HW10 Physics 311 Mechanics

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0.1 Problem 1

1. (10 points)
   Show that the total energy associated with each normal mode of oscillation is separately conserved.

SOLUTION:

The motion in each normal mode is de-coupled from each other mode. Each motion is a simple harmonic motion in terms of normal coordinates, and reduces to second order differential equation of the form

$$\ddot{\eta}_i + \omega_i^2 \eta_i = 0 \tag{1}$$

Where i ranges over the number of modes. The number of modes is equal to the number of independent degrees of freedoms in the system. Each mode oscillates at frequency ω_i. Since this is a simple harmonic motion, its energy is given by

$$E_i = \frac{1}{2} m_i \dot{\eta}_i^2 + \frac{1}{2} k_i \eta_i^2 \tag{2}$$

Where k_i is the effective stiffness of the mode and $\omega_i^2 = \frac{k_i}{m_i}$. Therefore $k_i = m_i \omega_i^2$.

To show that E is conserved, we need to show that $\frac{\partial E}{\partial t} = 0$. Hence from (2)

$$\frac{\partial E_i}{\partial t} = m_i \dot{\eta}_i \ddot{\eta}_i + \left(m_i \omega_i^2\right) \eta_i \dot{\eta}_i$$

But from (1) we see that $\ddot{\eta}_i = -\omega_i^2 \eta_i$. Substituting into the above gives

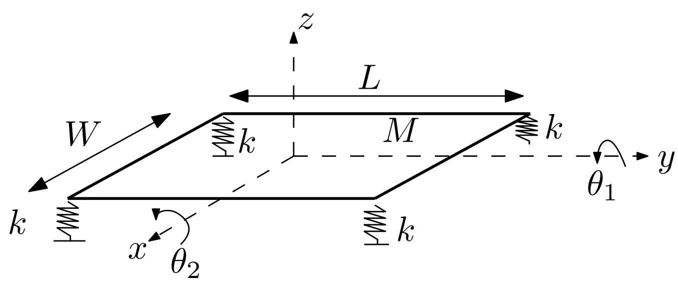
$$\begin{aligned}\frac{\partial E_i}{\partial t} &= m_i \dot{\eta}_i \left(-\omega_i^2 \eta_i\right) + \left(m_i \omega_i^2\right) \eta_i \dot{\eta}_i \\ &= 0\end{aligned}$$

Therefore energy in each mode is constant.

0.2 Problem 2

2. (10 points)
A uniform horizontal rectangular plate of mass M, length L, and width W rests with its corners on four similar vertical springs with spring constant k. Assume that the center of mass of the plate is restricted to move along a vertical line. Find the normal modes of vibration and prove that their frequencies are in the ratio $1 : \sqrt{3} : \sqrt{3}$. (This problem is simpler if you decide beforehand what the normal modes are and then use the appropriate generalized coordinates so that the equations of motion are decoupled from the start.)

SOLUTION:



degrees of freedom: z, θ_1, θ_2

Kinetic energy is

$$T = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2$$

Where I_1 is moment of inertia of plate around axis y, and I_2 is moment of inertia of plate around axis x. These are (from tables) :

$$I_1 = \frac{1}{12}MW^2$$
$$I_2 = \frac{1}{12}ML^2$$

The potential energy is

$$\begin{aligned} U &= 4\left(\frac{1}{2}Kz^2\right) + 4\left(\frac{1}{2}K\left(\frac{W}{2}\theta_1\right)^2\right) + 4\left(\frac{1}{2}K\left(\frac{L}{2}\theta_2\right)^2\right) \\ &= 2Kz^2 + 2K\left(\frac{W}{2}\theta_1\right)^2 + 2K\left(\frac{L}{2}\theta_2\right)^2 \\ &= 2Kz^2 + \frac{1}{2}KW^2\theta_1^2 + \frac{1}{2}KL^2\theta_2^2 \end{aligned}$$

Where small angle approximation is used in the above. Hence the Lagrangian is

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}M\dot{z}^2 + \frac{1}{2}I_1\dot{\theta}_1^2 + \frac{1}{2}I_2\dot{\theta}_2^2 - 2Kz^2 - \frac{1}{2}KW^2\theta_1^2 - \frac{1}{2}KL^2\theta_2^2 \end{aligned}$$

Equation of motion for z

$$\begin{aligned} \frac{\partial L}{\partial z} &= -4Kz \\ \frac{\partial L}{\partial \dot{z}} &= M\dot{z} \end{aligned}$$

Hence

$$M\ddot{z} + 4Kz = 0$$

Equation of motion for θ_1

$$\begin{aligned} \frac{\partial L}{\partial \theta_1} &= -KW^2\theta_1 \\ \frac{\partial L}{\partial \dot{\theta}_1} &= I_1\dot{\theta}_1 \end{aligned}$$

Hence

$$I_1\ddot{\theta}_1 + KW^2\theta_1 = 0$$

Similarly, we find

$$I_2\ddot{\theta}_2 + KL^2\theta_2 = 0$$

Therefore

$$[M]\,\ddot{\boldsymbol{q}} + [K]\,\boldsymbol{q} = 0$$

$$\begin{pmatrix} M & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_1 \end{pmatrix}\begin{pmatrix} \ddot{z} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} + \begin{pmatrix} 4K & 0 & 0 \\ 0 & KW^2 & 0 \\ 0 & 0 & KL^2 \end{pmatrix}\begin{pmatrix} z \\ \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Which leads to

$$\det\begin{pmatrix} 4K - M\omega^2 & 0 & 0 \\ 0 & KW^2 - I_1\omega^2 & 0 \\ 0 & 0 & KL^2 - I_2\omega^2 \end{pmatrix} = 0$$

$$4K^3L^2W^2 - MK^2L^2\omega^2W^2 - 4I_1K^2L^2\omega^2 - 4I_2K^2\omega^2W^2 + MI_1KL^2\omega^4 + MI_2K\omega^4W^2 + 4I_1I_2K\omega^4 - MI_1I_2\omega^6 = 0$$

$$\left(KL^2 - \omega^2 I_2\right)\left(KW^2 - \omega^2 I_1\right)\left(M\omega^2 - 4K\right) = 0$$

Therefore

$$\omega_1 = \sqrt{\frac{KL^2}{I_2}}$$

$$\omega_2 = \sqrt{\frac{KW^2}{I_1}}$$

$$\omega_3 = \sqrt{\frac{4K}{M}}$$

Using $I_1 = \frac{1}{12}MW^2, I_2 = \frac{1}{12}ML^2$, the above become

$$\omega_1 = \sqrt{12\frac{KL^2}{ML^2}} = 2\sqrt{3\frac{K}{M}}$$

$$\omega_2 = \sqrt{12\frac{KW^2}{MW^2}} = 2\sqrt{3\frac{K}{M}}$$

$$\omega_3 = \sqrt{\frac{4K}{M}} = 2\sqrt{\frac{K}{M}}$$

Hence $\frac{\omega_1}{\omega_2} = \frac{1}{1}, \frac{\omega_1}{\omega_3} = \sqrt{3}, \frac{\omega_2}{\omega_3} = \sqrt{3}$. Therefore

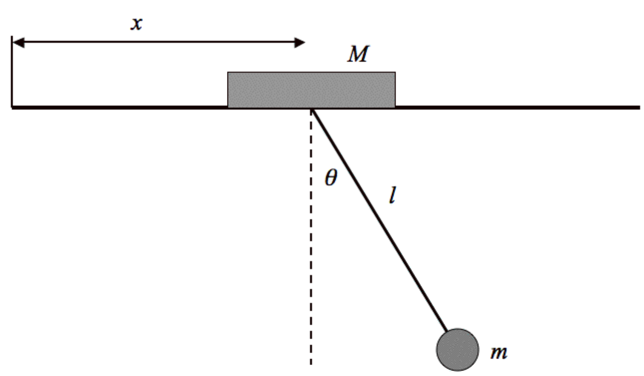
$$\omega_1 : \omega_2 : \omega_3 = 1 : 1 : \sqrt{3}$$

Or

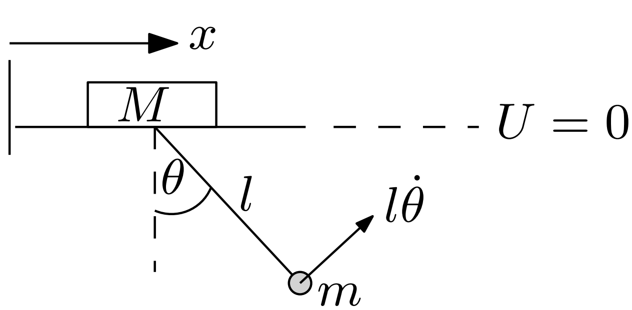
$$\omega_1 : \omega_2 : \omega_3 = \frac{1}{\sqrt{3}} : \frac{1}{\sqrt{3}} : 1$$

0.3 Problem 3

3. (15 points)
A pendulum of mass m and length l is attached to a support of mass M that can move on a frictionless horizontal track as shown on the figure below. Find the normal frequencies and the normal modes of (small) oscillations. Sketch the normal modes.



SOLUTION:



Kinetic energy is

$$
\begin{aligned}
T &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\left(\dot{x} + l\dot{\theta}\cos\theta\right)^2 + \left(l\dot{\theta}\sin\theta\right)^2\right) \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + l^2\dot{\theta}^2\cos^2\theta + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\sin^2\theta\right) \\
&= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right)
\end{aligned}
$$

And potential energy is

$$
U = -mgl\cos\theta
$$

Hence the Lagrangian

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\dot{\theta}\cos\theta + l^2\dot{\theta}^2\right) + mgl\cos\theta \end{aligned}$$

Now we find equations of motions. For θ

$$\begin{aligned} \frac{\partial L}{\partial \theta} &= -m\dot{x}l\dot{\theta}\sin\theta - mgl\sin\theta \\ \frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{2}m\left(2\dot{x}l\cos\theta + 2l^2\dot{\theta}\right) \\ &= m\left(\dot{x}l\cos\theta + l^2\dot{\theta}\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= m\left(\ddot{x}l\cos\theta - \dot{x}l\dot{\theta}\sin\theta + l^2\ddot{\theta}\right) \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \\ m\left(\ddot{x}l\cos\theta - \dot{x}l\dot{\theta}\sin\theta + l^2\ddot{\theta}\right) + m\dot{x}l\dot{\theta}\sin\theta + mgl\sin\theta &= 0 \\ m\ddot{x}l\cos\theta + ml^2\ddot{\theta} + mgl\sin\theta &= 0 \qquad (1) \end{aligned}$$

Now we find equation of motion for x

$$\begin{aligned} \frac{\partial L}{\partial x} &= 0 \\ \frac{\partial L}{\partial \dot{x}} &= M\dot{x} + m\left(\dot{x} + l\dot{\theta}\cos\theta\right) \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} &= M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) \end{aligned}$$

Hence

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \\ M\ddot{x} + m\left(\ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta\right) &= 0 \\ \ddot{x}\left(M+m\right) + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta &= 0 \qquad (2) \end{aligned}$$

Now we can write them in matrix form $[M]\ddot{\boldsymbol{q}} + [K]\boldsymbol{q} = 0$, from (1) and (2) we obtain, after using small angle approximation $\cos\theta \approx 1, \sin\theta \approx \theta$ and also $\dot{\theta}^2 \approx 0$

$$\begin{pmatrix} M+m & ml \\ ml & ml^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Now assuming solution is $\boldsymbol{q}(t) = \boldsymbol{a}e^{i\omega t}$, then the above can be rewritten as

$$\begin{pmatrix} -\omega^2\left(M+m\right) & -\omega^2 ml \\ -\omega^2 ml & mgl - ml^2\omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (1)$$

These have non-trivial solution when

$$\det\begin{pmatrix} -\omega^2(M+m) & -\omega^2 ml \\ -\omega^2 ml & mgl - ml^2\omega^2 \end{pmatrix} = 0$$
$$Ml^2 m\omega^4 - glm^2\omega^2 - Mglm\omega^2 = 0$$
$$\omega^2\left(Ml^2 m\omega^2 - glm^2 - Mglm\right) = 0$$

Hence $\omega = 0$ is one eigenvalue and $\omega = \sqrt{\frac{g}{l}\frac{m+M}{M}}$ is another.

$$\boxed{\omega_1 = 0}$$
$$\boxed{\omega_2 = \sqrt{\frac{g}{l}\frac{(M+m)}{M}}}$$

Now that we found ω_i we go back to (1) to find corresponding eigenvectors. For ω_1, (1) becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & mgl \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence from the second equation above

$$0a_{11} + mgla_{21} = 0$$

So a_{11} can be any value, and $a_{21} = 0$. So the following is a valid first eigenvector

$$\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$$

For ω_2 (1) becomes

$$\begin{pmatrix} -\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m) & -\left(\frac{g}{l}\frac{(M+m)}{M}\right)ml \\ -\left(\frac{g}{l}\frac{(M+m)}{M}\right)ml & mgl - ml^2\left(\frac{g}{l}\frac{(M+m)}{M}\right) \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation we find

$$-\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m)\,a_{12} - \left(\frac{g}{l}\frac{(M+m)}{M}\right)mla_{22} = 0$$
$$(M+m)\,a_{12} + mla_{22} = 0$$

Hence $a_{12} = -\frac{ml}{(M+m)}a_{22}$. So the following is a valid second eigenvector

$$\boldsymbol{a}_2 = \begin{pmatrix} -\frac{ml}{(M+m)}a_{22} \\ a_{22} \end{pmatrix}$$

Therefore

$$x = a_{11}\eta_1 + a_{12}\eta_2$$
$$\theta = a_{21}\eta_1 + a_{22}\eta_2$$

Where η_i are the normal coordinates. Using relation found earlier, then

$$x = a_{11}\eta_1 \tag{2}$$

$$\theta = -\frac{ml}{(M+m)}a_{22}\eta_1 + a_{22}\eta_2 \tag{3}$$

Hence from (2)

$$\eta_1 = -\frac{x}{a_{11}}$$

And now (3) can be written as

$$\theta = -\frac{ml}{(M+m)}a_{22}\frac{x}{a_{11}} + a_{22}\eta_2$$

Therefore

$$\eta_2 = \frac{\theta}{a_{22}} + \frac{1}{a_{11}}\frac{mlx}{(M+m)}$$

To sketch the mode shapes. Looking at $\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$ and $\boldsymbol{a}_2 = \begin{pmatrix} -\frac{ml}{(M+m)}a_{22} \\ a_{22} \end{pmatrix}$ and normalizing we can write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{ml}{(M+m)} \\ 1 \end{pmatrix}$$

So in the first mode shape, the mass M moves with the pendulum fixed to it in the same orientation all the time. So the whole system just slides along x with $\theta = 0$ all the time. In the second mode, x move by $\frac{-ml}{(M+m)}$ factor to θ motion. For example, for $M \ll m$, then mode 2 is $\begin{pmatrix} -l \\ 1 \end{pmatrix}$, hence antisymmetric mode. If $M = m$ then we get $\begin{pmatrix} -\frac{l}{2} \\ 1 \end{pmatrix}$ antisymmetric, but now the ratio changes. So the second mode shape is antisymmetric, but the ratio depends on the ratio of m to M.

0.3.1 Appendix to problem 3

This is extra and can be ignored if needed. I was not sure if we should use $s = l\theta$ as the generalized coordinate instead of θ in order to make all the coordinates of same units. So this is repeat of the above, but using $s = l\theta$ transformation. Starting with equations of motion

$$\ddot{x}(M+m) + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = 0$$

$$m\ddot{\theta} + m\ddot{x}\frac{\cos\theta}{l} + m\frac{g}{l}\sin\theta = 0$$

Will now use $s = l\theta$ transformation, and use s as the second degree of freedom, which is the small distance the pendulum mass swings by. This is so that both x and s has same units of length to make it easier to work with the shape functions. Hence the equations of motions become

$$\ddot{x}\left(M+m\right)+ml\frac{\ddot{s}}{l}\cos\left(\frac{s}{l}\right)-ml\frac{\dot{s}^2}{l^2}\sin\left(\frac{s}{l}\right)=0$$

$$m\frac{\ddot{s}}{l}+m\ddot{x}\frac{\cos\left(\frac{s}{l}\right)}{l}+m\frac{g}{l}\sin\left(\frac{s}{l}\right)=0$$

We first apply small angle approximation, which implies $\cos\frac{s}{l}\rightarrow 1, \sin\left(\frac{s}{l}\right)\rightarrow\frac{s}{l}$ and also $\frac{\dot{s}^2}{l^2}\rightarrow 0$, therefore the equations of motions becomes

$$\ddot{x}\left(M+m\right)+m\ddot{s}=0$$

$$m\frac{\ddot{s}}{l}+m\ddot{x}\frac{1}{l}+m\frac{g}{l}\frac{s}{l}=0$$

And now we write the matrix form

$$\begin{pmatrix}M+m & m\\ m & m\end{pmatrix}\begin{pmatrix}\ddot{x}\\ \ddot{s}\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & m\frac{g}{l}\end{pmatrix}\begin{pmatrix}x\\ s\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

Now assuming solution is $\boldsymbol{q}\left(t\right)=\boldsymbol{a}e^{i\omega t}$, then the above can be rewritten as

$$\begin{pmatrix}-\omega^2\left(M+m\right) & -\omega^2 m\\ -\omega^2 m & m\frac{g}{l}-m\omega^2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \tag{1}$$

These have non-trivial solution when

$$\det\begin{pmatrix}-\omega^2\left(M+m\right) & -\omega^2 m\\ -\omega^2 m & m\frac{g}{l}-m\omega^2\end{pmatrix}=0$$

$$-\frac{1}{l}\left(gm^2\omega^2-Mlm\omega^4+Mgm\omega^2\right)=0$$

$$\omega^2\left(\frac{gm^2}{l}-Mm\omega^2+M\frac{g}{l}m\right)=0$$

$$\omega^2\left(M\omega^2-\left(\frac{g}{l}\left(m+M\right)\right)\right)=0$$

Hence $\omega=0$ is one eigenvalue and $\omega=\sqrt{\frac{g}{l}\frac{(M+m)}{M}}$ is another.

$$\boxed{\omega_1=0}$$

$$\boxed{\omega_2=\sqrt{\frac{g}{l}\frac{(M+m)}{M}}}$$

Now that we found ω_i we go back to (1) to find corresponding eigenvectors. For ω_1, (1) becomes

$$\begin{pmatrix}0 & 0\\ 0 & m\frac{g}{l}\end{pmatrix}\begin{pmatrix}a_{11}\\ a_{21}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$0a_{11}+m\frac{g}{l}a_{21}=0$$

Hence from the second equation above

$$0a_{11} + m\frac{g}{l}a_{21} = 0$$

So a_{11} can be any value, and $a_{21} = 0$. So the following is a valid first eigenvector

$$\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$$

For ω_2 (1) becomes

$$\begin{pmatrix} -\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m) & -\left(\frac{g}{l}\frac{(M+m)}{M}\right)m \\ -\left(\frac{g}{l}\frac{(M+m)}{M}\right)m & m\frac{g}{l} - m\left(\frac{g}{l}\frac{(M+m)}{M}\right) \end{pmatrix}\begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From first equation we find

$$\begin{aligned} -\left(\frac{g}{l}\frac{(M+m)}{M}\right)(M+m)\,a_{12} - \left(\frac{g}{l}\frac{(M+m)}{M}\right)ma_{22} &= 0 \\ (M+m)\,a_{12} + ma_{22} &= 0 \end{aligned}$$

Hence $a_{12} = -\frac{m}{(M+m)}a_{22}$. So the following is a valid second eigenvector

$$\boldsymbol{a}_2 = \begin{pmatrix} -\frac{m}{(M+m)}a_{22} \\ a_{22} \end{pmatrix}$$

Therefore

$$\begin{aligned} x &= a_{11}\eta_1 + a_{12}\eta_2 \\ \theta &= a_{12}\eta_1 + a_{22}\eta_2 \end{aligned}$$

Where η_i are the normal coordinates. Using relation found earlier, then

$$x = a_{11}\eta_1 \tag{2}$$

$$\theta = -\frac{m}{(M+m)}a_{22}\eta_1 + a_{22}\eta_2 \tag{3}$$

Hence from (2)

$$\eta_1 = -\frac{x}{a_{11}}$$

And now (3) can be written as

$$\theta = -\frac{m}{(M+m)}a_{22}\frac{x}{a_{11}} + a_{22}\eta_2$$

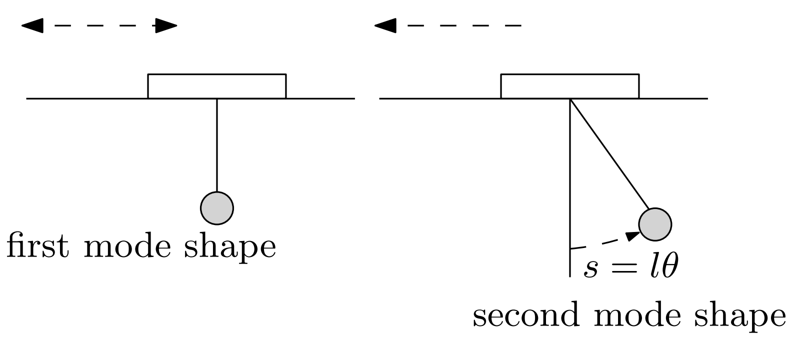
Therefore

$$\eta_2 = \frac{\theta}{a_{22}} + \frac{mx}{(M+m)}\frac{1}{a_{11}}$$

To sketch the mode shapes. Looking at $\boldsymbol{a}_1 = \begin{pmatrix} a_{11} \\ 0 \end{pmatrix}$ and $\boldsymbol{a}_2 = \begin{pmatrix} -\frac{m}{(M+m)}a_{22} \\ a_{22} \end{pmatrix}$ and normalizing we can write

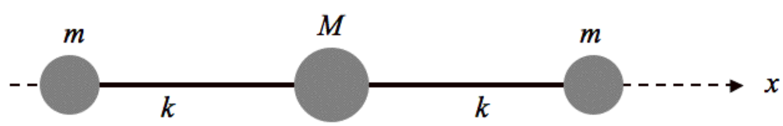
$$\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}-\frac{m}{(M+m)}\\1\end{pmatrix}$$

So in the first mode shape, the mass M moves with the pendulum fixed to it in the same orientation all the time. So the whole system just slides along x with $\theta = 0$ all the time. In the second mode, x move by $\frac{-m}{(M+m)}$ factor to θ motion. For example, for $M \ll m$, then mode 2 is $\begin{pmatrix}-1\\1\end{pmatrix}$, hence antisymmetric mode. If $M = m$ then we get $\begin{pmatrix}-\frac{1}{2}\\1\end{pmatrix}$ antisymmetric, but now the ratio changes. So the second mode shape is antisymmetric, but the ratio depends on the ratio of m to M.

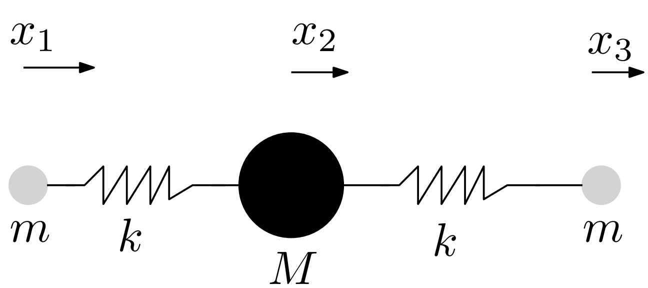


0.4 Problem 4

4. (15 points)
Consider the simple model for the carbon dioxide molecule CO_2 shown below. Two end particles of mass m are bound to the central particle M via a potential function that is equivalent to two springs with spring constant k. Consider motion in one dimension only, along the x-axis. Find the normal frequencies and the normal modes. Make a rough sketch of the normal modes.



SOLUTION:



Kinetic energy

$$T = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2$$

Potential energy

$$U = \frac{1}{2}k\left(x_2 - x_1\right)^2 + \frac{1}{2}k\left(x_3 - x_2\right)^2$$

Hence the Lagrangian

$$\begin{aligned} L &= T - U \\ &= \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}M\dot{x}_2^2 + \frac{1}{2}m\dot{x}_3^2 - \frac{1}{2}k(x_2 - x_1)^2 - \frac{1}{2}k(x_3 - x_2)^2 \end{aligned}$$

EQM for x_1

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= k(x_2 - x_1) \\ \frac{\partial L}{\partial \dot{x}_1} &= m\dot{x}_1 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} &= m\ddot{x}_1 \end{aligned}$$

Therefore

$$\begin{aligned} m\ddot{x}_1 - k(x_2 - x_1) &= 0 \\ m\ddot{x}_1 + kx_1 - kx_2 &= 0 \qquad (1) \end{aligned}$$

EQM for x_2

$$\begin{aligned} \frac{\partial L}{\partial x_2} &= -k(x_2 - x_1) + k(x_3 - x_2) \\ \frac{\partial L}{\partial \dot{x}_2} &= M\dot{x}_2 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} &= M\ddot{x}_2 \end{aligned}$$

Therefore

$$\begin{aligned} M\ddot{x}_2 + k(x_2 - x_1) - k(x_3 - x_2) &= 0 \\ M\ddot{x}_2 + kx_2 - kx_1 - kx_3 + kx_2 &= 0 \\ M\ddot{x}_2 + 2kx_2 - kx_1 - kx_3 &= 0 \qquad (2) \end{aligned}$$

EQM for x_3

$$\begin{aligned} \frac{\partial L}{\partial x_3} &= -k(x_3 - x_2) \\ \frac{\partial L}{\partial \dot{x}_3} &= m\dot{x}_3 \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_3} &= m\ddot{x}_3 \end{aligned}$$

Therefore

$$\begin{aligned} m\ddot{x}_3 + k(x_3 - x_2) &= 0 \\ m\ddot{x}_3 + kx_3 - kx_2 &= 0 \qquad (3) \end{aligned}$$

Now we can write equations (1,2,3) in matrix form $[M]\ddot{\boldsymbol{q}}+[K]\boldsymbol{q}=0$ to obtain

$$\begin{pmatrix} m & 0 & 0\\ 0 & M & 0\\ 0 & 0 & m\end{pmatrix}\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{pmatrix}+\begin{pmatrix} k & -k & 0\\ -k & 2k & -k\\ 0 & -k & k\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Now assuming solution is $\boldsymbol{q}(t)=\boldsymbol{a}e^{i\omega t}$, then the above can be rewritten as

$$\begin{pmatrix} k-m\omega^2 & -k & 0\\ -k & 2k-M\omega^2 & -k\\ 0 & -k & k-m\omega^2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix} \tag{4}$$

These have non-trivial solution when

$$\det\begin{pmatrix} k-m\omega^2 & -k & 0\\ -k & 2k-M\omega^2 & -k\\ 0 & -k & k-m\omega^2\end{pmatrix}=0$$

$$\omega^2\left(k-m\omega^2\right)\left(-Mm\omega^2+Mk+2km\right)=0$$

Hence we have 3 normal frequencies. One of them is zero.

$$\begin{aligned}\omega_1&=0\\ \omega_2&=\sqrt{\frac{k}{m}}\\ \omega_3&=\sqrt{k\frac{M+2m}{Mm}}\end{aligned}$$

For each normal frequency, there is a corresponding eigen shape vector. Now we find these eigen shapes. For ω_1, and from (4)

$$\begin{pmatrix} k & -k & 0\\ -k & 2k & -k\\ 0 & -k & k\end{pmatrix}\begin{pmatrix}a_1\\ a_2\\ a_3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

Hence

$$\begin{aligned}ka_1-ka_2+0a_3&=0\\ -ka_1+2ka_2-ka_3&=0\\ 0a_1-ka_2+ka_3&=0\end{aligned}$$

Or

$$\begin{aligned}a_1-a_2&=0\\ -a_1+2a_2-a_3&=0\\ -a_2+a_3&=0\end{aligned}$$

Hence $a_1=a_2$ and $a_2=a_3$. So $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is first eigenvector. Now we find the second one for ω_2.

From (4) and using $\omega = \sqrt{\frac{k}{m}}$

$$\begin{pmatrix} k - m\frac{k}{m} & -k & 0 \\ -k & 2k - M\frac{k}{m} & -k \\ 0 & -k & k - m\frac{k}{m} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -k & 0 \\ -k & 2k - M\frac{k}{m} & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} -ka_2 &= 0 \\ -ka_1 + \left(2k - M\frac{k}{m}\right)a_2 - ka_3 &= 0 \\ -ka_2 &= 0 \end{aligned}$$

Or

$$\begin{aligned} a_2 &= 0 \\ -a_1 + a_2\left(2 - \frac{M}{m}\right) - a_3 &= 0 \\ a_2 &= 0 \end{aligned}$$

hence $a_2 = 0$ and $a_1 = -a_3$. So $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is second eigenvector. Now we find the third one for ω_3.

From (4) and using $\omega = \sqrt{k\frac{M+2m}{Mm}}$

$$\begin{pmatrix} k - m\left(k\frac{M+2m}{Mm}\right) & -k & 0 \\ -k & 2k - M\left(k\frac{M+2m}{Mm}\right) & -k \\ 0 & -k & k - m\left(k\frac{M+2m}{Mm}\right) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} k - k\frac{M+2m}{M} & -k & 0 \\ -k & 2k - k\frac{M+2m}{m} & -k \\ 0 & -k & k - k\frac{M+2m}{M} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence

$$\begin{aligned} k\left(1 - \frac{M+2m}{M}\right)a_1 - ka_2 &= 0 \\ -ka_1 + k\left(2 - \frac{M+2m}{m}\right)a_2 - ka_3 &= 0 \\ -ka_2 + k\left(1 - \frac{M+2m}{M}\right)a_3 &= 0 \end{aligned}$$

Or

$$\left(1-\frac{M+2m}{M}\right)a_1 - a_2 = 0$$
$$-a_1 + \left(2-\frac{M+2m}{m}\right)a_2 - a_3 = 0$$
$$-a_2 + \left(1-\frac{M+2m}{M}\right)a_3 = 0$$

Solution is: $a_1 = a_3, a_2 = -\frac{2}{M}ma_3$ So $\begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$ is third eigevector. To sketch the mode shapes, will use the following diagram