HW7 Physics 311 Mechanics

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Contents

0.1 Problem 1

1. (10 points)
If a problem involves forces that cannot be derived from a potential (for example frictional forces), Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i \quad ,$$

where the Q_i are the generalized forces not derivable from a potential. The Q_i are defined through

$$Q_i = \vec{F} \cdot \frac{\partial \vec{r}}{\partial q_i} \quad .$$

Use this formalism for the following example.

A particle of mass m moves in a plane under the influence of a central force of potential $U(r)$ and also of a linear viscous drag $-mk(d\vec{r}/dt)$. Set up Lagrange's equations of motion and show that the angular momentum decays exponentially.

SOLUTION:

Using polar coordinates. The position vector of the particle is

$$\vec{r} = r\hat{r} + r\theta\hat{\theta} \tag{1}$$

We now find the Lagrangian

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \\ U &= V(r) \\ L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r) \end{aligned}$$

Since we are asked about the angular momentum part, we will just find the equation of motion for the θ generalized coordinates.

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \end{aligned}$$

Hence the EQM is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = Q_\theta$$

Where Q_θ is the generalized force corresponding to generalized coordinate θ. From (1)

$$d\vec{r} = dr\hat{r} + rd\theta\hat{\theta}$$

Hence

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$
$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

Therefore, the drag force can be written as

$$\vec{F} = -mk\frac{d\vec{r}}{dt}$$
$$= -mk\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \tag{2}$$

Applying the definition of $Q_\theta = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta}$ gives

$$Q_\theta = -mk\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \cdot \frac{\partial}{\partial \theta}\left(r\hat{r} + r\theta\hat{\theta}\right)$$
$$= -mk\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) \cdot \left(r\hat{\theta}\right)$$
$$= -mkr^2\dot{\theta} \tag{3}$$

Now that we found Q_θ, the EQM is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -mkr^2\dot{\theta}$$

We notice the same term on both sides (but for a constant k). The above is the same as

$$\frac{d}{dt}(Z) = -kZ$$

The solution must be exponential $Z = e^{-kt} + C$ where C is some constant. This means

$$mr^2\dot{\theta} = e^{-kt} + C$$

But $mr^2\dot{\theta}$ is the angular momentum. Hence, for positive k, the angular momentum decays exponentially with time.

0.2 Problem 2

2. (10 points)
In the lecture, we derived a formula for the percentage increase in speed necessary to transfer a spacecraft from low Earth orbit of radius r_0 to an elliptical orbit with the Moon at the apogee at distance r_1.

(1) Find the fractional change in the apogee $\delta r_1/r_1$ as a function of a small fractional change in the ratio of required perigee speed v_0 to circular orbit speed v_c, $\delta(v_0/v_c)/(v_0/v_c)$.

(2) If the speed ratio is 1% too great, by how much would the spacecraft miss the Moon?

SOLUTION:

0.2.1 Part (1)

From class notes, we found

$$\frac{v_o}{v_c} = \sqrt{\frac{2r_1}{r_1 + r_o}} = \sqrt{\frac{2}{1 + \frac{r_o}{r_1}}}$$

Where v_c is the velocity in the circular orbit just before speed boost, and v_o is the speed at the perigee of the ellipse just after the speed boost, and r_0 is the perigee distance and r_1 is the apogee distance. We need to find $\frac{\delta\left(\frac{v_o}{v_c}\right)}{\left(\frac{v_o}{v_c}\right)}$. To make the calculation easier, let $\frac{v_o}{v_c} = z$. Then we have

$$z = \left(\frac{2}{1 + \frac{r_o}{r_1}}\right)^{\frac{1}{2}}$$

Hence

$$\frac{\delta z}{\delta r_1} = \frac{1}{2}\frac{1}{\left(\frac{2}{1+\frac{r_o}{r_1}}\right)^{\frac{1}{2}}}\frac{\delta}{\delta r_1}\left(\frac{2}{1 + \frac{r_o}{r_1}}\right)$$

But $\left(\frac{2}{1+\frac{r_o}{r_1}}\right)^{\frac{1}{2}} = z$ so the above becomes

$$\begin{aligned}
\frac{\delta z}{\delta r_1} &= \frac{1}{2}\frac{1}{z}\frac{\delta}{\delta r_1}\left(\frac{2}{1+\frac{r_o}{r_1}}\right) \\
&= \frac{1}{2}\frac{1}{z}\left(2\frac{\delta}{\delta r_1}\left(1+\frac{r_o}{r_1}\right)^{-1}\right) \\
&= \frac{1}{2}\frac{1}{z}\left(2\left(-1\right)\left(1+\frac{r_o}{r_1}\right)^{-2}\frac{\delta}{\delta r_1}\left(\frac{r_o}{r_1}\right)\right) \\
&= \frac{1}{2}\frac{1}{z}\left(2\left(-1\right)\left(1+\frac{r_o}{r_1}\right)^{-2}\left(-r_o\right)r_1^{-2}\right) \\
&= \frac{1}{2}\frac{1}{z}\left(\frac{2}{\left(1+\frac{r_o}{r_1}\right)^2}\frac{r_o}{r_1^2}\right)
\end{aligned}$$

Since $\frac{2}{\left(1+\frac{r_o}{r_1}\right)} = z^2$ the above simplifies to

$$\begin{aligned}
\frac{\delta z}{\delta r_1} &= \frac{1}{2}\frac{1}{z}\left(z^2\frac{1}{\left(1+\frac{r_o}{r_1}\right)}\frac{r_o}{r_1^2}\right) \\
&= \frac{1}{2}z\frac{r_o}{r_1^2\left(1+\frac{r_o}{r_1}\right)} \\
&= \frac{1}{2}z\frac{r_o}{r_1\left(r_1+r_o\right)}
\end{aligned}$$

We want to find $\frac{\delta z}{z}$, therefore the above can be written as

$$\frac{\delta z}{z} = \frac{\delta r_1}{r_1}\frac{1}{2}\frac{r_o}{\left(r_1+r_o\right)}$$

Or in terms of $\frac{\delta r_1}{r_1}$ the above becomes

$$\frac{\delta r_1}{r_1} = \frac{\delta z}{z}\left(2\frac{\left(r_1+r_o\right)}{r_o}\right)$$

Since $z = \frac{v_o}{v_c}$, the reduces to

$$\boxed{\frac{\delta r_1}{r_1} = \frac{\delta\left(\frac{v_o}{v_c}\right)}{\left(\frac{v_o}{v_c}\right)}\left(2\frac{\left(r_1+r_o\right)}{r_o}\right)}$$

0.2.2 Part (2)

For $\frac{\delta\left(\frac{v_o}{v_c}\right)}{\left(\frac{v_o}{v_c}\right)} = 0.01$ then

$$\frac{\delta r_1}{r_1} = 0.01\left(2\frac{(r_1 + r_o)}{r_o}\right)$$

Using $r_0 = \frac{1}{60}r_1$ in the above gives

$$\begin{aligned}\frac{\delta r_1}{r_1} &= 0.01\left(2\frac{\left(r_1 + \frac{1}{60}r_1\right)}{\frac{1}{60}r_1}\right)\\ &= 1.22\end{aligned}$$

This means that δr_1 is 22% of r_1. The spacecraft will miss the moon by 22% of r_1. (This seems like a big miss for such small speed boost error)

0.3 Problem 3

3. (10 points)
A particle of mass m moves in a circular orbit of radius $r = a$ under the influence of the central attractive force $F(r) = -c\exp(-br)/r^2$, where c and b are positive constants.

(1) What is the effective potential energy in terms of r and the angular momentum ℓ? (Your answer may contain an integral.)

(2) Write down the Lagrangian of the system. Derive the equation of motion.

(3) For what values of b will this orbit be stable?

(4) Find the apsidal angle Ψ for nearly circular orbits in this field.

SOLUTION:

0.3.1 Part (1)

One way to find $U_{eff}(r)$ is to find the Largrangian L and pick the terms in it that have r without time derivative in them.

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

To find $U(r)$, since we are given $f(r)$ and since $f(r) = -\frac{\partial U(r)}{\partial r}$, then

$$\begin{aligned} U(r) &= -\int f(r)\,dr \\ &= \int \frac{ce^{-rb}}{r^2}dr \end{aligned}$$

Hence

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \int \frac{ce^{-rb}}{r^2}dr \end{aligned}$$

Hence

$$U_{eff}(r) = \frac{1}{2}mr^2\dot{\theta}^2 - \int \frac{ce^{-rb}}{r^2}dr$$

In terms of $l = mr^2\dot{\theta}$, the above can be written as

$$U_{eff}(r) = \frac{1}{2}l\dot{\theta} - \int \frac{ce^{-rb}}{r^2}dr$$

Or, it can also be written, as done in class notes, as

$$U_{eff}(r) = \frac{1}{2}\frac{l^2}{mr^2} - \int \frac{ce^{-rb}}{r^2}dr$$

0.3.2 Part (2)

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \int \frac{ce^{-rb}}{r^2}dr$$

Hence

$$\begin{aligned} \frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - \frac{ce^{-rb}}{r^2} \\ \frac{\partial L}{\partial \dot{r}} &= m\dot{r} \end{aligned}$$

The equation of motion for r is

$$\begin{aligned} m\ddot{r} - \left(mr\dot{\theta}^2 - \frac{ce^{-rb}}{r^2}\right) &= 0 \\ m\ddot{r} - mr\dot{\theta}^2 + \frac{ce^{-rb}}{r^2} &= 0 \\ m\ddot{r} - mr\dot{\theta}^2 &= F(r) \end{aligned}$$

Written in terms of angular momentum, since $\dot{\theta} = \frac{l}{mr^2}$ (integral of motion) where l is the angular momentum, the above becomes

$$m\ddot{r} - \frac{l^2}{mr^3} = F(r) \tag{1}$$

For θ,

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= 0 \\ \frac{\partial L}{\partial \dot{\theta}} &= mr^2\dot{\theta} \end{aligned}$$

The equation of motion for θ is

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = C$$

Where C is some constant. The full EQM for θ is

$$\begin{aligned} m\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right) &= 0 \\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} &= 0 \end{aligned}$$

0.3.3 Part (3)

To check for stability, since this is circular orbit, the radius is constant, say a. Then we perturb it by replacing a by $x + a$ where $x \ll a$ in the equation of motion $m\ddot{r} - \frac{l^2}{mr^3} = F(r)$ and it becomes

$$\begin{aligned} m\ddot{x} - \frac{l^2}{m(x+a)^3} &= F(x+a) \\ m\ddot{x} &= \frac{l^2(x+a)^{-3}}{m} + F(a+x) \end{aligned}$$

Since $x \ll a$, we expand $(x+a)^{-3}$ in Binomial and obtain

$$\begin{aligned} m\ddot{x} &= \frac{l^2}{ma^3}\left(1+\frac{x}{a}\right)^{-3} + F(a+x) \\ &\approx \frac{l^2}{ma^3}\left(1-\frac{3x}{a}+\cdots\right) + \overbrace{F(a) + xF'(a) + \cdots}^{\text{Taylor expansion}} \end{aligned}$$

Since circular orbit, then $\ddot{r}=0$ and the EQM motion becomes $-\frac{l^2}{ma^3} = F(a)$. Using this to replace $\frac{l^2}{ma^3}$ with in the above expression we find

$$\begin{aligned} m\ddot{x} &\approx -F(a)\left(1-\frac{3x}{a}\right) + F(a) + xF'(a) \\ &= -F(a) + F(a)\frac{3x}{a} + F(a) + xF'(a) \\ &= F(a)\frac{3x}{a} + xF'(a) \end{aligned}$$

Hence

$$\begin{aligned} m\ddot{x} + \left(-F(a)\frac{3x}{a} - xF'(a)\right) &= 0 \\ m\ddot{x} + \left(-\frac{3}{a}F(a) - F'(a)\right)x &= 0 \end{aligned}$$

This perturbation motion is stable if $\left(-\frac{3}{a}F(a) - F'(a)\right) > 0$. But $F(a) = -\frac{ce^{-ba}}{a}$ and $F'(a) = \frac{ce^{-ab}}{a^2} + \frac{bce^{-ab}}{a}$, hence

$$\begin{aligned} \Delta &= -\frac{3}{a}F(a) - F'(a) \\ &= -\frac{3}{a}\left(-\frac{ce^{-ba}}{a}\right) - \left(\frac{ce^{-ab}}{a^2} + \frac{bce^{-ab}}{a}\right) \end{aligned}$$

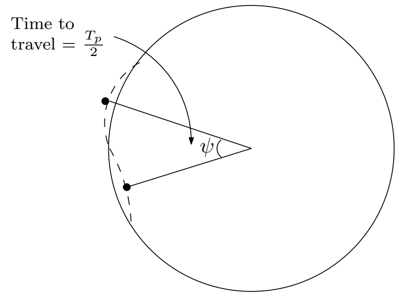
We want the above to be positive for stability. Simplifying gives

$$\begin{aligned} \Delta &= \frac{3ce^{-ba}}{a^2} - \frac{ce^{-ab}}{a^2} - \frac{bce^{-ab}}{a} \\ &= \frac{2ce^{-ba}}{a^2} - \frac{bce^{-ab}}{a} \\ &= \frac{2ce^{-ba} - abce^{-ab}}{a^2} \\ &= \frac{ce^{-ba}}{a^2}(2-ab) \end{aligned}$$

Therefore, we want $(2-ab) > 0$ or $2 > ab$ or

$$\boxed{b < \frac{2}{a}}$$

0.3.4 Part (4)



The angle ψ is found from

$$\psi = \frac{T_p}{2}\dot{\theta} \tag{1}$$

Where T_p is the period of oscillation due to the perturbation from the exact circular orbit, and $\dot{\theta}$ is the angular velocity on the circular orbit. But

$$\dot{\theta} \approx \frac{l}{ma^2} \tag{2}$$

But from part(3) we found that

$$-\frac{l^2}{ma^3} = F(a)$$
$$l = \sqrt{-F(a)\,ma^3}$$

Therefore (2) becomes

$$\dot{\theta} \approx \frac{1}{ma^2}\sqrt{-F(a)\,ma^3}$$
$$= \sqrt{\frac{-F(a)}{ma}}$$

We now find T_p. Since the perturbation equation of motion, from part (3) is $m\ddot{x}+\left(-\frac{3}{a}F(a) - F'(a)\right)x = 0$, which is of the form

$$\ddot{x} + \overbrace{\left(\frac{-\frac{3}{a}F(a) - F'(a)}{m}\right)}^{\omega_0^2} x = 0$$

Then, the natural frequency is $\omega = \sqrt{\frac{\left(-\frac{3}{a}F(a)-F'(a)\right)}{m}}$, therefore

$$\frac{2\pi}{T_p} = \sqrt{\frac{-\frac{3}{a}F(a) - F'(a)}{m}}$$

$$T_p = 2\pi\sqrt{\frac{m}{-\frac{3}{a}F(a) - F'(a)}}$$

Equation (1) now becomes

$$\begin{aligned}\psi &= \frac{T_p}{2}\dot{\theta} \\ &= \pi\sqrt{\frac{m}{-\frac{3}{a}F(a) - F'(a)}}\sqrt{\frac{-F(a)}{ma}} \\ &= \pi\sqrt{\frac{-F(a)}{-3F(a) - aF'(a)}} \\ &= \pi\sqrt{\frac{F(a)}{3F(a) + aF'(a)}}\end{aligned}$$

But $F(a) = -\frac{ce^{-ba}}{a^2}$ and $F'(a) = \frac{ce^{-ab}}{a^2} + \frac{bce^{-ab}}{a}$ then the above becomes

$$\begin{aligned}\psi &= \pi\sqrt{\frac{-\frac{ce^{-ba}}{a^2}}{3F(a) + aF'(a)}} \\ &= \pi\sqrt{\frac{-\frac{ce^{-ba}}{a^2}}{3\left(-\frac{ce^{-ba}}{a^2}\right) + a\left(\frac{ce^{-ab}}{a^2} + \frac{bce^{-ab}}{a}\right)}} \\ &= \pi\sqrt{\frac{-\frac{ce^{-ba}}{a^2}}{-3\frac{ce^{-ba}}{a^2} + \left(\frac{ce^{-ab}+abce^{-ab}}{a}\right)}} \\ &= \pi\sqrt{\frac{-ce^{-ba}}{-3ce^{-ba} + \left(ace^{-ab} + a^2bce^{-ab}\right)}} \\ &= \pi\sqrt{\frac{-1}{-3 + a + a^2b}}\end{aligned}$$

Hence

$$\boxed{\psi = \pi\sqrt{\frac{1}{3-a(1+ab)}}}$$

0.4 Problem 4

4. (10 points)
A ball is dropped from a height h onto a horizontal pavement. If the coefficient of restitution is ϵ, show that the total vertical distance the ball goes before the rebounds end is $h(1+\epsilon^2)/(1-\epsilon^2)$. What is the total length of time that the ball bounces?

SOLUTION:

The first time the ball falls from height h it will have speed of $v_1 = \sqrt{2gh}$ just before hitting the platform, which is found using

$$mgh = \frac{1}{2}mv_1^2$$

On bouncing back, it will have speed of $v_1' = \varepsilon\sqrt{2gh}$. It will then travel up a distance of $h_1 = \varepsilon^2 h$ which is found by solving for $h_1$from

$$mgh_1 = \frac{1}{2}m\left(v_1'\right)^2$$

The second time it it falls back it will have speed of $v_2 = \varepsilon\sqrt{2gh_1}$. When it bounces back up, it will have speed $v_2' = \varepsilon^2\sqrt{2gh_1}$ and now it will travel up a distance of $h_2 = \varepsilon^4 h$ which is found by solving for h_2 from

$$mgh_2 = \frac{1}{2}m\left(v_2'\right)^2$$

This process will continue until the ball stops. We see that the distance travelled at each bouncing is

$$\Delta = \left\{h, 2\varepsilon^2 h, 2\varepsilon^4 h, 2\varepsilon^6 h, \cdots, 2\varepsilon^{2n} h\right\}$$

We added 2 to each bounce after the first one to count for going up and then coming down the same distance. The first time it will only have one h. We now can calculate total distance travelled Δ as

$$\begin{aligned}\Delta &= h + 2\varepsilon^2 h + 2\varepsilon^4 h + \cdots \\ &= h\left(1 + 2\varepsilon^2 + 2\varepsilon^4 + \cdots\right)\end{aligned}$$

The above can be written as

$$\Delta = h\left(2 + 2\varepsilon^2 + 2\varepsilon^4 + \cdots\right) - h \tag{1}$$

But since $\varepsilon \leq 1$ the series sum is

$$2 + 2\varepsilon^2 + 2\varepsilon^4 + \cdots = 2\sum_{n=0}^{\infty}\varepsilon^{2n} = 2\frac{1}{1-\varepsilon^2}$$

Therefore (1) becomes

$$\begin{aligned}\Delta &= \frac{2h}{1-\varepsilon^2} - h\\ &= \frac{2h - h\left(1-\varepsilon^2\right)}{1-\varepsilon^2}\\ &= \frac{2h - h + h\varepsilon^2}{1-\varepsilon^2}\end{aligned}$$

Hence total distance is

$$\boxed{\frac{h\left(1+\varepsilon^2\right)}{1-\varepsilon^2}}$$

To find the total time of all ball bounces, we need to find the time it takes to travel in each bounce. The time it takes to fall distance h is $\sqrt{\frac{2h}{g}}$, using the information we found about each h_i from above, we now set up the sequence of times we we did for distances

$$\Delta_{time} = \left\{\sqrt{\frac{2h}{g}}, 2\sqrt{\frac{2\varepsilon^2 h}{g}}, 2\sqrt{\frac{2\varepsilon^4 h}{g}}, 2\sqrt{\frac{2\varepsilon^6 h}{g}}, \cdots\right\}$$

Adding the times gives

$$\begin{aligned}\Delta &= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2\varepsilon^2 h}{g}} + 2\sqrt{\frac{2\varepsilon^4 h}{g}} + 2\sqrt{\frac{2\varepsilon^6 h}{g}}\\ &= \sqrt{\frac{2h}{g}}\left(1 + 2\varepsilon + 2\varepsilon^2 + 2\varepsilon^3 + 2\varepsilon^4 \cdots\right)\\ &= \sqrt{\frac{2h}{g}}\left(2 + 2\varepsilon + 2\varepsilon^2 + 2\varepsilon^3 + 2\varepsilon^4 \cdots\right) - \sqrt{\frac{2h}{g}}\\ &= \sqrt{\frac{2h}{g}} \sum_{n=0}^{\infty} 2\varepsilon^n - \sqrt{\frac{2h}{g}}\end{aligned}$$

But $2\sum_{n=0}^{\infty} \varepsilon^n = 2\frac{1}{1-\varepsilon}$, hence the above becomes

$$\begin{aligned}\Delta &= \sqrt{\frac{2h}{g}}\frac{2}{1-\varepsilon} - \sqrt{\frac{2h}{g}}\\ &= \sqrt{\frac{2h}{g}}\left(\frac{2}{1-\varepsilon} - 1\right)\\ &= \sqrt{\frac{2h}{g}}\left(\frac{2-(1-\varepsilon)}{1-\varepsilon}\right)\end{aligned}$$

Hence total time is

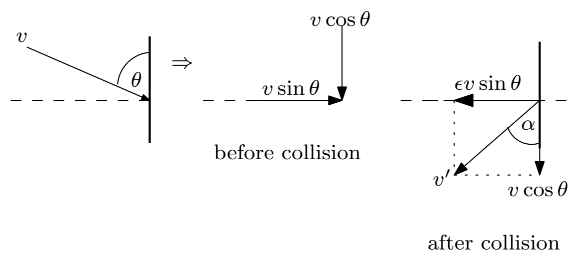
$$\boxed{\sqrt{\frac{2h}{g}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)}$$

0.5 Problem 5

5. (10 points)
A particle of mass m strikes a wall at an angle θ with respect to the normal. The collision is inelastic with coefficient of restitution ϵ. Find the rebound angle of the particle after collision with the wall.

SOLUTION:

First we make a diagram showing the geometry involved



We resolve the incoming velocity into its x, y components and apply conservation of linear momentum to each part. The vertical component remain the same after collision since it is parallel to the wall. Hence

$$v'_y = v_y = v\cos\theta$$

While the x component will change to

$$v'_x = \varepsilon v_x = \varepsilon v\sin\theta$$

By definition of ε. Therefore we see that after collision

$$\begin{aligned}\tan\alpha &= \frac{\varepsilon v\sin\theta}{v\cos\theta}\\ &= \varepsilon\tan\theta\end{aligned}$$

Hence

$$\boxed{\alpha = \arctan(\varepsilon\tan\theta)}$$