HW1, Math 319, Fall 2016

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Contents

Note on plots: Some of these problems requires plotting. These were done both by hand and also by the computer but only the computer version of the plot was included.

0.1 Section 1.2 problem 1

Solve each of the following and plot the solution for different y_0 values.

0.1.1 part a

$\frac{dy}{dt} = -y + 5, y(0) = y_0$

$$\frac{dy}{dt} + y = 5$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 1, g(t) = 5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

The Integrating factor is $e^{\int dt} = e^t$. Multiplying both sides by e^t gives

$$\frac{d}{dt}\left(ye^t\right) = 5e^t$$

Integrating

$$\begin{aligned} ye^t &= 5\int e^t dt + c \\ &= 5e^t + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{-t} \tag{1}$$

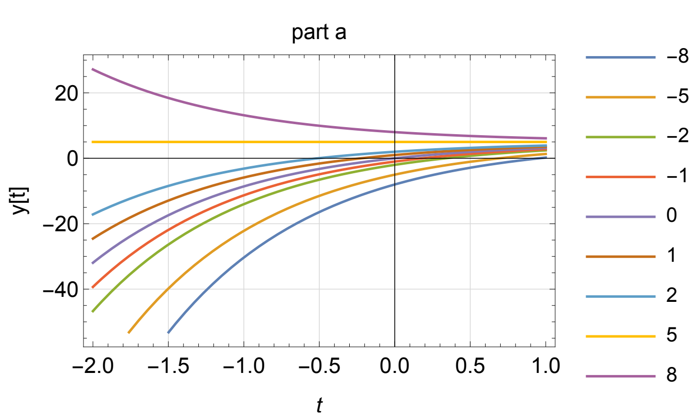
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{-t} \qquad t \in \Re$$

As $t \to \infty$ the solution approaches $y(t) = 5$.The following plot gives the solution $y(t)$ for few values of y_0



0.1.2 part b

$\frac{dy}{dt} = -2y + 5, y(0) = y_0$

$$\frac{dy}{dt} + 2y = 5$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 2, g(t) = 5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{2\int dt} = e^{2t}$. Multiplying both sides by e^{2t} gives

$$\frac{d}{dt}\left(ye^{2t}\right) = 5e^{2t}$$

Integrating

$$\begin{aligned} ye^{2t} &= 5\int e^{2t}dt + c \\ &= \frac{5}{2}e^{2t} + c \end{aligned}$$

Hence

$$y(t) = \frac{5}{2} + ce^{-2t} \tag{1}$$

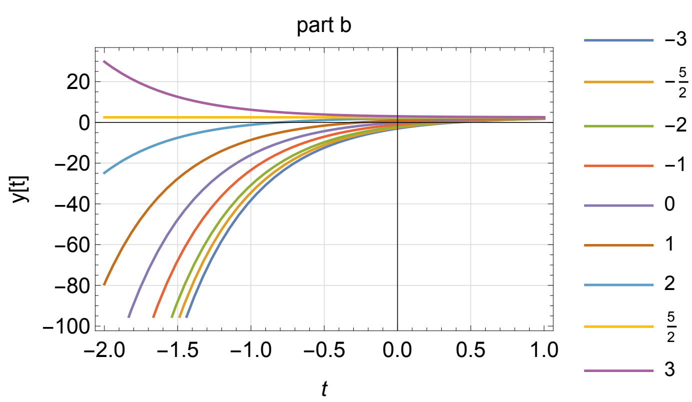
Applying initial conditions gives

$$\begin{aligned} y_0 &= \frac{5}{2} + c \\ c &= y_0 - \frac{5}{2} \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 2.5 + \left(y_0 - 2.5\right)e^{-2t} \qquad t \in \Re$$

As $t \to \infty$ the solution approaches $y(t) = 2.5$. The following plot gives the solution $y(t)$ for few values of y_0



0.1.3 part c

$\frac{dy}{dt} = -2y + 10, y(0) = y_0$

$$\frac{dy}{dt} + 2y = 10$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 2, g(t) = 10$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{2\int dt} = e^{2t}$. Multiplying both sides by e^{2t} gives

$$\frac{d}{dt}\left(ye^{2t}\right) = 10e^{2t}$$

Integrating

$$\begin{aligned} ye^{2t} &= 10\int e^{2t}dt + c \\ &= 5e^{2t} + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{-2t} \tag{1}$$

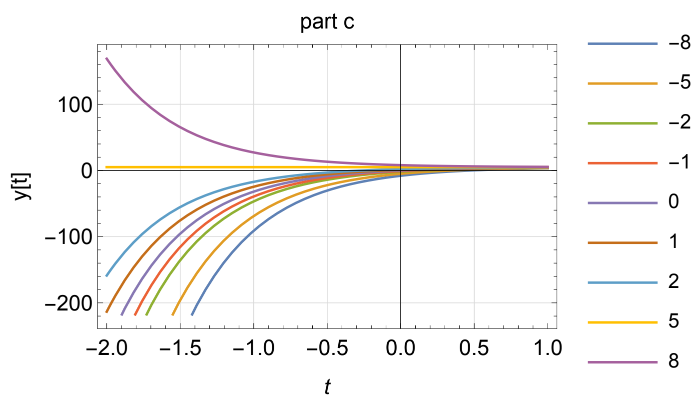
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{-2t} \qquad t \in \Re$$

As $t \to \infty$ the solution approaches $y(t) = 5$. The following plot gives the solution $y(t)$ for few values of y_0



Discussion of differences In all solutions the term with e^{-t} and e^{-2t} in it will vanish as $t \to +\infty$. Hence for $t > 0$ all solution approach a constant value as $t \to \infty$, which is 5 for part (a) and (c) and 2.5 for part (b). Since part(b,c) has e^{-2t} term, these will approach the asymptote faster (converges faster) than part (a) which has e^{-t} term.

0.2 Section 1.2, problem 2

0.2.1 part (a)

$\frac{dy}{dt} = y - 5, y(0) = y_0$

$$\frac{dy}{dt} - y = -5$$

This is first order, linear ODE of the form $y' + p(t)\,y = g(t)$ where $p(t) = -1, g(t) = -5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{-\int dt} = e^{-t}$. Multiplying both sides by e^{-t} gives

$$\frac{d}{dt}\left(ye^{-t}\right) = -5e^{-t}$$

Integrating

$$\begin{aligned} ye^{-t} &= -5\int e^{-t}dt + c \\ &= 5e^{-t} + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{t} \tag{1}$$

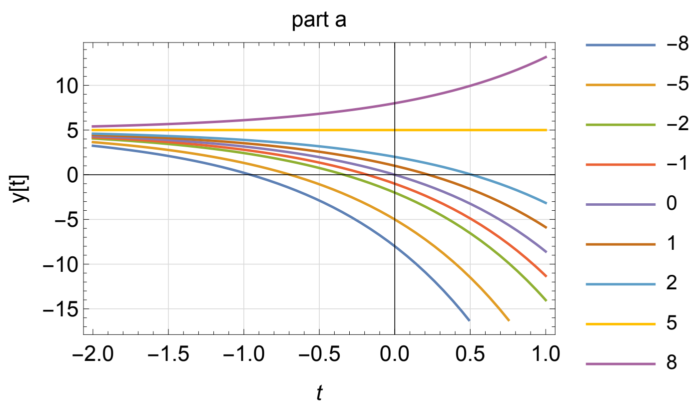
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{t} \qquad t \in \Re$$

The following plot gives the solution $y(t)$ for few values of y_0



0.2.2 part (b)

$\frac{dy}{dt} = 2y - 5, y(0) = y_0$

$$\frac{dy}{dt} - 2y = -5$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -2, g(t) = -5$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{-2\int dt} = e^{-2t}$. Multiplying both sides by e^{-2t} gives

$$\frac{d}{dt}\left(ye^{-2t}\right) = -5e^{-2t}$$

Integrating

$$\begin{aligned} ye^{-2t} &= -5\int e^{-2t}dt + c \\ &= 2.5e^{-2t} + c \end{aligned}$$

Hence

$$y(t) = 2.5 + ce^{2t} \tag{1}$$

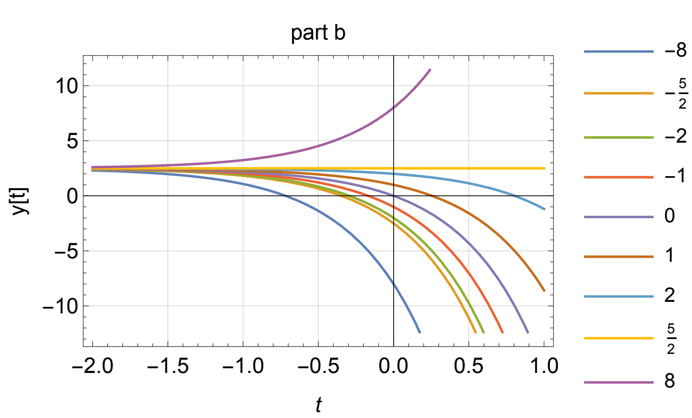
Applying initial conditions gives

$$\begin{aligned} y_0 &= 2.5 + c \\ c &= y_0 - 2.5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 2.5 + \left(y_0 - 2.5\right)e^{2t} \qquad t \in \Re$$

The following plot gives the solution $y(t)$ for few values of y_0



0.2.3 part (c)

$\frac{dy}{dt} = 2y - 10, y(0) = y_0$

$$\frac{dy}{dt} - 2y = -10$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -2, g(t) = -10$.

Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is $e^{-2\int dt} = e^{-2t}$. Multiplying both sides by e^{-2t} gives

$$\frac{d}{dt}\left(ye^{-2t}\right) = -10e^{-2t}$$

Integrating

$$\begin{aligned} ye^{-2t} &= -10\int e^{-2t}dt + c \\ &= 5e^{-2t} + c \end{aligned}$$

Hence

$$y(t) = 5 + ce^{2t} \tag{1}$$

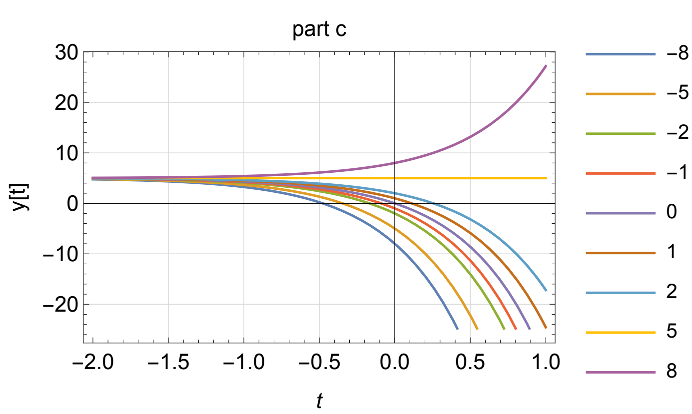
Applying initial conditions gives

$$\begin{aligned} y_0 &= 5 + c \\ c &= y_0 - 5 \end{aligned}$$

The complete solution from (1) becomes

$$y(t) = 5 + \left(y_0 - 5\right)e^{2t} \qquad t \in \Re$$

The following plot gives the solution $y(t)$ for few values of y_0



<u>Discussion of differences</u> In all solutions the term with e^t and e^{2t} in it will vanish as $t \to -\infty$. Hence for $t < 0$ all solution approach a constant value as $t \to -\infty$, which is 5 for part (a) and (c) and 2.5 for part (b). Since part(b,c) has e^{2t} term, these will diverge faster for large t than part (a) which has e^t term.

0.3 Section 1.3, problem 7

In each of the problems below, verify that each given function is the solution to the ODE

$y'' - y = 0; y_1(t) = e^t; y_2(t) = \cosh(t)$.

For $y_1(t)$, taking derivatives of y_1 gives $y_1' = e^t, y_1'' = e^t$. Substituting into the ODE gives

$$e^t - e^t = 0$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

For $y_2(t)$, taking derivatives of y_2 gives $y_2' = \sinh(t), y_2'' = \cosh(t)$. Substituting into the ODE gives

$$\cosh(t) - \cosh(t) = 0$$

Which is the RHS in the original ODE. Hence $y_2(t)$ is solution to the ODE.

0.4 Section 1.3, problem 8

$y'' + 2y' - 3y = 0; y_1(t) = e^{-3t}; y_2(t) = e^t$.

For $y_1(t)$: Taking derivatives of y_1 gives $y_1' = -3e^{-3t}, y_1'' = 9e^{-3t}$. Substituting into the ODE gives

$$\begin{aligned} 9e^{-3t} + 2\left(-3e^{-3t}\right) - 3\left(e^{-3t}\right) &= 9e^{-3t} - 6e^{-3t} - 3e^{-3t} \\ &= 0 \end{aligned}$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

For $y_2(t)$: Taking derivatives of y_2 gives $y_2' = e^t, y_2'' = e^t$. Substituting into the ODE gives

$$e^t + 2e^t - 3e^t = 0$$

Which is the RHS in the original ODE. Hence $y_2(t)$ is solution to the ODE.

0.5 Section 1.3, problem 9

$ty' - y = t^2; y_1(t) = 3t + t^2$

Taking derivative of y_1 gives $y_1' = 3 + 2t$. Substituting into the ODE gives

$$\begin{aligned} t(3 + 2t) - \left(3t + t^2\right) &= 3t + 2t^2 - 3t - t^2 \\ &= t^2 \end{aligned}$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

0.6 Section 1.3, problem 10

$y^{(4)} + 4y''' + 3y = t; y_1(t) = \frac{t}{3}; y_2(t) = e^{-t} + \frac{t}{3}$

For y_1: Taking derivative of y_1 gives $y_1' = \frac{1}{3}, y_1'' = 0, y_1''' = 0, y_1^{(4)} = 0$. Substituting into the ODE gives

$$0 + 0 + 3\left(\frac{t}{3}\right) = t$$

Which is the RHS in the original ODE. Hence $y_1(t)$ is solution to the ODE.

For y_2: Taking derivatives of y_2 gives $y_2' = -e^{-t} + \frac{1}{3}, y_2'' = e^{-t}, y_2''' = -e^{-t}, y_2^{(4)} = e^{-t}$ .Substituting into the ODE gives

$$\begin{aligned} e^{-t} - 4e^{-t} + 3\left(e^{-t} + \frac{t}{3}\right) &= e^{-t} - 4e^{-t} + 3e^{-t} + t \\ &= t \end{aligned}$$

Which is the RHS in the original ODE. Hence $y_2(t)$ is solution to the ODE.

0.7 Section 1.3, problem 15

Determine the value of r for which the given ODE has solution in the form $y = e^{rt}$

$$y' + 2y = 0$$

Assume the solution is of the form Ae^{rt} where A is arbitrary constant. Substituting this into the ODE gives

$$Are^{rt} + 2Ae^{rt} = 0$$

Since $e^{rt} \neq 0$ and $A \neq 0$ (else trivial solution), then dividing thought by Ae^{rt} gives

$$r + 2 = 0$$

Hence

$$\boxed{r = -2}$$

The solution is

$$y(t) = Ae^{-2t}$$

0.8 Section 1.3, problem 16

Determine the value of r for which the given ODE has solution in the form $y = e^{rt}$

$$y'' - y = 0$$

Assume the solution is of the form Ae^{rt} where A is arbitrary constant. Substituting this into the ODE gives

$$Ar^2e^{rt} - Ae^{rt} = 0$$

Since $e^{rt} \neq 0$ and $A \neq 0$ (else trivial solution), then dividing thought by Ae^{rt} gives

$$r^2 - 1 = 0$$

Hence

$$\boxed{r = \pm 1}$$

The solution is

$$y(t) = c_1 e^{-t} + c_2 e^{t}$$

0.9 Section 2.1 problem 1

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \to \infty$

$y' + 3y = t + e^{-2t}$

0.9.1 Part (a)

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 3, g(t) = t + e^{-2t}$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique.

First the ODE is written such that y' is on one side, and everything else on the other side.

$$\begin{aligned} y' &= -3y + t + e^{-2t} \\ &= f(t, y) \end{aligned}$$

<u>Global view:</u> For fixed y, as $t \to \infty, y' \to \infty$ and for $t \to -\infty, y' \to \infty$. At $t = 0, y' = -3y + 1$.

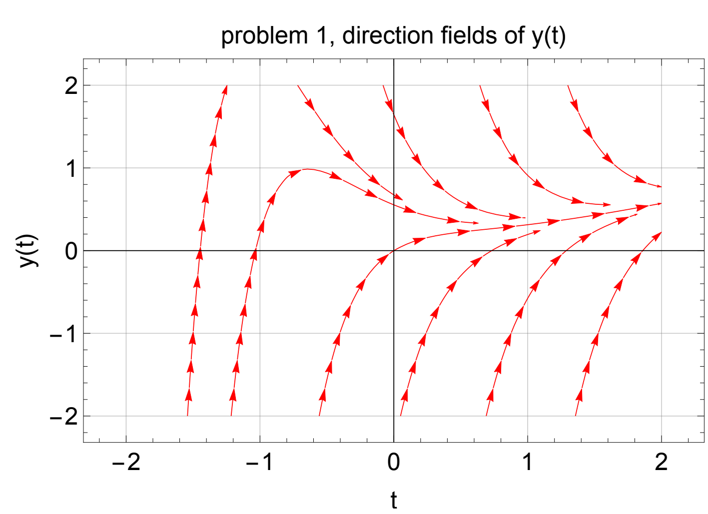
For each value of $y = \{-1, 0, 1\}$ and for each $t = \{-1, 0, 1\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t = -1$	$t = 0$	$t = 1$
$y = -1$	$y' = 3 - 1 + e^2 \approx 9$	$y' = 3 + 0 + e^0 = 4$	$y' = 3 + 1 + e^{-2} \approx 4$
$y = 0$	$y' = -1 + e^2 \approx 6$	$y' = 0 + e^0 = 1$	$y' = 0 + 1 + e^{-2} \approx 1$
$y = 1$	$y' = -3 - 1 + e^2 \approx 3$	$y' = -3 + 0 + e^0 = -2$	$y' = -3 + 1 + e^{-2} \approx -2$

The above data gives y' at at coordinates

$$\{\{-1,-1\},\{0,-1\},\{1,-1\},\{-1,0\},\{0,0\},\{1,0\},\{-1,1\},\{0,1\},\{1,1\}\}$$

A sketch was now made by hand as well using the computer. The computer version is given below.



0.9.2 Part (b)

The solutions for large positive t appear to approach an asymptote straight line with positive slope. This is confirmed by next part.

0.9.3 Part (c)

$$y' + 3y = t + e^{-2t}$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = 3, g(t) = t + e^{-2t}$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is e^{3t}, and multiplying both sides by this results in

$$\frac{d}{dt}\left(e^{3t}y\right) = te^{3t} + e^{t}$$

Integrating

$$\begin{aligned} e^{3t}y &= \int te^{3t}dt + \int e^{t}dt + c \\ &= e^{3t}\left(\frac{t}{3} - \frac{1}{9}\right) + e^{t} + c \end{aligned}$$

Therefore

$$y = \left(\frac{t}{3} - \frac{1}{9}\right) + e^{-2t} + ce^{-3t} \qquad t \in \Re$$

For large positive t, the term $\frac{t}{3}$ dominates and $y(t) \approx \frac{1}{3}t$. Hence the solution as $t \to \infty$ approaches asymptote line with slope $\frac{1}{3}$. For $t \to -\infty$ the solution grows exponentially in the negative half plane. The sign of c determines which direction the solution grows to since e^{-3t} increases faster than e^{-2t} for negative t.

0.10 Section 2.1 problem 2

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \to \infty$

$y' - 2y = te^{-2t}$

0.10.1 Part (a)

First the ODE is written such that y' is on one side, and everything else on the other side.

$$y' = 2y + te^{-2t}$$

Global view: As $t \to \infty, y' \to \infty$ and as $t \to -\infty, y' \to \infty$. And $y' = 0$ at point $t = 0, y = -\frac{1}{2}$.

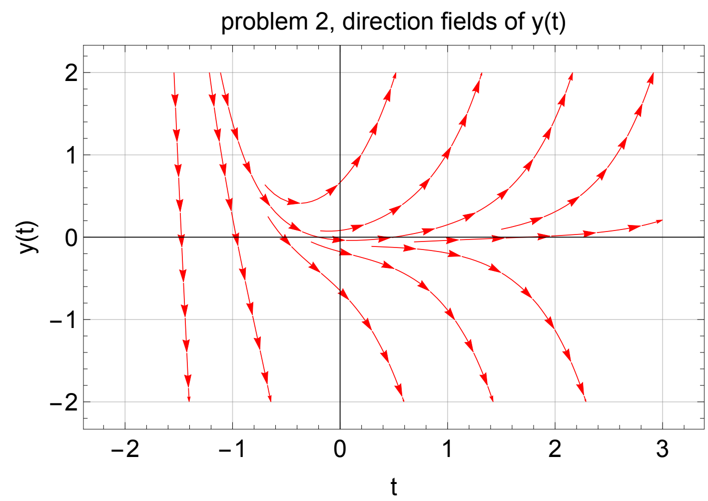
For each value of $y = \{-1, 0, 1\}$ and for each $t = \{-1, 0, 1\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t = -1$	$t = 0$	$t = 1$
$y = -1$	$y' = -2 - e^2 \approx -9$	$y' = -2 + 0 = -2$	$y' = -2 + e^{-2} \approx -1.8$
$y = 0$	$y' = -e^{2t} \approx -7$	$y' = 0$	$y' = 0 + e^{-2} \approx 0.1$
$y = 1$	$y' = 2 - e^2 \approx -5$	$y' = 2$	$y' = 2 + e^{-2} \approx 2.1$

The above data gives y' at at coordinates

$$\{\{-1,-1\},\{0,-1\},\{1,-1\},\{-1,0\},\{0,0\},\{1,0\},\{-1,1\},\{0,1\},\{1,1\}\}$$

A sketch was now made by hand as well using the computer. The computer version is given below.



0.10.2 Part (b)

The solutions for large positive t appear to grow exponentially. This is confirmed by next part.

0.10.3 Part (c)

$$y' - 2y = te^{-2t}$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -2, g(t) = te^{-2t}$. Since both $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is e^{-2t}, and multiplying both sides by this results in

$$\frac{d}{dt}\left(e^{-2t}y\right) = te^{-4t}$$

Integrating

$$\begin{aligned} e^{-2t}y &= \int te^{-4t}dt + c \\ &= e^{-4t}\left(-\frac{t}{4} - \frac{1}{16}\right) + c \end{aligned}$$

Hence

$$y = e^{-2t}\left(-\frac{t}{4} - \frac{1}{16}\right) + ce^{2t}$$

For large positive t, the term $e^{-2t}\left(\frac{t}{4} - \frac{1}{16}\right) \rightarrow 0$ and what is left is $e^{2t}c$ which grows exponentially.

$$\lim_{t\rightarrow\infty} y(t) = ce^{2}t$$

For large negative t, the solution grows exponentially in the negative half plane.

0.11 Section 2.1 problem 3

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \rightarrow \infty$

$y' + y = te^{-t} + 1$

0.11.1 Part (a)

First the ODE is written such that y' is on one side, and everything else on the other side.

$$y' = -y + te^{-t} + 1$$

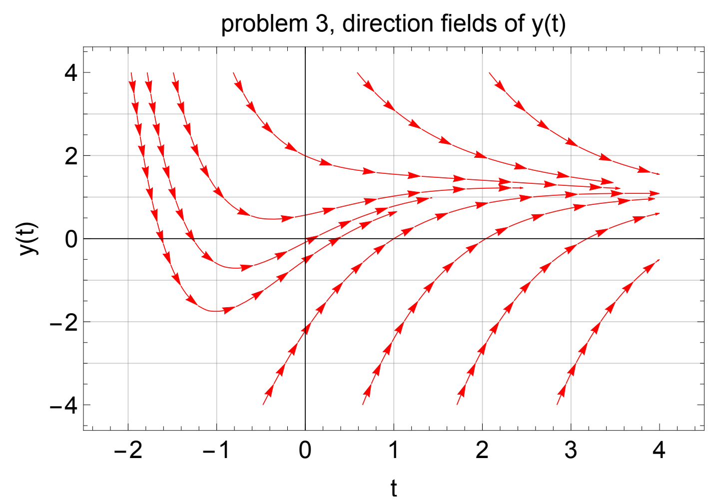
For each value of $y = \{-1, 0, 1\}$ and for each $t = \{-1, 0, 1\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t=-1$	$t=0$	$t=1$
$y=-1$	$y'=1-e^{t}+1\approx -0.7$	$y'=1+1=2$	$y'=1+e^{-1}+1\approx 2.3$
$y=0$	$y'=0-e^{t}+1\approx -1.7$	$y'=1$	$y'=0+e^{-1}+1\approx 1.3$
$y=1$	$y'=-1-e^{1}+1\approx -0.7$	$y'=-1+1=0$	$y'=1+e^{-1}+1\approx 0.3$

The above data gives y' at at coordinates

$$\{\{-1,-1\},\{0,-1\},\{1,-1\},\{-1,0\},\{0,0\},\{1,0\},\{-1,1\},\{0,1\},\{1,1\}\}$$

A sketch was now made by hand as well using the computer. The computer version is given below.



0.11.2 Part (b)

The solutions for large positive t appear to approach an asymptote line $y(t)=1$. This is confirmed by next part.

0.11.3 Part (c)

$$y'+y=te^{-t}+1$$

This is first order, linear ODE of the form $y'+p(t)y=g(t)$ where $p(t)=1,g(t)=te^{-2t}+1$. Since both $p(t),g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique. Now the ODE is solved.

Integrating factor is e^{t}, and multiplying both sides by this results in

$$\frac{d}{dt}\left(e^{t}y\right)=t+e^{t}$$

Integrating

$$\begin{aligned} e^{t}y &= \int t\,dt+\int e^{t}dt+c \\ &= \frac{1}{2}t^{2}+e^{t}+c \end{aligned}$$

Hence

$$\begin{aligned} y &= \frac{1}{2}t^{2}e^{-t}+1+ce^{-t} \\ &= e^{-t}\left(\frac{1}{2}t^{2}+c\right)+1 \end{aligned}$$

For large positive t, the term $e^{-t}\left(\frac{1}{2}t^{2}+c\right)\rightarrow 0$ and what is left is 1. Hence the solution as $t\rightarrow\infty$ approaches asymptote line $y(t)=1$.

$$\lim_{t\rightarrow\infty}y(t)=1$$

0.12 Section 2.1 problem 4

draw direction field for the given ODE. Based on inspection, describe how the solutions behave for large t. Find general solution to the ODE and use to determine how solution behaves as $t \to \infty$

$y' + \frac{y}{t} = 3\cos 2t$ for $t > 0$

0.12.1 Part (a)

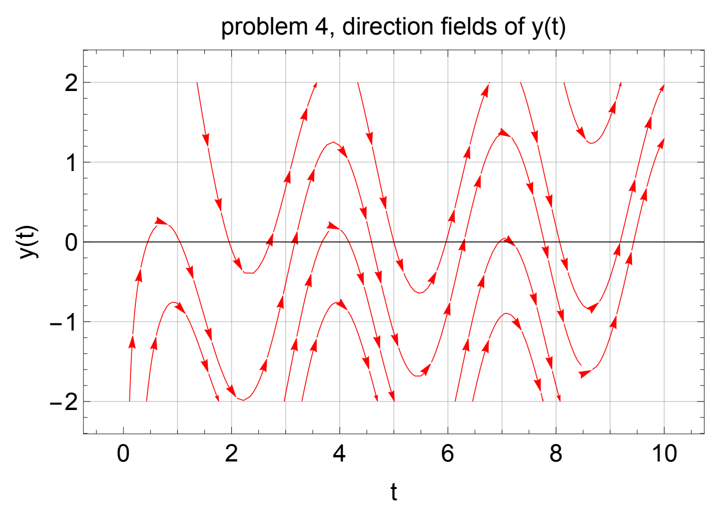
First the ODE is written such that y' is on one side, and everything else on the other side.

$$y' = -\frac{y}{t} + 3\cos 2t$$

For each value of $y = \{-1, 0, 1\}$ and for each $t = \{1, 2, 3\}$ the RHS is calculated and the slope y' is drawn as tangent at that point.

	$t = 1$	$t = 2$	$t = 3$
$y = -1$	$y' = 1 + 3\cos 2 \approx -0.25$	$y' = \frac{1}{2} + 3\cos 4 \approx -1.5$	$y' = \frac{1}{3} + 3\cos 6 \approx 3.2$
$y = 0$	$y' = 3\cos 2 \approx -1.25$	$y' = 0 + 3\cos 4 \approx -2$	$y' = 0 + 3\cos 6 \approx 2.9$
$y = 1$	$y' = -1 + 3\cos 2 \approx -2.25$	$y' = -\frac{1}{2} + 3\cos 4 = -2.5$	$y' = -\frac{1}{3} + 3\cos 2t \approx 2.5$

A sketch was now made by hand as well using the computer. The computer version is given below.



0.12.2 Part (b)

The solutions for large positive t appear to oscillate, but it is hard to see that from the few points above, as more points is needed and only after using the computer plot and solving it did this become more clear.

0.12.3 Part (c)

$$y' + \frac{y}{t} = 3\cos 2t$$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = \frac{1}{t}, g(t) = 3\cos 2t$. $p(t)$ is singular at $t = 0$ (not continuous at that point) while $g(t)$ is continuous on the whole real line, then by theorem 1, a solution exists and is unique only if the initial condition is not at $t_0 = 0$. The solution found is valid on an interval that excludes $t = 0$ but includes t_0. The problem says to solve this on $t > 0$ which bypasses $t = 0$. Now the ODE is solved.

Integrating factor is $e^{\int \frac{1}{t} dt} = e^{\ln t} = t$, and multiplying both sides by this results in

$$\frac{d}{dt}\left(ty\right) = 3t\cos 2t$$

Integrating

$$ty = 3\int t\cos(2t)\,dt + c \tag{1}$$

Using $\int udv = uv - \int vdv$, let $u = t, dv = \cos(2t) \rightarrow du = 1, v = \frac{1}{2}\sin(2t)$, hence

$$\begin{aligned}\int t\cos(2t)\,dt &= \frac{1}{2}t\sin(2t) - \int \frac{1}{2}\sin(2t)\,dt\\ &= \frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t)\end{aligned}$$

Equation (1) becomes

$$\begin{aligned}ty &= 3\left(\frac{1}{2}t\sin(2t) + \frac{1}{4}\cos(2t)\right) + c\\ y &= \frac{3}{2}\sin(2t) + \frac{3}{4}\frac{\cos(2t)}{t} + \frac{c}{t} \qquad t > 0\end{aligned}$$

In the limit as $t \rightarrow \infty$ the terms $\frac{c}{t} \rightarrow 0$ and $\frac{\cos(2t)}{t} \rightarrow 0$, therefore

$$\lim_{t\rightarrow\infty} y(t) = \frac{3}{2}\sin(2t)$$

Hence the solution is sinusoidal at large t.

0.13 Section 2.1 problem 13

Find the solution to the given initial value problem. $y' - y = 2te^{2t}$ with $y(0) = 1$

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = -1, g(t) = 2te^{2t}$. Since $p(t), g(t)$ are continuous on the real line, then by theorem 1, a solution exists and is unique.

Integrating factor is $e^{\int -dt} = e^{-t}$. Multiplying both sides by this results in

$$\frac{d}{dt}\left(ye^{-t}\right) = 2te^{t}$$

Integrating

$$ye^{-t} = 2\int te^{t}dt + c \tag{1}$$

Using $\int udv = uv - \int vdv$, let $u = t, dv = e^{t} \rightarrow du = 1, v = e^{t}$, hence

$$\begin{aligned}\int te^{t}dt &= te^{t} - \int e^{t}dt\\ &= te^{t} - e^{t}\end{aligned}$$

Therefore (1) becomes

$$\begin{aligned}ye^{-t} &= 2\left(te^{t} - e^{t}\right) + c\\ y &= 2\left(te^{2t} - e^{2t}\right) + ce^{t}\\ &= 2e^{2t}(t-1) + ce^{t}\end{aligned}$$

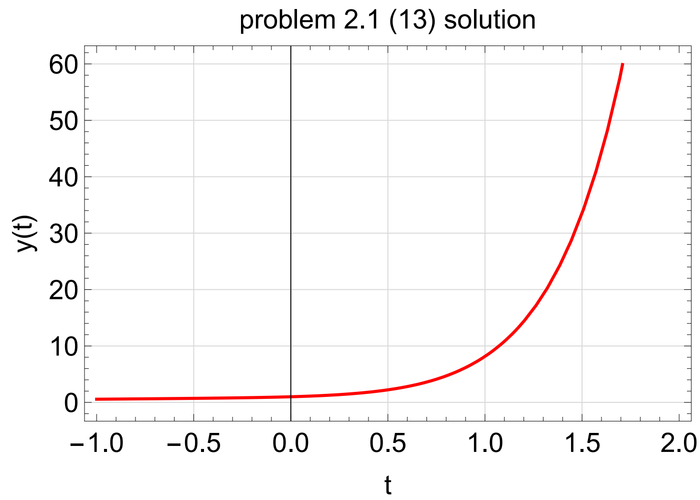
Applying initial conditions gives

$$\begin{aligned}&1 = 2e^{0}(0-1) + ce^{0}\\ &1 = -2 + c\\ &c = 3\end{aligned}$$

Hence the general solution is

$$y = 2e^{2t}(t-1) + 3e^{t} \qquad t \in \Re$$

Here is a plot of the solution



0.14 Section 2.1 problem 14

Find the solution to the given initial value problem. $y' + 2y = te^{-2t}$ with $y(1) = 0$

This is first order, linear ODE of the form $y' + p(t)\,y = g(t)$ where $p(t) = 2, g(t) = te^{2t}$. Since $p(t), g(t)$ are continuous on the real line, then by theorem 1 a solution exists and is unique.

Integrating factor is $e^{2\int dt} = e^{2t}$. Multiplying both sides by e^{2t} gives

$$\frac{d}{dt}\left(ye^{2t}\right) = t$$

Integrating

$$\begin{aligned} ye^{2t} &= \frac{1}{2}t^2 + c \\ y &= \frac{1}{2}t^2e^{-2t} + ce^{-2t} \end{aligned} \tag{1}$$

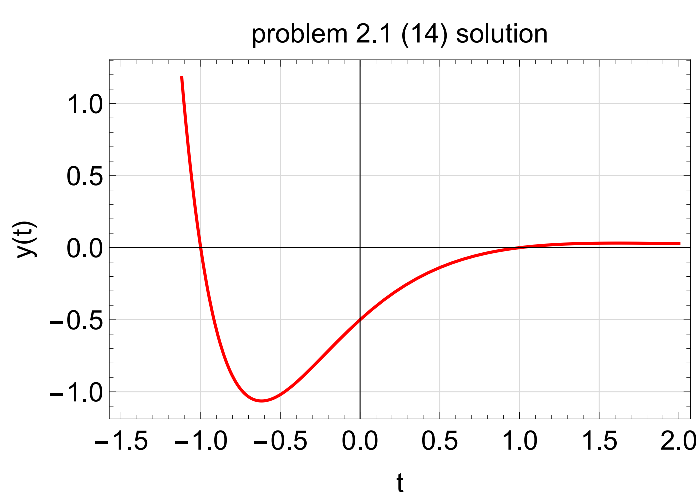
Applying initial conditions

$$\begin{aligned} 0 &= \frac{1}{2}e^{-2} + ce^{-2} \\ c &= -\frac{1}{2} \end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned} y &= \frac{1}{2}t^2e^{-2t} - \frac{1}{2}e^{-2t} \\ &= \frac{1}{2}e^{-2t}\left(t^2 - 1\right) \qquad t \in \Re \end{aligned}$$

Here is a plot of the solution



0.15 Section 2.1, problem 29

Consider $y' + \frac{1}{4}y = 3 + 2\cos(2t)$ with $y(0) = 0$. (a) find the solution and describe its behavior for large t. (b) Determine t for which the solution first intersects the line $y = 12$.

0.15.1 Part (a)

This is first order, linear ODE of the form $y' + p(t)y = g(t)$ where $p(t) = \frac{1}{4}, g(t) = 3 + 2\cos(2t)$. Since $p(t), g(t)$ are continuous on the real line, then by theorem 1 a solution exists and is unique.

Integrating factor is $e^{\frac{1}{4}\int dt} = e^{\frac{t}{4}}$. Multiplying both sides by $e^{\frac{1}{4}t}$ gives

$$\frac{d}{dt}\left(ye^{\frac{t}{4}}\right) = 3e^{\frac{t}{4}} + 2e^{\frac{t}{4}}\cos 2t$$

Integrating

$$ye^{\frac{t}{4}} = 3\int e^{\frac{t}{4}}dt + 2\int e^{\frac{t}{4}}\cos(2t)\,dt + c \tag{1}$$

$\int e^{\frac{t}{4}}dt = 4e^{\frac{t}{4}}$. For the second integral, integration by parts is used. Using $\int udv = uv - \int vdv$, let $u = \cos(2t), dv = e^{\frac{t}{4}} \to du = -2\sin(2t), v = 4e^{\frac{t}{4}}$, hence

$$\begin{aligned}
I &= \int e^{\frac{t}{4}}\cos(2t)\,dt \\
&= 4\cos(2t)\,e^{\frac{t}{4}} - \int (-2\sin(2t))\,4e^{\frac{t}{4}}dt \\
&= 4\cos(2t)\,e^{\frac{t}{4}} + 8\int \sin(2t)\,e^{\frac{t}{4}}dt
\end{aligned}$$

Applying integration by parts again on $\int \sin(2t)\,e^{\frac{t}{4}}dt$. Let $u = \sin(2t), dv = e^{\frac{t}{4}}$, hence $du = 2\cos(2t), v = 4e^{\frac{t}{4}}$. Therefore the above becomes

$$\begin{aligned}
I &= 4\cos(2t)\,e^{\frac{t}{4}} + 8\left(4\sin(2t)\,e^{\frac{t}{4}} - \int 2\cos(2t)\,4e^{\frac{t}{4}}dt\right) \\
&= 4\cos(2t)\,e^{\frac{t}{4}} + 32\sin(2t)\,e^{\frac{t}{4}} - 64\int \cos(2t)\,e^{\frac{t}{4}}dt
\end{aligned}$$

But $I = \int e^{\frac{t}{4}}\cos(2t)\,dt$, hence the above is

$$I = 4\cos(2t)\,e^{\frac{t}{4}} + 32\sin(2t)\,e^{\frac{t}{4}} - 64I$$

Solving for I

$$\begin{aligned}
65I &= 4\cos(2t)\,e^{\frac{t}{4}} + 32\sin(2t)\,e^{\frac{t}{4}} \\
I &= \frac{4}{65}\cos(2t)\,e^{\frac{t}{4}} + \frac{32}{65}\sin(2t)\,e^{\frac{t}{4}}
\end{aligned}$$

Putting these results back into (1) gives

$$\begin{aligned}
ye^{\frac{t}{4}} &= 3\int e^{\frac{t}{4}}dt + 2\int e^{\frac{t}{4}}\cos(2t)\,dt + c \\
&= 3\left(4e^{\frac{t}{4}}\right) + 2\left(\frac{4}{65}\cos(2t)\,e^{\frac{t}{4}} + \frac{32}{65}\sin(2t)\,e^{\frac{t}{4}}\right) + c
\end{aligned}$$

Hence

$$\begin{aligned}
y &= 12 + 2\left(\frac{4}{65}\cos(2t) + \frac{32}{65}\sin(2t)\right) + ce^{-4t} \\
&= 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) + ce^{-4t}
\end{aligned}$$

Applying initial conditions

$$
\begin{aligned}
0 &= 12 + \frac{8}{65}\cos(0) + \frac{64}{65}\sin(0) + ce^{0} \\
&= 12 + \frac{8}{65} + c \\
c &= -12 - \frac{8}{65} \\
&= -\frac{788}{65}
\end{aligned}
$$

Hence the general solution is

$$
y(t) = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}e^{-4t} \qquad t \in \Re \tag{2}
$$

As t becomes very large, the term $\frac{788}{65}e^{-4t} \to 0$ and the solution only contains sinusoidal.

$$
\lim_{t\to\infty} y(t) = 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t)
$$

0.15.2 Part (b)

Solving for t when $y = 12$ results in

$$
\begin{aligned}
12 &= 12 + \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}e^{-4t} \\
0 &= \frac{8}{65}\cos(2t) + \frac{64}{65}\sin(2t) - \frac{788}{65}e^{-4t}
\end{aligned}
$$

It is not clear what method is supposed to be used to solve the above for t since it is non-linear. So $y(t)$ was first plotted and by inspection $y(t)$ cross the line $y = 12$ at about $t = 10$. Then using computer root finding with search starting at $t = 10$ the required value of t was found to be

$$
\boxed{t = 10.0658}
$$

Here is a plot of the solution given in (2)