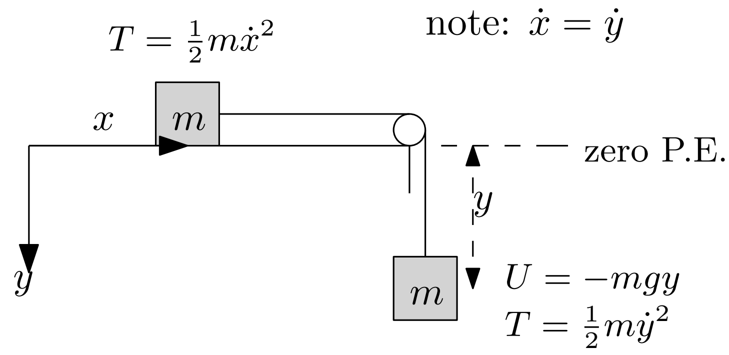
0.1 Problem 1

1. (5 points)
Two blocks of equal mass m are connected by an extensionless uniform string of length l. One block is placed on a smooth horizontal table, the other block hangs over the edge, the string passing over a frictionless pulley. Determine the Lagrangian of the system and find the acceleration of the blocks, assuming the mass of the string is negligible.

SOLUTION



$$L = T - U$$

Where U is the potential energy of the whole system and T is the kinetic energy of the whole system. The two masses will have the same speed since the string does not stretch. This means $\dot{x} = \dot{y}$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2$$

Since $\dot{x} = \dot{y}$, we can write the above as

$$T = m\dot{y}^2$$

The potential energy U, using zero as the level shown in the above diagram is

$$U = -mgy$$

Hence the Lagrangian is $L = T - U$ or

$$\boxed{L = m\dot{y}^2 + mgy}$$

To find equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0$$

But $\frac{\partial L}{\partial y} = mg$ and $\frac{d}{dt}\frac{\partial L}{\dot{y}} = \frac{d}{dt}\left(2m\dot{y}\right) = 2m\ddot{y}$, hence the above becomes

$$2m\ddot{y} - mg = 0$$

Or

$$\boxed{\ddot{y} = \frac{g}{2}}$$

This is an acceleration in the downward direction as down was taken positive as shown in the diagram. Since both masses move with same acceleration (magnitude is the same, but direction is ofcourse is as shown in the diagram), then the acceleration of the top mass is also the same

$$\boxed{\ddot{x} = \frac{g}{2}}$$

0.2 Problem 2

2. (5 points)
Use the Euler-Lagrange equation to show that the shortest path between two points in a plane is a straight line. *Hint:* An element of length in a plane is $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$.

SOLUTION

$$ds = \sqrt{dx^2 + dy^2}$$

Therefore we want to minimize

$$J = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int \sqrt{1 + \left(y'\right)^2} dx$$

Hence

$$f = \sqrt{1 + \left(y'\right)^2}$$

And the Euler Lagrangian equation is $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$, but

$$\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2} \frac{2y'}{\sqrt{1 + \left(y'\right)^2}}$$

And since $\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$ then this mean that $\frac{\partial f}{\partial y'} = c$ where c is some constant. Hence

$$\frac{1}{2}\frac{2y'}{\sqrt{1+\left(y'\right)^2}} = c$$
$$y' = c\sqrt{1+\left(y'\right)^2}$$

Squaring both sides

$$\left(y'\right)^2 = c_1\left(1+\left(y'\right)^2\right)$$

Where c_1 is new constant. Hence

$$\left(y'\right)^2 = c_1 + c_1\left(y'\right)^2$$
$$\left(y'\right)^2 = \frac{c_1}{1-c_1} = c_2$$

Where c_2 is new constant. Therefore

$$y' = \pm c_3$$

Where c_3 is new constant. So the above says that $\frac{dy}{dx}$ is constant. In other words, a line, since line has constant slope. The solution to the above is

$$y = m \pm c_3 x$$

Where m is some constant and c_3 is the slope. This is the equation of a line.

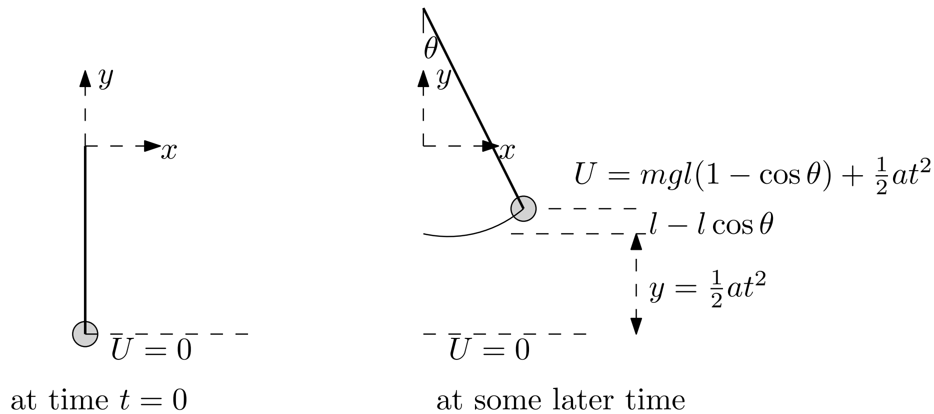
0.3 Problem 3

3. (10 points)
The point of support of a simple pendulum is being elevated at a constant acceleration a. Use Lagrange's method to find the differential equation of motion and show that for small oscillations, the period T of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{g+a}}\quad.$$

SOLUTION

The coordinate system is as shown below. $U = 0$ is taken when the pendulum is hanging in the vertical position before the base starts moving upwards.



Therefore,

$$U = mgl\,(1 - \cos\theta) + \frac{1}{2}at^2$$

Where $y = \frac{1}{2}at^2$ is the distance the pendulum moves upwards in time t since it has constant acceleration. We now need to obtain the kinetic energy. Resolving the velocity of the pendulum bob in the horizontal and in the vertical direction gives

$$\begin{aligned}\dot{x} &= l\dot{\theta}\cos\theta \\ \dot{y} &= l\dot{\theta}\sin\theta + at\end{aligned}$$

Therefore

$$\begin{aligned}v^2 &= \dot{x}^2 + \dot{y}^2 \\ &= \left(l\dot{\theta}\right)^2\cos^2\theta + \left(l\dot{\theta}\right)^2\sin^2\theta + a^2t^2 + 2atl\dot{\theta}\sin\theta \\ &= l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\end{aligned}$$

Hence

$$\begin{aligned}T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m\left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\right)\end{aligned}$$

Now that U and T are determined, the Lagrangian L is computed

$$\begin{aligned}L &= T - U \\ &= \frac{1}{2}m\left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\right) - mgl\,(1 - \cos\theta) + \frac{1}{2}at^2\end{aligned}$$

Hence

$$\frac{\partial L}{\partial \theta} = matl\dot{\theta}\cos\theta - mgl\sin\theta$$

And

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + matl\sin\theta$$

Hence

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2\ddot{\theta} + mal\sin\theta + \dot{\theta}matl\cos\theta$$

Therefore the Euler Lagrangian equation is

$$\begin{aligned}
\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= 0 \\
matl\dot{\theta}\cos\theta - mgl\sin\theta - \left(ml^2\ddot{\theta} + mal\sin\theta + \dot{\theta}matl\cos\theta\right) &= 0 \\
-mgl\sin\theta - ml^2\ddot{\theta} - mal\sin\theta &= 0
\end{aligned}$$

Hence

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{a}{l}\sin\theta = 0}$$

For small oscillations $\sin\theta \approx \theta$ and the above becomes

$$\ddot{\theta} + \theta\left(\frac{g+a}{l}\right) = 0$$

Which is now in the form $\ddot{\theta} + \omega_n^2\theta = 0$ where $\omega_n = \frac{2\pi}{T}$ is the undamped natural radian frequency, and T is the period of oscillation in seconds. hence

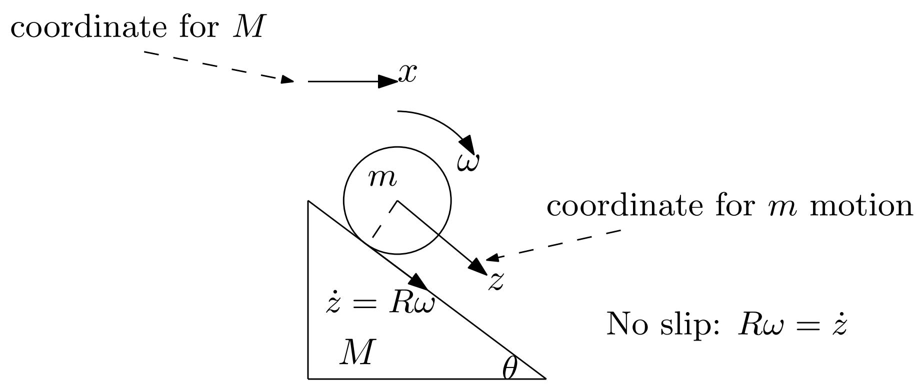
$$\begin{aligned}
T &= \frac{2\pi}{\omega_n} \\
&= \frac{2\pi}{\sqrt{\left(\frac{g+a}{l}\right)}} \\
&= 2\pi\sqrt{\left(\frac{l}{g+a}\right)}
\end{aligned}$$

0.4 Problem 4

4. (10 points)
A ball of mass m, radius R, and moment of inertia $I = \frac{2}{5}mR^2$ rolls down a moveable wedge of mass M without slipping. The angle of the wedge is θ and it is free to slide without friction on a smooth horizontal surface. Find the acceleration of the wedge.

SOLUTION

There are 2 generalized coordinates in this problem. One for the motion of center of mass of m and one for the motion of the wedge M itself. The positive directions are taken as shown in this diagram



2 generalized coordinates: x and z

The first step is to determine the kinetic energy T and potential energy U of the whole system. For mass M

$$T_M = \frac{1}{2} M \dot{x}^2$$

For the rolling mass m since it has both rotational motion and translation motion then

$$T_m = \frac{1}{2} m \left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2 \right] + \frac{1}{2} I \omega^2 \tag{1}$$

Where in the above the term $(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2$ is the translation velocity of the rolling mass. Since the motion is without slip, then we can now relate ω to z using

$$R\omega = \dot{z}$$

Hence (1) becomes

$$T_m = \frac{1}{2} m \left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2 \right] + \frac{1}{2} I \left(\frac{\dot{z}}{R} \right)^2$$

But $I = \frac{2}{5} R^2 m$, hence the above reduces to

$$T_m = \frac{1}{2} m \left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2 \right] + \frac{1}{5} m \dot{z}^2$$

Now that the overall T is found from

$$\begin{aligned}
T &= T_M + T_m \\
&= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[(\dot{x} + \dot{z}\cos\theta)^2 + (\dot{z}\sin\theta)^2 \right] + \frac{1}{5} m \dot{z}^2 \\
&= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left[\dot{x}^2 + \dot{z}^2 \cos^2\theta + 2\dot{x}\dot{z}\cos\theta + \dot{z}^2 \sin^2\theta \right] + \frac{1}{5} m \dot{z}^2 \\
&= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}^2 + \dot{z}^2 + 2\dot{x}\dot{z}\cos\theta \right) + \frac{1}{5} m \dot{z}^2 \\
&= \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{x}^2 + m\dot{x}\dot{z}\cos\theta + \frac{7}{10} m \dot{z}^2
\end{aligned}$$

Now we find U. The potential energy comes from the rolling mass losing U as it moves down. Assuming zero U is at top of the wedge, the distance it moves it $z\sin\theta$. Hence

$$U = -mgz\sin\theta$$

Now the Lagrangian is found $L = T - U$, hence

$$\boxed{L = \left(\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{z}\cos\theta + \frac{7}{10}m\dot{z}^2\right) + mgz\sin\theta}$$

Let us find the equation of motion for m, which has acceleration $\ddot{z}$ first, then find the equation of motion for M which is the required acceleration $\ddot{x}$

$$\begin{aligned}
\frac{\partial L}{\partial z} &= mg\cos\theta \\
\frac{\partial L}{\partial \dot{z}} &= m\dot{x}\cos\theta + \frac{7}{5}m\dot{z} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} &= m\ddot{x}\cos\theta + \frac{7}{5}m\ddot{z}
\end{aligned}$$

Therefore, using Euler-Lagrangian equation

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} &= 0 \\
m\ddot{x}\cos\theta + \frac{7}{5}m\ddot{z} - mg\cos\theta &= 0
\end{aligned}$$

Hence

$$\boxed{\ddot{z} = \frac{5}{7}\left(g\sin\theta - \ddot{x}\cos\theta\right)} \tag{2}$$

We now apply Euler-Lagrangian equation to find $\ddot{x}$

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial \dot{x}} &= M\dot{x} + m\dot{x} + m\dot{z}\cos\theta \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= M\ddot{x} + m\ddot{x} + m\ddot{z}\cos\theta
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\
M\ddot{x} + m\ddot{x} + m\ddot{z}\cos\theta &= 0 \\
\ddot{x}\,(M + m) &= -m\ddot{z}\cos\theta
\end{aligned}$$

But we found $\ddot{z}$ earlier. Hence using (2) into the above gives

$$
\begin{aligned}
\ddot{x}(M+m) &= -m\frac{5}{7}\left(g\sin\theta - \ddot{x}\cos\theta\right)\cos\theta \\
\ddot{x}(M+m) &= \frac{5}{7}m\ddot{x}\cos^2\theta - \frac{5}{7}mg\sin\theta\cos\theta \\
\ddot{x}(M+m) - \frac{5}{7}m\ddot{x}\cos^2\theta &= -m\frac{5}{7}g\sin\theta\cos\theta \\
\ddot{x}\left((M+m) - \frac{5}{7}m\cos^2\theta\right) &= -\frac{5}{7}mg\sin\theta\cos\theta \\
\ddot{x} &= \frac{-\frac{5}{7}mg\sin\theta\cos\theta}{\left((M+m) - \frac{5}{7}m\cos^2\theta\right)} \\
&= \frac{-5mg\sin\theta\cos\theta}{7(M+m) - 5m\cos^2\theta}
\end{aligned}
$$

Hence

$$\boxed{\ddot{x} = \frac{5g\sin\theta\cos\theta}{5\cos^2\theta - 7\left(\frac{M+m}{m}\right)}}$$

0.5 Problem 5

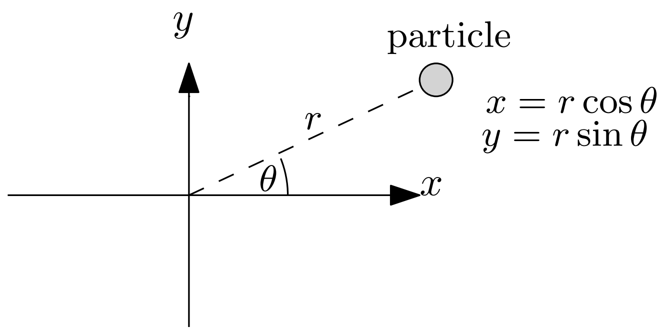
5. (10 points)
Use Lagrange's equations to determine the equations of motion of a particle constrained to move in a plane in a central force field. Show that the angular momentum of the particle is conserved.

SOLUTION

In a central force field, the force on the particle depends only on the magnitude of the direct distance r between the particle and the center of the force. Let the force be located at the origin, then the force on the particle depends only on the magnitude of the position vector $\boldsymbol{r}$ of the particle and not on the angular position of the particle.

$$\boldsymbol{F} = F(r)\,\hat{\boldsymbol{r}}$$

Where $\hat{\boldsymbol{r}}$ is a unit vector pointing in the direction of the force. If the force $\boldsymbol{F}$ causes the distance r between the particle and the origin (where the source of force is assumed) to become smaller, then this force is attractive and it is assigned a negative sign. There are 2 degrees of freedom, hence there are two generalized coordinates. It is easier to use polar coordinates (r,θ) where r is the distance of the particle from the origin, and θ is the angle from the x axis



The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

But $x = r\cos\theta$, hence $\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$ and $y = r\sin\theta$, hence $\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$, therefore

$$\begin{aligned}
\dot{x}^2 + \dot{y}^2 &= \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)^2 + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)^2 \\
&= \left(\dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta\right) + \left(\dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta\right) \\
&= \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta + \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta \\
&= \dot{r}^2 + r^2\dot{\theta}^2
\end{aligned}$$

Hence in polar coordinates

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

And

$$U(r) = V(r)$$

Therefore the Lagrangian

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r)
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial L}{\partial r} &= mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r} \\
\frac{\partial L}{\partial \dot{r}} &= m\dot{r} \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} &= m\ddot{r}
\end{aligned}$$

Hence the equation of motion for the linear (radial) coordinate r is

$$\left(mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r}\right) - m\ddot{r} = 0$$

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{\partial V(r)}{\partial r}$$

But $-\frac{\partial V(r)}{\partial r} = f(r)$ then

$$\boxed{m\ddot{r} = mr\dot{\theta}^2 + f(r)} \tag{1}$$

Now the equation of motion in the θ coordinate is found.

$$\frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(mr^2\dot{\theta}\right)$$

Hence, since $\frac{\partial L}{\partial \theta} = 0$ then $\frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$ or

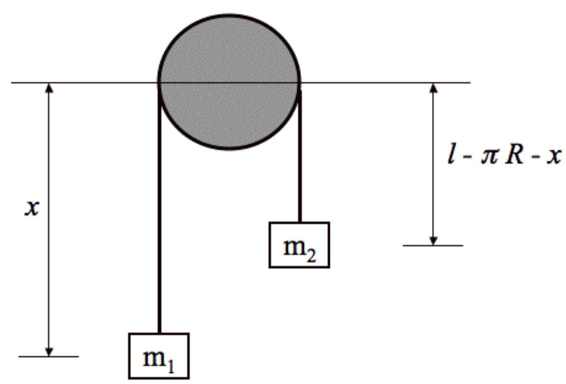
$$\boxed{mr^2\dot{\theta} = \text{constant}} \tag{2}$$

Therefore (2) shows that the angular momentum $I\omega$ is conserved (where I is mr^2, the moment of inertia). This is called the integral of motion.

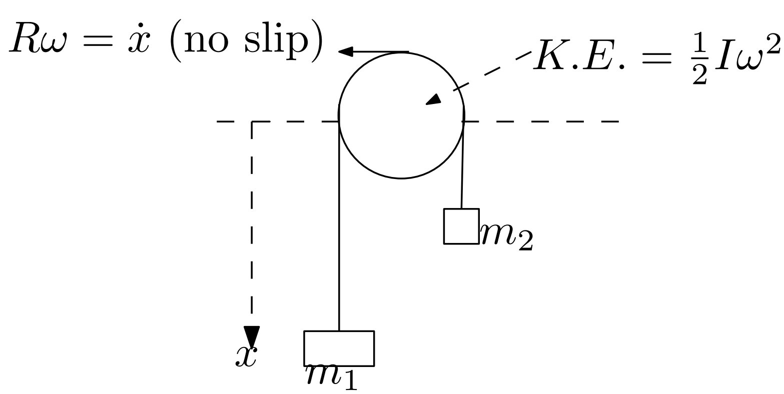
0.6 Problem 6

6. (10 points)
Atwood's machine consists of two weights of mass m_1 and m_2 connected by an ideal massless string of length l that passes over a frictionless pulley of radius R and moment of inertia I. Show that the acceleration of the system is

$$\ddot{x} = \frac{(m_1 - m_2)\, g}{m_1 + m_2 + I/R^2} \quad .$$



SOLUTION



Since both masses will move with same speed $\dot{x}$, then the total kinetic energy of the system is

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I\omega^2$$

Assuming no slip, we can relate ω to $\dot{x}$ using $R\omega = \dot{x}$, hence the above becomes

$$\begin{aligned} T &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\left(\frac{\dot{x}}{R}\right)^2 \\ &= \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{R^2}\right) \end{aligned}$$

Using $U = 0$ as the level shown where the pulley is located, then

$$V = -m_1xg - m_2(l - \pi R - x)g$$

Hence the Lagrangian L is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{R^2}\right) - (-m_1x - m_2(l - \pi R - x))g \\ &= \frac{1}{2}\dot{x}^2\left(m_1 + m_2 + \frac{I}{R^2}\right) + (m_1x + m_2l - m_2\pi R - xm_2)g \end{aligned}$$

Hence

$$\frac{\partial L}{\partial x} = (m_1 - m_2)g$$

And

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= \dot{x}\left(m_1 + m_2 + \frac{I}{R^2}\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= \ddot{x}\left(m_1 + m_2 + \frac{I}{R^2}\right) \end{aligned}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ \ddot{x}\left(m_1 + m_2 + \frac{I}{R^2}\right) - (m_1 - m_2)g &= 0 \end{aligned}$$

Therefore

$$\boxed{\ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{I}{R^2}}}$$