HW6 Physics 311 Mechanics

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Contents

0.1 Problem 1

1. (5 points)
An Earth satellite has a speed of 28,070 km/h when it is at its perigee of 220 km above Earth's surface. Find the apogee distance, its speed at apogee, and its period of revolution.

SOLUTION:

From the vis-viva relation

$$v_{perigee} = \sqrt{\frac{\alpha}{m}\left(\frac{2}{r_p} - \frac{1}{a}\right)} \tag{1}$$

Where m is the reduced mass and $\alpha = GM_{earth}m_{satt}$, which reduces to GM_{earth} and known constant called the Standard gravitational parameter which for earth is (From table)

$$\frac{\alpha}{m} = 398600 \text{ km}^3/s^2$$

And

$$\begin{aligned} r_p &= 220 + 6378 \\ &= 6598 \text{ km} \end{aligned}$$

Where 6378 is the equatorial radius of earth. And $v_{perigee} = 28070$ km/h. Therefore, we use (1) to solve for a, the length of the semimajor axes of the elliptical orbit of the satellite around the earth. From (1), by squaring both sides

$$\begin{aligned} v_p^2 &= \frac{\alpha}{m}\left(\frac{2}{r_p} - \frac{1}{a}\right) \\ \left(\frac{28070}{60 \times 60}\right)^2 &= 398600\left(\frac{2}{220 + 6378} - \frac{1}{a}\right) \end{aligned}$$

Solving for a gives

$$a = 6640 \text{ km}$$

Hence the apogee distance is

$$2a = 13280 \text{ km}$$

We can also find

$$\begin{aligned} r_a &= 2a - r_p \\ &= 13280 - 6598 \\ &= 6682 \text{ km} \end{aligned}$$

When the satellite is at the apogee, it will be above the earth at height of

$$\begin{aligned} h_a &= r_a - r_{earth} \\ &= 6682 - 6378 \\ &= 304 \text{ km} \end{aligned}$$

The period T is given by

$$\begin{aligned} T &= 2\pi\sqrt{\frac{a^3}{\frac{\alpha}{m}}} \\ &= 2\pi\sqrt{\frac{6640^3}{398600}} \\ &= 5385 \text{ sec} \\ &= \frac{5385}{60 \times 60} = 1.496 \text{ hr} \end{aligned}$$

0.2 Problem 2

2. (5 points)
A spacecraft is in circular orbit 200 km above Earth's surface. What minimum velocity kick must be applied to let the spacecraft escape from Earth's influence? What is the spacecraft's escape trajectory with respect to Earth?

SOLUTION:

The total energy is

$$E = \frac{1}{2} m\dot{r}^2 + U_{effective}$$

The escape velocity is when $U_{effective} = 0$, therefore

$$0 = -U + \frac{l^2}{2mr^2}$$

But angular momentum $l = mrv$ and $U = \frac{GM_e m}{r}$, hence the above becomes

$$\begin{aligned} 0 &= -\frac{GM_e m}{r} + \frac{m^2 r^2 v^2}{2mr^2} \\ &= -\frac{GM_e m}{r} + \frac{mv^2}{2} \\ &= -\frac{GM_e}{r} + \frac{v^2}{2} \end{aligned} \tag{1}$$

Now we are given that the satellite was at $r = 200 + 6378 = 6578$ km (this is r_p for the new orbit as well). Using $GM_e = 398600 \text{ km}^3/s^2$ from tables then we solve now for v in (1), which will be the new velocity. Hence

$$\begin{aligned} 0 &= -\frac{398600}{6578} + \frac{v^2}{2} \\ v &= 11.009 \text{ km/sec} \end{aligned}$$

Before this, the spacecraft was in circular orbit. So its speed was

$$\begin{aligned} v_c &= \sqrt{\frac{\alpha}{m}\frac{1}{r}} \\ &= \sqrt{\frac{398600}{6578}} \\ &= 7.784 \text{ km/sec} \end{aligned}$$

The difference is the minimum speed kick needed, which is

$$11.009 - 7.784 = 3.225 \text{ km/sec}$$

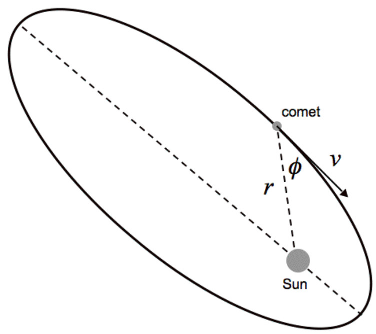
This orbit is ***parabolic*** since $U_{effective} = 0$ as seen on the $U_{effective}$ vs. r graph. parabolic is the first orbit beyond elliptic that do not contain turn points. The next orbit is hyperbolic.

0.3 Problem 3

3. (15 points)
A comet is observed to have a speed v when it is at a distance r from the Sun. Its direction of motion makes an angle ϕ with the radius vector from the Sun.

(1) Find the eccentricity of the comet's orbit.

(2) If the velocity of the comet is expressed as q times the Earth's velocity and its distance to the Sun as d astronomical units, show that the orbit of the comet is hyperbolic, parabolic, or elliptic, depending on whether the quantity $q^2 d$ is greater than, equal to, or less than 2, respectively.



SOLUTION:

0.3.1 Part (1)

Eccentricity is defined as (for all conic sections)

$$e = \sqrt{1 + \frac{2El^2}{m\alpha^2}} \tag{1}$$

Where $\alpha = GM_{sun}m$ and l is the angular momentum

$$\begin{aligned} l &= m\,|\boldsymbol{r} \times \boldsymbol{v}| \\ &= mrv\sin\phi \end{aligned}$$

Therefore (1) becomes

$$e = \sqrt{1 + \frac{2E\left(rv\sin\phi\right)^2}{m\left(GM_{sun}\right)^2}}$$

The energy of the comet is given by $E=\frac{1}{2}mv^2-\frac{GM_{sun}m}{r}$, then the above becomes

$$\begin{aligned} e &= \sqrt{1+\frac{2\left(\frac{1}{2}mv^2-\frac{GM_{sun}m}{r}\right)\left(rv\sin\phi\right)^2}{m\left(GM_{sun}\right)^2}} \\ &= \sqrt{1+\left(\frac{2\left(\frac{1}{2}mv^2-\frac{GM_{sun}m}{r}\right)}{m}\right)\left(\frac{rv\sin\phi}{GM_{sun}}\right)^2} \\ &= \sqrt{1+\left(v^2-\frac{2GM_{sun}}{r}\right)\left(\frac{rv\sin\phi}{GM_{sun}}\right)^2} \end{aligned}$$

0.3.2 Part (2)

Let $v=qv_e$ where v_e is earth velocity around the sun and let $r=dr_e$ where r_e is the astronomical unit (the distance between the earth and sun) then result of part (1) becomes

$$e=\sqrt{1+\left(\left(qv_e\right)^2-\frac{2GM_{sun}}{dr_e}\right)\left(\frac{dr_eqv_e\sin\phi}{GM_{sun}}\right)^2} \tag{2}$$

Looking at the earth/sun system, we know that

$$\begin{aligned} \frac{GM_{sun}m_{earth}}{r_e^2} &= \frac{m_{earth}v_e^2}{r_e} \\ \frac{GM_{sun}}{r_e} &= v_e^2 \\ GM_{sun} &= r_ev_e^2 \end{aligned}$$

Replacing GM_{sun} in (2) by the above result gives

$$\begin{aligned} e &= \sqrt{1+\left(\left(qv_e\right)^2-\frac{2r_ev_e^2}{dr_e}\right)\left(\frac{dr_eqv_e\sin\phi}{r_ev_e^2}\right)^2} \\ &= \sqrt{1+\left(\left(qv_e\right)^2-\frac{2v_e^2}{d}\right)\left(\frac{dq\sin\phi}{v_e}\right)^2} \\ &= \sqrt{1+\left(q^2-\frac{2}{d}\right)\left(dq\sin\phi\right)^2} \\ &= \sqrt{1+\left(\frac{q^2d-2}{d}\right)\left(dq\sin\phi\right)^2} \end{aligned}$$

We are now ready to answer the final part. If $q^2d=2$ then $e=1$ which means it is parabolic. If $q^2d>2$ then $\left(\frac{q^2d-2}{d}\right)$ is positive and the expression inside $\sqrt{.}$ is larger than one, and hence $e>1$, which means the orbit is hyperbolic. Finally, if $q^2d<2$ then $\left(\frac{q^2d-2}{d}\right)$ is negative, and the expression inside $\sqrt{.}$ is less than one, which means $e<1$ and hence the orbit is elliptic.

0.4 Problem 4

4. (10 points)
If the minimum and maximum velocities of a moon rotating around a planet are $v_{min} = v - v_0$ and $v_{max} = v + v_0$, show that the eccentricity is given by

$$e = \frac{v_0}{v} \quad .$$

SOLUTION:

The angular momentum l is constant. At perigee, where the speed is maximum, we have

$$l_p = m v_{\max} r_p$$

And at apogee, where the speed is minimum, we have

$$l_a = m v_{\min} r_a$$

Since l is constant, then

$$\begin{aligned} m v_{\max} r_p &= m v_{\min} r_a \\ v_{\max} r_p &= v_{\min} r_a \end{aligned} \tag{1}$$

But

$$\begin{aligned} r_a &= a(1+e) \\ r_p &= a(1-e) \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} v_{\max} a (1-e) &= v_{\min} a (1+e) \\ v_{\max} (1-e) &= v_{\min} (1+e) \\ v_{\max} - e v_{\max} &= v_{\min} + e v_{\min} \\ v_{\max} - v_{\min} &= e (v_{\min} + v_{\max}) \\ e &= \frac{v_{\max} - v_{\min}}{v_{\min} + v_{\max}} \end{aligned}$$

Replacing $v_{\max} = v + v_0$ and $v_{\min} = v - v_0$ gives

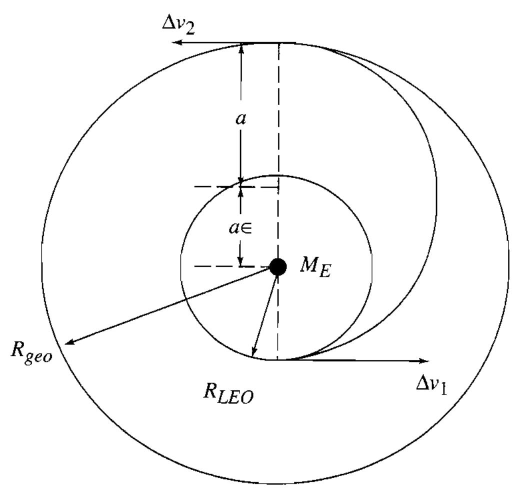
$$\begin{aligned} e &= \frac{(v + v_0) - (v - v_0)}{(v + v_0) + (v - v_0)} \\ &= \frac{2 v_0}{2 v} \\ &= \frac{v_0}{v} \end{aligned}$$

0.5 Problem 5

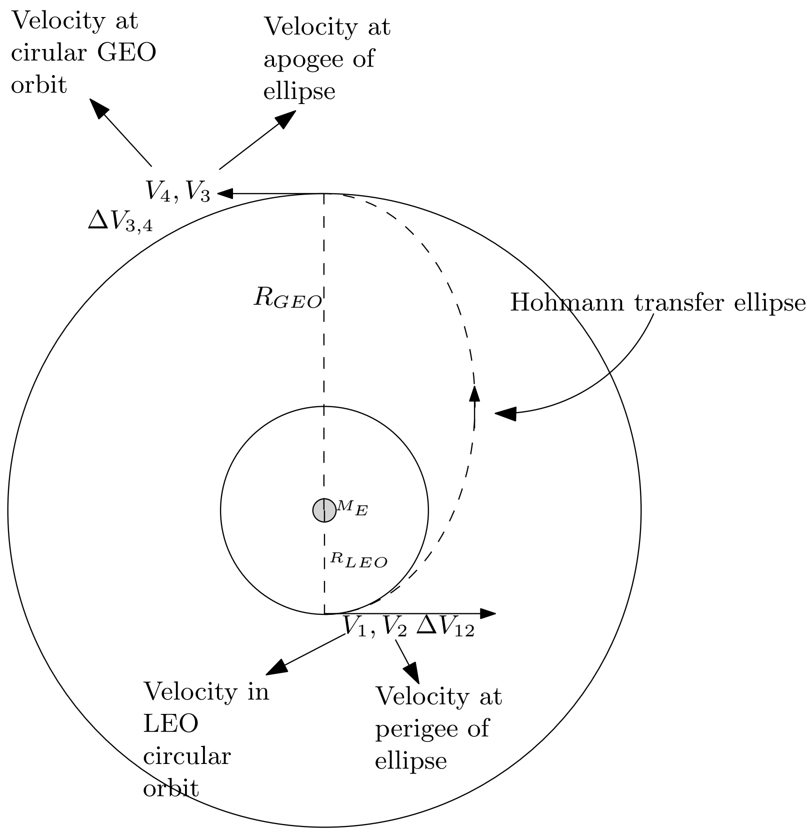
5. (15 points)
When a spacecraft is placed into geosynchronous orbit, it is first launched, along with a propulsion stage, into a near circular low Earth orbit (LEO) using a booster rocket. Then the propulsion stage is fired and the spacecraft is transferred to an elliptical "transfer" orbit designed to take it to geosynchronous altitude at orbital apogee. At apogee, the propulsion stage is fired again to take it out of the elliptical orbit back into a circular (now geosynchronous) orbit.

(1) Calculate the required velocity boost Δv_1 to move the satellite from its circular low Earth orbit into the elliptical transfer orbit.

(2) Calculate the required velocity boost Δv_2 to move the satellite from the elliptical transfer orbit into the geosynchronous circular orbit.



SOLUTION:



0.5.1 Part (1)

In this calculation, the standard symbol μ is used for GM_{earth} which is the Standard gravitational parameter (in class, we used $\frac{\alpha}{m}$ for this same parameter). For earth

$$\mu = 398600 \text{ km}^3/s^2$$

The first step is to find a for the transfer ellipse. This is given by

$$a = \frac{R_{LEO} + R_{GEO}}{2}$$

Next, we first find V_1, which is velocity in the LEO circular orbit just before initial kick to V_2. Since this is circular, the speed is given by

$$V_1 = \sqrt{\frac{\mu}{R_{LEO}}}$$

Next step is to find V_2, which is the speed at the perigee of the ellipse (the transfer orbit). This is given by the standard vis-viva relation

$$V_2 = \sqrt{\mu\left(\frac{2}{R_{LEO}} - \frac{1}{a}\right)} \tag{1}$$

Where $R_{LEO} = r_{perigee}$ for the ellipse. Now that we found V_2 and V_1, then

$$\begin{aligned}\Delta V_{12} &= V_2 - V_1 \\ &= \sqrt{\mu\left(\frac{2}{R_{LEO}} - \frac{1}{a}\right)} - \sqrt{\frac{\mu}{R_{LEO}}}\end{aligned}$$

0.5.2 Part (2)

When at the apogee of the transfer ellipse, the speed is given by

$$V_3 = \sqrt{\mu\left(\frac{2}{R_{GEO}} - \frac{1}{a}\right)}$$

We now want to be of GEO circular orbit, hence

$$V_4 = \sqrt{\frac{\mu}{R_{GEO}}}$$

And therefore, the speed boost is

$$\begin{aligned}\Delta V_{34} &= V_4 - V_3 \\ &= \sqrt{\frac{\mu}{R_{GEO}}} - \sqrt{\mu\left(\frac{2}{R_{GEO}} - \frac{1}{a}\right)}\end{aligned}$$