Problem 1

Consider a small ball of radius s and moment of inertia I rolling off of a sphere of radius R. At what angle does the ball leave the surface of the sphere if it is gently displaced from the top (i.e. total energy is equal to potential energy of a stationary ball at the top)?

A lot of the new difficulty of this problem (relative to the particle sliding off of a sphere) comes from setting up the constraints correctly.

The Lagrangian (without constraints) is given by:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}I\dot{\phi}^2 - mgr\cos\theta$$

The distance from the center of the big sphere to the center of the small sphere is $R + s$, so the natural constraint for that is $r = R + s$. We also need to constrain the rolling of the ball relative to the motion of the ball along the sphere. The simplest constraint that will work is setting the arclength along the ball to the arclength along the surface of the sphere, i.e. $R\theta = s\phi$ (I was being overly cautious when I said that this wouldn't work in discussion). So the constraint function is given by $\lambda_1(r - R - s) + \lambda_2(R\theta - s\phi)$ and now our equations of motion become:

$$m\ddot{r} - mr\dot{\theta}^2 + mg\cos\theta + \lambda_1 = 0$$

$$m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) + mgr\sin\theta + \lambda_2 R = 0$$

$$I\ddot{\phi} - \lambda_2 s = 0$$

With some substitutions from the constraint equations and their time derivatives we can reduce this to:

$$-m(R + s)\dot{\theta}^2 + mg\cos\theta + \lambda_1 = 0$$

$$m(R + s)^2\ddot{\theta} + mg(R + s)\sin\theta + (\frac{R}{s})^2 I\ddot{\theta} = 0$$

And to completely solve this problem we need to use conservation of energy. The Hamiltonian (total energy) of the system is given by

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}I\dot{\phi}^2 + mgr\cos\theta = \frac{1}{2}m(R+s)^2\dot{\theta}^2 + \frac{1}{2}I(\frac{R}{s}\dot{\theta})^2 + mg(R+s)\cos\theta$$

And since the ball has been 'gently pushed' from the top of the sphere we have

$$\left[\frac{1}{2}m(R + s)^2 + \frac{1}{2}I(\frac{R}{s})^2\right]\dot{\theta}^2 + mg(R + s)\cos\theta = mg(R + s)$$

The ball will leave the surface of the sphere when the constraint force (corresponding to the normal force) that keeps the radius fixed changes signs, i.e. when $\lambda_1 = 0$. So we have

$$m(R+s)\dot{\theta}^2 = mg\cos\theta$$

Putting these two together we have

$$\left[\frac{1}{2}m(R+s)^2 + \frac{1}{2}I(\frac{R}{s})^2\right]\frac{g\cos\theta}{R+s} + mg(R+s)\cos\theta = mg(R+s)$$

$$\left[m(R+s)^2 + I(\frac{R}{s})^2 + 2m(R+s)^2\right]\cos\theta = 2m(R+s)^2$$

$$\cos\theta = \frac{2m(R+s)^2}{3m(R+s)^2 + I(\frac{R}{s})^2}$$

which you can see reduces to $\frac{2}{3}$ when $I = 0$, consistent with the simpler version of the problem.

Problem 2

The Lagrangian of a free particle in a magnetic field is given by $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q(A_x\dot{x} + A_y\dot{y})$, where A is the magnetic vector potential (whose curl is the magnetic field). Consider the field given by $A_x = \alpha y$, $A_y = 0$. Find the equations of motion and solve them. Find an integral of motion that is not energy and confirm that it is conserved.

The Lagrangian in this case is given by

$$\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + q\alpha y\dot{x}$$

So the equations of motion are given by

$$m\ddot{x} + q\alpha\dot{y} = 0$$

$$m\ddot{y} - q\alpha\dot{x} = 0$$

Let $\beta = \frac{q\alpha}{m}$ and note that we have $\dddot{x} + \beta\ddot{y} = 0$ and therefore

$$\dddot{x} + \beta^2\dot{x} = 0$$

Which is the equation of a harmonic oscillator in $\dot{x}$. The same equation can be derived for $\dot{y}$, so we know the solution must have the form

$$\dot{x} = A\cos(\beta t + \phi)$$

$$\dot{y} = B\cos(\beta t + \psi)$$

Plugging these into the original equations constrains A, ϕ, B, and ψ relative to each other. Assume without loss of generality that $\phi = 0$, then you can show that the solution must be of the form

$$\dot{x} = A\cos\beta t$$

$$\dot{y} = A\sin\beta t$$

So integrating gives the full solution:

$$x = x_0 + \frac{A}{\beta}\sin\beta t$$

$$y = y_0 - \frac{A}{\beta}\cos\beta t$$

For the integral of motion notice that the Lagrangian has no x dependence, therefore the corresponding generalized momentum $\frac{\partial L}{\partial \dot{x}}$ must be conserved.

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + q\alpha y$$

Plugging in the solution we got gives

$$mA\cos\beta t + q\alpha(y_0 - \frac{A}{\beta}\cos\beta t) = q\alpha y_0$$

which is in fact a conserved quantity. There actually is an analogous generalized momentum for y but it is less obvious why it should be conserved.

Problem 4

Consider an anharmonic (or nonlinear) spring with potential energy $V = \frac{1}{2}kr^2 + \frac{1}{4}\alpha r^4$ $(k, \alpha > 0)$ spinning at some fixed angular frequency ω_0 with a mass at the end. What are the equilibrium positions of the system as a function of ω_0 and which equilibria are stable?

The coordinates in this problem are given by

$$x = r\cos\omega_0 t$$

$$y = r\sin\omega_0 t$$

with derivatives

$$\dot{x} = \dot{r}\cos\omega_0 t - r\omega_0\sin\omega_0 t$$

$$\dot{y} = \dot{r}\sin\omega_0 t + r\omega_0\cos\omega_0 t$$

So our kinetic energy is given by

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\omega_0^2)$$

$U = \frac{1}{2}kr^2 + \frac{1}{4}\alpha r^4$

And our Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\omega_0^2) - \frac{1}{2}kr^2 - \frac{1}{4}\alpha r^4$$

Giving equation of motion

$$m\ddot{r} = -(k - \omega_0^2)r - \alpha r^3$$

This is at equilibrium when $\ddot{r} = 0$ or in other words $(k - \omega_0^2)r + \alpha r^3 = 0$. This is always solved by $r = 0$, but it is also solved by $r = \pm\sqrt{\frac{\omega_0^2 - k}{\alpha}}$. If $\omega_0^2 < k$ then these solutions are imaginary and unphysical. Although it's a little unusal relative to polar coordinates the way we set up the coordinate system allows negative r, so both of the equilibria are physical once $k < \omega_0^2$, although they look very similar. The stability of the equilibrium is determined by the derivative of the force as a function of position, which is $\frac{\partial}{\partial r}(-(k - \omega_0^2)r - \alpha r^3) = -(k - \omega_0^2) - 3\alpha r^2$. At $r = 0$ this is negative (and therefore stable) when $\omega_0^2 < k$ and positive (and therefore unstable) when $k < \omega_0^2$. At the other two equilibria we have $-(k - \omega_0^2) - 3\alpha\frac{\omega_0^2 - k}{\alpha} = 2(k - \omega_0^2)$. So these equilibria are stable only if the $r = 0$ equilibrium is unstable, i.e. when $\omega_0^2 < k$.

For the critical $\omega_0^2 = k$ case we have $m\ddot{r} = -\alpha r^3$ for the equations of motion. The second derivative test fails to determine stability, since it gives 0, so we need to consider the fourth derivative of the energy (the third derivative of the force) which is -6α, which is always negative and therefore stable.

Problem 3

Consider a double pendulum (i.e. a rod attached to another rod by a hinge) with both rods the same length ℓ, where the inner rod is constrained to rotate at a fixed angular velocity ω_0. What is the frequency of small oscillations of the system if there is no gravity?

While it would be possible with constraints it would be simpler to set this problem up directly in terms of the coordinates. The coordinates are given by (where θ is the angle of the second pendulum relative to some fixed vertical axis)

$$x = \ell(\cos\omega_0 t + \cos\theta)$$

$$y = \ell(\sin\omega_0 t + \sin\theta)$$

The time derivatives of these are

$$\dot{x} = -\ell(\omega_0 \sin\omega_0 t + \dot{\theta}\sin\theta)$$

$$\dot{y} = \ell(\omega_0 \cos\omega_0 t + \dot{\theta}\cos\theta)$$

So our kinetic energy is (using the trig identity $\sin a \sin b + \cos a \cos b = \cos(a-b)$)

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \omega_0^2 + 2\omega_0\dot{\theta}\cos(\theta - \omega_0 t))$$

And there is no potential energy since the system is somewhere where there's no gravity (like space). So now we have the equation of motion is

$$m\ell^2\ddot{\theta} - 2m\ell^2\omega_0\sin(\theta-\omega_0 t)(\dot{\theta}-\omega_0) + 2m\ell^2\omega_0\dot{\theta}\sin(\theta-\omega_0 t) = m\ell^2\ddot{\theta} + 2m\ell^2\omega_0^2\sin(\theta-\omega_0 t) = 0$$

Now since we're free to change coordinate systems, a more transparent coordiante system would be $\phi = \theta - \omega_0 t$, $\dot{\phi} = \dot{\theta} - \omega_0$, $\ddot{\phi} = \ddot{\theta}$. In these coordinate we have

$$m\ell^2\ddot{\phi} + 2m\ell^2\omega_0^2\sin\phi = 0$$

Which we know from experience is the equation of motion of a pendulum. In particular in the small ϕ approximation this becomes

$$m\ell^2\ddot{\phi} + 2m\ell^2\omega_0^2\phi = 0$$

$$\ddot{\phi} + 2\omega_0^2\phi = 0$$

So the frequency of small oscillations is given by $\omega = \sqrt{2}\omega_0$. This form makes sense in terms of dimensional analysis. We could have figured out at the beginning that that answer needed to be of the form $\omega = \#\omega_0$ for some fixed number $\#$.