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \tag{2}$$

Where $V(t)$ is the volume of the whole mixture at time t. This is different from earlier problems where volume was constant. This is because in this problem the rate of pouring into the tank is larger than the rate of flow out of the tank. The volume at time t can easily be found as

$$\begin{aligned} V(t) &= 200\,[\text{gallon}] + 3\left[\frac{\text{gallon}}{\text{min}}\right]t\,[\text{min}] - 2\left[\frac{\text{gallon}}{\text{min}}\right]t[\text{min} \\ &= (200+t)\,[\text{gallon}] \end{aligned}$$

This means at any time t, there will be $200+t$ gallons of mixture in the tank. This value is now used in (2) above to complete the solution. Note that the tank will overflow when $200+t=500$ since 500 is the maximum size of the tank. Going back to (2) now it becomes

$$\begin{aligned} R_{out} &= \frac{y(t)}{200+t}\frac{[\text{lb}]}{[\text{gallon}]} \times 2\left[\frac{\text{gallon}}{\text{min}}\right] \\ &= \frac{2y(t)}{200+t} \end{aligned}$$

Therefore (1) becomes

$$y' = 3 - \frac{2y}{200+t}$$

This is linear ODE of first order of the form $y' + p(t)y = g(t)$ where $p(t) = \frac{2}{200+t}$ and $g(t)$. Both are continuous for $t \geq 0$ hence there will be a unique solution for $t \geq 0$. Now the ODE is solved using an integration factor

$$\frac{dy}{dt} + \frac{2y}{200+t} = 3$$

The integrating factor is $e^{2\int \frac{1}{200+t}dt}$. To evaluate $\int \frac{1}{200+t}dt$, let $u = 200+t$, hence $\frac{du}{dt} = 1$ and the integral becomes $\int \frac{1}{u}du = \ln|u|$ Therefore $\int \frac{1}{200+t}dt = \ln|200+t|$ and the integrating factor is $e^{2\ln|200+t|} = |200+t|^2 = (200+t)^2$. Therefore now that the integrating is found, the solution can be written as

$$\frac{d}{dt}\left(y(200+t)^2\right) = 3(200+t)^2$$

Integrating both sides gives

$$y(200+t)^2 = 3\int (200+t)^2\,dt$$

Let $u=200+t$, then $\frac{du}{dt}=t$, hence $\int(200+t)^2\,dt=\int u^2du=\frac{u^3}{3}+c_1=\frac{(200+t)^3}{3}+c_1$. Therefore the above becomes

$$\begin{aligned} y(200+t)^2 &= 3\left(\frac{(200+t)^3}{3}+c_1\right) \\ &= (200+t)^3+c \end{aligned}$$

Solving for $y(t)$ gives

$$\boxed{y(t)=(200+t)+\frac{c}{(200+t)^2}} \tag{3}$$

Now c is found from initial conditions. Given that $y(0)=100$, then from the above

$$\begin{aligned} 100 &= 200+\frac{c}{(200)^2} \\ &= 200+\frac{c}{40000} \\ c &= (-100)(40000) \\ &= -4\times10^6 \end{aligned}$$

Therefore the solution (3) becomes

$$y(t)=(200+t)-\frac{4\times10^6}{(200+t)^2} \tag{4}$$

Now the above ODE is only valid until the tank overflows. This value of time is found by solving $200+t=500$ for t, which gives $t=300$. Hence (4) becomes

$$\boxed{y(t)=(200+t)-\frac{4\times10^6}{(200+t)^2}\qquad 0\le t\le 300} \tag{5}$$

At $t=300$ minutes, the mass of salt in lbs is therefore $y(300)$ which is

$$\begin{aligned} y(300) &= (200+300)-\frac{4\times10^6}{(200+300)^2} \\ &= 484\ \ [\text{lbs}] \end{aligned}$$

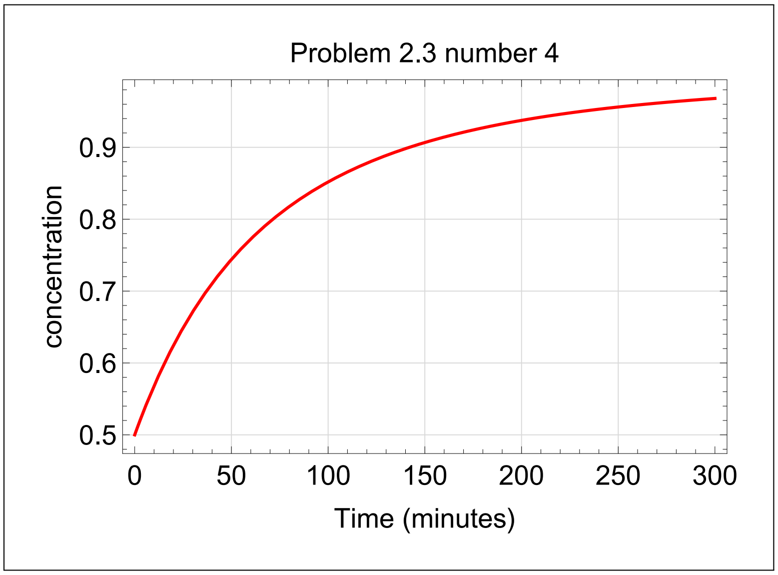
And since the volume now is 500 gallons, then the concentration at time $t=300$ minutes is

$$\frac{484}{500}\left[\frac{\text{lbs}}{\text{gallon}}\right]=0.968\left[\frac{\text{lbs}}{\text{gallon}}\right]$$

If the tank had infinite capacity, then using the solution found in (5) and dividing by current volume, which was found before to be $200+t$ and then taking the limit $t\to\infty$ gives the answer. Let $\rho(t)$ be now the concentration in $\left[\frac{\text{lbs}}{\text{gallon}}\right]$ at any time t. Then

$$\begin{aligned} \rho(t) &= \frac{(200+t)-\frac{4\times10^6}{(200+t)^2}}{V(t)}=\frac{(200+t)-\frac{4\times10^6}{(200+t)^2}}{200+t} \\ &= 1-\frac{4\times10^6}{200+t} \end{aligned}$$

As $t\to\infty$ then $\rho(t)\to1$. Therefore at 300 minutes the concentration is 96.8% of the theoretical limit. The following is a plot of $\rho(t)$ as function of time. At $t=0$ the concentration is 0.5 since this is the initial condition.



0.13 Section 2.4 problem 1

Determine an interval which the given initial value problem is valid. $(t-3)y' + \ln(t)y = 2t$ with $y(1) = 2$.

This is linear first order ODE. In standard form it becomes $y' + \frac{\ln(t)}{t-3}y = \frac{2t}{t-3}$, and comparing to $y' + p(t)y = g(t)$ then

$$p(t) = \frac{\ln(t)}{t-3}$$
$$g(t) = \frac{2t}{t-3}$$

$p(t)$ is not not continuous at $t = 3$ and also at $t = 0$ since $\ln(0) = -\infty$. $g(t)$ is not continuous at $t = 3$. Therefore the region must include initial point, which is $t = 1$ but not include $t = 3$ nor $t = 0$. Hence

$$0 < t < 3$$

And for forward only ODE the region is

$$1 \leq t < 3$$

0.14 Section 2.4 problem 2

Determine an interval which the given initial value problem is valid. $t(t-4)y' + y = 2t$ with $y(2) = 1$.

This is linear first order ODE. In standard form it becomes $y' + \frac{1}{t(t-4)}y = \frac{2}{(t-4)}$, and comparing to $y' + p(t)y = g(t)$ then

$$p(t) = \frac{1}{t(t-4)}$$
$$g(t) = \frac{2}{(t-4)}$$

$p(t)$ is not continuous at $t = 0$ and $t = 4$ while $g(t)$ is not continuous at $t = 4$. Therefore the region must include initial point, which is $t = 2$ but not include $t = 4$ nor $t = 0$. Hence

$$0 < t < 4$$

And for forward only ODE the region is

$$2 \leq t < 4$$

0.15 Section 2.4 problem 3

Determine an interval which the given initial value problem is valid. $y' + \tan(t)\, y = \sin(t)$ with $y(\pi) = 0$.

This is linear first order ODE. Comparing to $y' + p(t)\, y = g(t)$ then

$$p(t) = \tan(t)$$
$$g(t) = \sin(t)$$

$g(t)$ is continuous everywhere but $p(t)$ is not continuous at $\left\{\cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots\right\}$ therefore the region must be between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ since the initial point π is inside this region. Hence

$$\frac{\pi}{2} < t < 1.5\pi$$

0.16 Section 2.4 problem 4

Determine an interval which the given initial value problem is valid. $\left(4 - t^2\right) y' + 2ty = 3t^2$ with $y(-3) = 1$.

This is linear first order ODE. In standard form it becomes $y' + \frac{2t}{\left(4-t^2\right)} y = \frac{3t^2}{\left(4-t^2\right)}$, and comparing to $y' + p(t)\, y = g(t)$ then

$$p(t) = \frac{2t}{\left(4 - t^2\right)}$$
$$g(t) = \frac{3t^2}{\left(4 - t^2\right)}$$

$p(t)$ is not not continuous at $t^2 = 4$ or $t = \pm 2$ and the same for $g(t)$. Therefore the region must include initial point, which is $t = -3$ but not include $t = \pm 2$. Hence

$$-\infty < t < -2$$

And for forward only ODE the region is

$$-3 \leq t < -2$$

0.17 Section 2.4 problem 5

Determine an interval which the given initial value problem is valid. $\left(4 - t^2\right) y' + 2ty = 3t^2$ with $y(1) = -3$.

This is linear first order ODE. In standard form it becomes $y' + \frac{2t}{\left(4-t^2\right)} y = \frac{3t^2}{\left(4-t^2\right)}$, and comparing to $y' + p(t)\, y = g(t)$ then

$$p(t) = \frac{2t}{\left(4 - t^2\right)}$$
$$g(t) = \frac{3t^2}{\left(4 - t^2\right)}$$

$p(t)$ is not not continuous at $t^2 = 4$ or $t = \pm 2$ and the same for $g(t)$. Therefore the region must include initial point, which is $t = 1$ but not include $t = \pm 2$. Hence

$$-2 < t < 2$$

And for forward only ODE the region is

$$1 \leq t < 2$$

0.18 Section 2.4 problem 6

Determine an interval which the given initial value problem is valid. $\ln(t) y' + y = \frac{1}{\tan(t)}$ with $y(2) = 3$.

This is linear first order ODE. In standard form it becomes $y' + \frac{1}{\ln(t)} y = \frac{1}{\tan(t)\ln(t)}$, and comparing to $y' + p(t) y = g(t)$ then

$$p(t) = \frac{1}{\ln(t)}$$
$$g(t) = \frac{1}{\tan(t)\ln(t)}$$

When $t = 1$ then $\ln(t) = 0$ and $p(t)$ becomes unbounded. And since for real t then t must remain positive, else $\ln(t)$ becomes complex. Then $p(t)$ says that $t \geq 0$ and $t \neq 1$. Looking at $g(t)$ then $\tan(t) = 0$ when $t = \{\cdots, -\pi, \pi, \cdots\}$ hence the region that includes initial point $t_0 = 2$ must be inside these. Therefore the singular points are $t = 1, -\pi, \pi$ and $t \geq 0$. Putting all these together, the region is

$$1 < t < \pi$$

And for forward only ODE the region is

$$2 \leq t < \pi$$