HW2 ECE 332 Feedback Control

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Electrical engineering department
University of Wisconsin, Madison

Instructor: Professor B Ross Barmish

By

Nasser M. Abbasi

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0.1 Problem 1

Problem 1: Determine the step, ramp and parabolic steady-state errors of the following unity-feedback control systems. The forward-path transfer functions are given

(a) $G(s)H(s) = \dfrac{1000}{(1+0.1s)(1+10s)}$

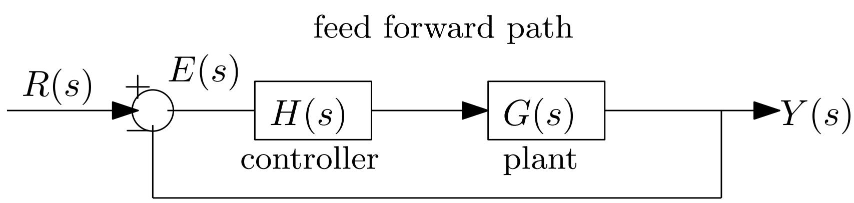
(b) $G(s)H(s) = \dfrac{1000}{s(s+10)(s+100)}$

(c) $G(s)H(s) = \dfrac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}$

(d) What relationships can you find between the number of poles of G(s) at the origin and the type of input signal for which there is a constant steady-state error ($\neq 0$)? If there is a relation, state it; if there is no relation, give the evidence to support your claim.

SOLUTION:

In all of these systems, the feedback block diagram is configured as follows



$$\frac{E(s)}{R(s)} = \frac{1}{1+H(s)G(s)}$$

Since we are looking at steady state, we need to obtain the transfer function between $E(s)$ and $R(s)$. Given that $E(s) = R(s) - Y(s)$ and $Y(s) = E(s)G(s)H(s)$ then we solve these two equations for $E(s)$ by eliminating $Y(s)$ giving

$$E(s) = R(s) - E(s)G(s)H(s)$$
$$E(s)(1 + G(s)H(s)) = R(s)$$
$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}$$

The above is is the transfer function used for the different $R(s)$ signals: unit step $u(t)$, ramp t, and parabolic t^2.

0.1.1 part (a)

The open loop transfer function is $\frac{1000}{(1+0.1s)(1+10s)}$. Since the number of poles at zero is zero, *the system type*[1] *is zero.*

[1]The system type is the number of poles at zero of the open loop transfer function $G(s)H(s)$.

When the input is a unit step $u(t)$, then $R(s) = \frac{1}{s}$. Using the steady state error transfer function found above gives

$$\begin{aligned}
E(s) &= \frac{R(s)}{1 + G(s)H(s)} \\
&= \frac{1}{s}\frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}} = \frac{1}{s}\frac{\left(1 + \frac{1}{10}s\right)(1 + 10s)}{\left(1 + \frac{1}{10}s\right)(1 + 10s) + 1000} = \frac{1}{s}\frac{\left(1 + \frac{1}{10}s\right)(1 + 10s)}{s^2 + \frac{101}{10}s + 1001}
\end{aligned}$$

We see that the poles are located at $s = 0, s = -5.05 \pm 31.233i$. Therefore this is stable $E(s)$ as the real parts of the poles are negative. We are allowed one pole at the origin. Applying the final value theorem gives

$$\begin{aligned}
e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} \frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}} \\
&= \frac{1}{1 + \frac{1000}{\lim_{s\to 0}(1+0.1s)(1+10s)}} \\
&= \frac{1}{1 + 1000}
\end{aligned}$$

Hence

$$e_{ss} = \frac{1}{1001}$$

When the input is a ramp, then $R(s) = \frac{1}{s^2}$, therefore

$$\begin{aligned}
E(s) &= \frac{R(s)}{1 + G(s)H(s)} \\
&= \frac{1}{s^2}\frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}}
\end{aligned}$$

There are two poles at the origin and the other two poles are the same as above at $s = -5.05 \pm 31.233i$. Since there are two poles at the origin, the final value is not defined (taken from now on as infinity in order to be compatible with the text book result and notation).

Finally, when the input is t^2, then $R(s) = \frac{2}{s^3}$ and

$$\begin{aligned}
E(s) &= \frac{2}{s^3}\frac{1}{1 + G(s)H(s)} \\
&= \frac{2}{s^3}\frac{1}{1 + \frac{1000}{(1+0.1s)(1+10s)}} \\
&= \frac{2(1 + 0.1s)(1 + 10s)}{s^3(1 + 0.1s)(1 + 10s) + 1000s^3} \\
&= \frac{2(1 + 0.1s)(1 + 10s)}{s^3\left(s^2 + 10.1s + 1001\right)}
\end{aligned}$$

There are now three poles at the origin $s=0$. As above, this means the final value is taken as infinity.

0.1.2 part (b)

The open loop transfer function is $\frac{1000}{s(1+10)(s+100)}$. There is one pole at the origin which means *the system type is one.*

When the input is a unit step, then $R(s)=\frac{1}{s}$ and

$$\begin{aligned}
E(s) &= \frac{1}{s}\frac{R(s)}{1+G(s)H(s)}\\
&= \frac{1}{s}\frac{1}{1+\frac{1000}{s(1+10)(s+100)}} = \frac{1}{s+\frac{1000}{(1+10)(s+100)}} = \frac{(1+10)(s+100)}{s(1+10)(s+100)+1000} = \frac{(1+10)(s+100)}{11s^2+1100s+1000}
\end{aligned}$$

The poles are at $s=-0.9175, s=-99.08$. This is stable $E(s)$ and we can now apply the final value theorem

$$\begin{aligned}
e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)\\
&= \lim_{s\to 0} s\frac{\frac{1}{s}}{1+\frac{1000}{s(1+10)(s+100)}}\\
&= \frac{1}{1+\lim_{s\to 0}\frac{1000}{s(1+10)(s+100)}}
\end{aligned}$$

Hence

$$e_{ss}=0$$

When the input is a ramp, then $R(s)=\frac{1}{s^2}$ and

$$\begin{aligned}
E(s) &= \frac{R(s)}{1+G(s)H(s)}\\
&= \frac{1}{s^2}\frac{1}{1+\frac{1000}{s(1+10)(s+100)}}\\
&= \frac{1}{s^2+\frac{1000s}{(1+10)(s+100)}}\\
&= \frac{(1+10)(s+100)}{s^2(1+10)(s+100)+1000s}\\
&= \frac{(1+10)(s+100)}{s\left(11s^2+1100s+1000\right)}
\end{aligned}$$

There is one pole at the origin $s=0$ and the other two poles are the same at $s=-0.9175, s=$

−99.08. This is stable. Applying the final value theorem gives

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{1}{s} \frac{1}{1+\frac{1000}{s(1+10)(s+100)}} \\ &= \lim_{s\to 0} \frac{1}{s+\frac{1000}{(1+10)(s+100)}} \\ &= \lim_{s\to 0} \frac{(1+10)(s+100)}{s(1+10)(s+100)+1000} \\ &= \frac{(1+10)(100)}{1000} \end{aligned}$$

Hence

$$e_{ss} = 1.1$$

When the input is a t^2, then $R(s) = \frac{2}{s^3}$ and

$$\begin{aligned} E(s) &= \frac{R(s)}{1+G(s)H(s)} \\ &= \frac{2}{s^3} \frac{1}{1+\frac{1000}{s(1+10)(s+100)}} \\ &= \frac{2}{s^2+\frac{1000s^2}{(1+10)(s+100)}} \\ &= \frac{(2)(11)(s+100)}{s^2(11s+2100)} \end{aligned}$$

There are now two poles at the origin. Therefore final value is taken as infinity.

0.1.3 part (c)

The open loop transfer function is $\frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}$. There are two poles at the origin which means *the system type is 2.*

When the input is a unit step, then $R(s) = \frac{1}{s}$ and

$$\begin{aligned} E(s) &= \frac{R(s)}{1+G(s)H(s)} \\ &= \frac{1}{s}\frac{1}{1+\frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}} \\ &= \frac{1}{s+\frac{K(1+2s)(1+4s)}{s(s^2+s+1)}} \\ &= \frac{s\left(s^2+s+1\right)}{s^2\left(s^2+s+1\right)+K(1+2s)(1+4s)} \end{aligned}$$

We have to now assume that $E(s)$ is stable to be able to apply the final value theorem as this depends on the value of k which is not given in the problem. Therefore

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\left(\frac{1}{s}\frac{1}{1+\frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}}\right) \\ &= \lim_{s\to 0} \frac{s^2\left(s^2+s+1\right)}{s^2\left(s^2+s+1\right)+K(1+2s)(1+4s)} \\ &= \frac{0}{K} \end{aligned}$$

Which means

$$e_{ss} = 0$$

When the input is a ramp, then $R(s) = \frac{1}{s^2}$. Applying the final value theorem gives

$$\begin{aligned} e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\left(\frac{1}{s^2}\frac{1}{1+\frac{K(1+2s)(1+4s)}{s^2(s^2+s+1)}}\right) \\ &= \lim_{s\to 0} s\left(\frac{1}{s^2+\frac{K(1+2s)(1+4s)}{(s^2+s+1)}}\right) \\ &= \lim_{s\to 0} s\left(\frac{\left(s^2+s+1\right)}{s^2\left(s^2+s+1\right)+K(1+2s)(1+4s)}\right) \\ &= \lim_{s\to 0} s\left(\frac{1}{K}\right) \end{aligned}$$

Hence

$$e_{ss} = 0$$

When the input is t^2 then $R(s) = \frac{2}{s^3}$ and

$$
\begin{aligned}
e_{ss} &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\left(\frac{2}{s^3}\frac{1}{1+\frac{K(1+2s)(1+4s)}{s^2\left(s^2+s+1\right)}}\right) \\
&= \lim_{s\to 0} s\left(\frac{2}{s}\frac{\left(s^2+s+1\right)}{s^2\left(s^2+s+1\right)+K(1+2s)(1+4s)}\right) \\
&= \lim_{s\to 0}\left(2\frac{1}{k}\right)
\end{aligned}
$$

Hence

$$
e_{ss} = \frac{2}{K}
$$

0.1.4 Part (d)

Summary of results from the above parts is

	$G(s)H(s)$	system type (number of poles at origin)	e_{ss} step	e_{ss} ramp	e_{ss} t^2
part(a)	$\frac{1000}{(1+0.1s)(1+10s)}$	0	$\frac{1}{1001}$	∞	∞
part(b)	$\frac{1000}{s(1+10)(s+100)}$	1	0	$\frac{11}{10}$	∞
part(c)	$\frac{K(1+2s)(1+4s)}{s^2\left(s^2+s+1\right)}$	2	0	0	$\frac{2}{K}$

From the above table, we see that as the system type (number of poles at origin of the open loop $G(s)H(s)$) increases, then the system can handle more signal types while still producing zero steady state error (this is good). The input signal that gives constant (non zero) steady state error per system type is summarized below.

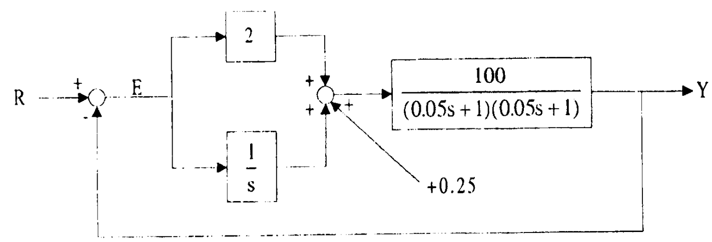
System type (open loop number of poles at origin)	Input that gives constant nonzero e_{ss}
0	step $\left(t^0\right)$
1	ramp $\left(t^1\right)$
2	parabolic $\left(t^2\right)$

So the relation between number of poles at origin of open loop and the type of signal that gives constant non zero steady state error can be written as

if the system type is m then nonzero constant e_{ss} is generated by signal t^m.

0.2 Problem 2

Problem 2: Consider the linear control system shown



and let R(t)=1.5t. What is the steady state error?

SOLUTION:

Let the first input $R(s)$ be $U_1(s)$ and the second input (the constant 0.25) be $U_2(s)$, then

$$Y(s) = \left[\left(2 + \frac{1}{s}\right) E(s) + U_2(s)\right] G(s)$$

And

$$E(s) = U_1(s) - Y(s)$$

Hence

$$\begin{aligned} E(s) &= U_1(s) - \left[\left(2 + \frac{1}{s}\right) E(s) + U_2(s)\right] G(s) \\ &= U_1(s) - \left(2 + \frac{1}{s}\right) E(s) G(s) - U_2(s) G(s) \\ E(s)\left(1 + \left(2 + \frac{1}{s}\right) G(s)\right) &= U_1(s) - U_2(s) G(s) \\ E(s) &= \frac{U_1(s) - U_2(s) G(s)}{1 + \left(2 + \frac{1}{s}\right) G(s)} \end{aligned} \tag{1}$$

To obtain the error transfer function from $E(s)$ to $U_1(s)$, the input $U_2(s)$ is set to zero. To obtain the error transfer function from $E(s)$ to $U_2(s)$, the input $U_1(s)$ is set to zero. Applying these to (1) gives

$$\begin{aligned} \left.\frac{E(s)}{U_1(s)}\right|_{U_2=0} &= \frac{1}{1 + \left(2 + \frac{1}{s}\right) G(s)} \\ \left.\frac{E(s)}{U_2(s)}\right|_{U_1=0} &= \frac{-G(s)}{1 + \left(2 + \frac{1}{s}\right) G(s)} \end{aligned}$$

In Matrix form,

$$E(s) = \begin{pmatrix} \frac{1}{1+\left(2+\frac{1}{s}\right)G(s)} & \frac{-G(s)}{1+\left(2+\frac{1}{s}\right)G(s)} \end{pmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}$$

But $U_1(s) = \frac{1.5}{s^2}$ and $U_2(s) = \frac{0.25}{s}$, and the above becomes

$$E(s) = \begin{pmatrix} \frac{1}{1+\left(2+\frac{1}{s}\right)\frac{100}{(0.05s+1)^2}} & \frac{-\frac{100}{(0.05s+1)^2}}{1+\left(2+\frac{1}{s}\right)\frac{100}{(0.05s+1)^2}} \end{pmatrix} \begin{pmatrix} \frac{1.5}{s^2} \\ \frac{0.25}{s} \end{pmatrix}$$

Hence

$$E(s) = \overbrace{\frac{1.5}{s} \frac{(0.05s+1)^2}{0.0025s^3 + 0.1s^2 + 201s + 100.0}}^{E_1(s)} - \overbrace{\frac{10000}{s^3 + 40s^2 + 80400s + 40000}}^{E_2(s)}$$

The poles of the first term are $-19.751 \pm 282.83i$,$s = -0.497, s = 0$, Hence this is stable and have at most one pole at origin. Then using F.V.T. gives

$$\begin{aligned} e_{ss_1} &= \lim_{s\to 0} sE_1(s) \\ &= \lim_{s\to 0} 1.5 \frac{(0.05s+1)^2}{0.0025s^3 + 0.1s^2 + 201s + 100} \\ &= 0.015 \end{aligned}$$

For $E_2(s)$, the poles are at , $s = -19.75 \pm 282.83i, s = -0.498$, Hence this is stable. Therefore using F.V.T. gives

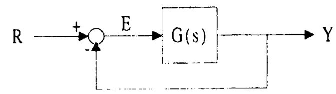
$$\begin{aligned} e_{ss_2} &= \lim_{s\to 0} sE_2(s) \\ &= \lim_{s\to 0} \frac{10000s}{s^3 + 40s^2 + 80400s + 40000} \\ &= 0 \end{aligned}$$

Hence

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= 0.015 - 0 \\ &= 0.015 \end{aligned}$$

0.3 Problem 3

Problem 3: Consider the closed loop system



and assume the following:

(i) The steady state error for a step input is zero.

(ii) The denominator of the closed loop transfer function $\frac{Y(s)}{R(s)}$ (also called the characteristic polynomial of the closed loop system) is $s^3 + 4s^2 + 6s + 4$.

Find the transfer function G(s). Also find the steady state error if the input is a unit ramp.

(Hint: Let n(s) and d(s) be the numerator and denominator of G(s). Express the closed loop transfer function as a function of n(s) and d(s))

SOLUTION:

Let $G(s) = \frac{N(s)}{D(s)}$. The closed loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\frac{N(s)}{D(s)}}{1 + \frac{N(s)}{D(s)}} = \frac{N(s)}{D(s) + N(s)}$$

We are given that $D(s) + N(s) = s^3 + 4s^2 + 6s + 4$. The error transfer function is

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)} = \frac{1}{1 + \frac{N(s)}{D(s)}} = \frac{D(s)}{D(s) + N(s)}$$

Substituting for $D(s) + N(s)$ in the above with the given polynomial results in

$$\frac{E(s)}{R(s)} = \frac{D(s)}{s^3 + 4s^2 + 6s + 4} \tag{1}$$

We are told that $\lim_{s\to 0} sE(s) = 0$ when $R(s) = \frac{1}{s}$. Applying this to (1) gives

$$E(s) = \frac{1}{s}\frac{D(s)}{s^3 + 4s^2 + 6s + 4}$$

$E(s)$ above is stable since the poles are at $-2, -1 \pm i$ with another pole at zero. Hence F.V.T. can be applied to $E(s)$

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} s\left(\frac{1}{s}\frac{D(s)}{s^3 + 4s^2 + 6s + 4}\right) \\ &= \lim_{s\to 0} \frac{D(s)}{4} \end{aligned}$$

We are also told that the above is zero. Hence

$$0 = \lim_{s\to 0} \frac{D(s)}{4}$$

The above implies that $D(s)$ must contain only s terms and no constant terms, since we want $D(s) = 0$ when $s = 0$.

Assuming proper transfer function $G(s)$ where degree of $N(s) \leq$ degree of $N(s)$, then $D(s)$ can be s^3 or $s^3 + 4s^2$, or $s^3 + 4s^2 + 6s$, since any of these will give $\lim_{s\to 0} \frac{D(s)}{4} = 0$. But $D(s)$ can not be s^2 for example, else $G(s)$ will not proper $G(s)$.

There are actually an infinite number of $D(s)$ polynomials which meets this condition (if we use fractions for the coefficients). Below is an example of two possible $D(s)$ choices and the corresponding $G(s)$

$$D_1(s) = s^3 + 4s^2 + 6s$$

Then

$$\boxed{G(s) = \frac{4}{s^3+4s^2+6s}}$$

For steady state when input is ramp, using the above $G(s)$ gives

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\left(\frac{1}{s^2}\frac{s^3+4s^2+6s}{s^3+4s^2+6s+4}\right) \\ &= \lim_{s\to 0}\frac{s^2+4s+6}{s^3+4s^2+6s+4} \end{aligned}$$

Hence

$$e_{ss} = 1.5$$

Another choice is $D_2(s) = s^3 + 4s^2$. Using this, $G(s) = \frac{6s+4}{s^3+4s^2}$. Using this, and when the input is ramp then

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s\left(\frac{1}{s^2}\frac{s^3+4s^2}{s^3+4s^2+6s+4}\right) \\ &= \lim_{s\to 0}\frac{s^2+4s}{s^3+4s^2+6s+4} \\ &= 0 \end{aligned}$$

So the steady state error for ramp depends on which $G(s)$ is used.

0.4 Problem 4

Problem 4: Two feedback systems are shown in Figure 1 and Figure 2.

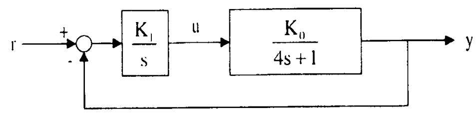


Figure 1: Feedback System 1

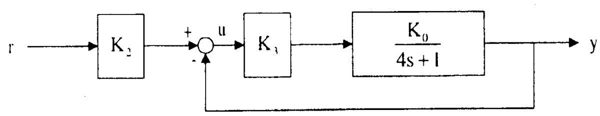


Figure 2: Feedback System 2

(a) Let $K_0 = 1$. Determine the values of K_1 for system 1 and K_2 and K_3 for system 2 so that both of the systems exhibit zero steady error to step inputs and such the steady state error to a unit ramp is 1 in both cases.

(b) Suppose K_0 changes from 1 to $1+\delta$. Show that the steady state error with this perturbed K_0 is still zero to a unit step input, for Figure 1. Also show that this is not the case for Figure 2.

(c) *A control engineer would prefer the system in Figure 1 to the one in Figure 2.* Do you agree with this statement? Justify.

SOLUTION:

0.4.1 Part(a)

For system 1. Using $K_0 = 0$ we first obtain expression for $E(s)$ and $Y(s)$

$$E(s) = R(s) - Y(s)$$

$$Y(s) = E(s)\left(\frac{K_1}{s}\right)\left(\frac{1}{4s+1}\right)$$

Solving for $E(s)$ from the above two equations gives

$$E(s) = R(s) - \left(E(s)\frac{K_1}{s(4s+1)}\right)$$

$$E(s)\left(1 + \frac{K_1}{s(4s+1)}\right) = R(s)$$

$$E(s) = R(s)\frac{s(4s+1)}{s(4s+1)+K_1}$$

When $R(s) = \frac{1}{s}$ we want $e_{ss} = 0$, therefore

$$\begin{aligned} e_{ss} = 0 &= \lim_{s\to 0} sE(s) \\ 0 &= \lim_{s\to 0} \frac{s(4s+1)}{s(4s+1)+K_1} \\ &= \lim_{s\to 0} \frac{s(4s+1)}{K_1} \\ &= \frac{0}{K_1} \end{aligned}$$

The above is true for any K_1 since the numerator is already zero. Considering now the ramp input. When $R(s) = \frac{1}{s^2}$ we want $e_{ss} = 1$, hence

$$\begin{aligned} e_{ss} = 1 &= \lim_{s\to 0} sE(s) \\ 1 &= \lim_{s\to 0} s\frac{1}{s^2}\frac{s(4s+1)}{s(4s+1)+K_1} \\ &= \lim_{s\to 0} \frac{4s+1}{s(4s+1)+K_1} \\ &= \frac{1}{K_1} \end{aligned}$$

Therefore

$$\boxed{K_1 = 1}$$

For system 2

$$E(s) = R(s) - Y(s) \tag{1}$$

But

$$Y(s) = U(s)K_3\frac{1}{4s+1} \tag{2}$$

And

$$U(s) = R(s)K_2 - Y(s)$$

Hence (2) becomes

$$\begin{aligned} Y(s) &= (R(s)K_2 - Y(s))\frac{K_3}{4s+1} \\ Y(s)\left(1+\frac{K_3}{4s+1}\right) &= R(s)\frac{K_2K_3}{4s+1} \\ Y(s) &= R(s)\frac{\frac{K_2K_3}{4s+1}}{1+\frac{K_3}{4s+1}} \\ &= R(s)\frac{K_2K_3}{4s+1+K_3} \end{aligned} \tag{3}$$

Substituting (3) into (1) gives

$$E(s) = R(s) - R(s)\frac{K_2K_3}{4s+1+K_3}$$
$$E(s) = R(s)\left(1 - \frac{K_2K_3}{4s+1+K_3}\right) \tag{4}$$

When $R(s) = \frac{1}{s}$ we want $e_{ss} = 0$, hence

$$e_{ss} = 0 = \lim_{s\to 0} sE(s)$$
$$0 = \lim_{s\to 0} s\frac{1}{s}\left(1 - \frac{K_2K_3}{4s+1+K_3}\right)$$
$$= 1 - \frac{K_2K_3}{1+K_3}$$

For the above to be true, then

$$\boxed{\frac{K_2K_3}{1+K_3} = 1} \tag{5}$$

We now obtain a second equation from the ramp condition. When $R(s) = \frac{1}{s^2}$ we want $e_{ss} = 1$, hence

$$e_{ss} = 1 = \lim_{s\to 0} sE(s)$$
$$1 = \lim_{s\to 0} s\frac{1}{s^2}\left(1 - \frac{K_2K_3}{4s+1+K_3}\right)$$
$$= \lim_{s\to 0} \frac{1}{s}\left(1 - \frac{K_2K_3}{4s+1+K_3}\right)$$
$$= \lim_{s\to 0} \frac{1}{s}\left(\frac{4s+1+K_3-(K_2K_3)}{4s+1+K_3}\right)$$

Replacing K_2K_3 in the above with $1+K_3$ found in (5) gives

$$1 = \lim_{s\to 0} \frac{1}{s}\left(\frac{4s+1+K_3-(1+k_3)}{4s+1+K_3}\right)$$
$$= \lim_{s\to 0} \frac{1}{s}\left(\frac{4s}{4s+1+K_3}\right)$$
$$= \frac{4}{\lim_{s\to 0}(4s+1+K_3)}$$
$$1 = \frac{4}{1+K_3}$$

Hence $1 + K_3 = 4$ or

$$\boxed{K_3 = 3}$$

Now that we found K_3 we go back to (5) and solve for K_2

$$\begin{aligned}\frac{K_2K_3}{1+K_3} &= 1\\ K_2 &= \frac{1+K_3}{K_3}\\ &= \frac{1+3}{3}\end{aligned}$$

Hence

$$\boxed{K_2 = \tfrac{4}{3}}$$

Summary

	K_1	K_2	K_3
system 1	1	N/A	N/A
system 2	N/A	$\frac{4}{3}$	3

0.4.2 Part (b)

For system 1.

$$\begin{aligned}E(s) &= R(s) - Y(s)\\ Y(s) &= E(s)\frac{K_1}{s}\frac{(1+\delta)}{4s+1}\end{aligned}$$

Hence

$$\begin{aligned}E(s) &= R(s) - \left(E(s)\frac{K_1}{s}\frac{1+\delta}{4s+1}\right)\\ E(s)\left(1+\frac{K_1}{s}\frac{1+\delta}{4s+1}\right) &= R(s)\\ E(s) &= R(s)\frac{1}{1+\frac{K_1}{s}\frac{1+\delta}{4s+1}}\\ &= R(s)\frac{s(4s+1)}{s(4s+1)+K_1(1+\delta)}\end{aligned}$$

When $R(s) = \frac{1}{s}$ then

$$\begin{aligned}e_{ss} &= \lim_{s\to 0} sE(s)\\ &= \lim_{s\to 0}\frac{s(4s+1)}{s(4s+1)+K_1(1+\delta)}\\ &= \frac{\lim_{s\to 0} s(4s+1)}{K_1(1+\delta)}\\ &= \frac{0}{K_1(1+\delta)}\end{aligned}$$

The above is zero for any K_1 and any perturbation δ since the numerator is already zero. This is the same condition we found in part(a). Perturbing K_0 has no effect on the result of e_{ss} for step input.

For system 2

$$E(s) = R(s) - Y(s) \tag{1}$$

But

$$Y(s) = U(s) K_3 \frac{1+\delta}{4s+1} \tag{2}$$

And

$$U(s) = R(s) K_2 - Y(s)$$

Replacing $U(s)$ into (2)

$$\begin{aligned}
Y(s) &= (R(s) K_2 - Y(s)) K_3 \frac{1+\delta}{4s+1} \\
Y(s)\left(1 + \frac{K_3(1+\delta)}{4s+1}\right) &= R(s) \frac{K_2 K_3 (1+\delta)}{4s+1} \\
Y(s) &= R(s) \frac{\frac{K_2K_3(1+\delta)}{4s+1}}{1+\frac{K_3(1+\delta)}{4s+1}} \\
&= R(s) \frac{K_2 K_3 (1+\delta)}{4s+1+K_3(1+\delta)} \qquad (3)
\end{aligned}$$

Substituting $Y(s)$ from (3) into (1) gives

$$\begin{aligned}
E(s) &= R(s) - R(s) \frac{K_2 K_3 (1+\delta)}{4s+1+K_3(1+\delta)} \\
&= R(s)\left(1 - \frac{K_2 K_3 (1+\delta)}{4s+1+K_3(1+\delta)}\right) \qquad (4)
\end{aligned}$$

When $R(s) = \frac{1}{s}$ and using the F.V.T. gives

$$\begin{aligned}
e_{ss} &= \lim_{s\to 0} sE(s) \\
&= \lim_{s\to 0} s\frac{1}{s}\left(1 - \frac{K_2 K_3 (1+\delta)}{4s+1+K_3(1+\delta)}\right) \\
&= \lim_{s\to 0}\left(1 - \frac{K_2 K_3 (1+\delta)}{4s+1+K_3(1+\delta)}\right) \\
&= 1 - \frac{K_2 K_3 (1+\delta)}{1+K_3(1+\delta)}
\end{aligned}$$

For the above e_{ss} to be zero, then the condition is that

$$\frac{K_2 K_3 (1+\delta)}{1+K_3(1+\delta)} = 1$$

or

$$K_2K_3(1+\delta) = 1 + K_3(1+\delta)$$
$$K_2K_3 - K_3 = \frac{1}{1+\delta}$$

Using $K_2 = \frac{4}{3}$ and $K_3 = 3$ found in part (a) then the above becomes

$$\left(\frac{4}{3}\right)3 - 3 = \frac{1}{1+\delta}$$
$$1 = \frac{1}{1+\delta}$$

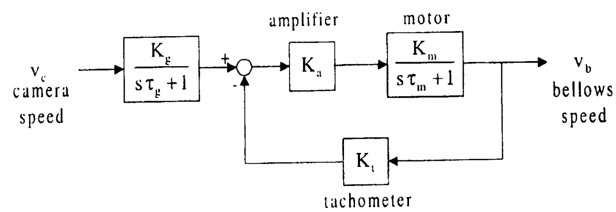
But this is **impossible** since the RHS must be either larger than one or smaller than one (depending on the sign of δ). This means if K_0 is perturbed from unity, then it is no longer possible to obtain zero steady state error for a step input with the same k_2, k_3.

0.4.3 Part(c)

I agree. For first system, it gives $e_{ss} = 0$ for a step input regardless of the value of K_1 or K_0 as was shown in part (b) above. But for system two, $e_{ss} = 0$ for step input only when using specific values of K_i. Any small change in K_0, the steady state error is no longer zero. In other words, system one is more robust in this regard to changes in K_0 and it is therefore the preferred system.

0.5 Problem 5

Problem 5: An important problem for television camera systems is the jumping and wobbling of the picture due to movement of the camera. This effect occurs when the camera is mounted on a moving truck or airplane. A system has been designed (shown below) which is intended to reduce the effect of rapid scanning motion. A maximum scanning motion of 25% is expected.



(a) Determine the steady state error of the system for a step input $V_c(s) = \frac{25}{s}$. Assume that τ_g is "negligible" and $K_g = K_t = 1$.

(b) Determine the necessary loop gain $K_a K_m$ when a $1^{\circ}/\text{sec}$ steady state error is allowable. (Same assumptions as Part (a))

(c) Show that the step response of the system is of the form

$$v_b(t) = \frac{k}{q}\{1 - e^{-qt}\}$$

under the assumptions in Part (a). Express k and q in terms of the system parameters.

(d) The settling time is defined as the time it takes for the step response to be within 2% of the steady state value. Given the expression of the step response determined in Part (c), derive the expression for the settling time of v_b. Also, find the loop gain $K_a K_m$ so that the settling time of v_b is less than or equal to 0.04 sec. Take $\tau_m = 0.4$ sec as the motor time constant.

SOLUTION:

0.5.1 Part (a)

We first need to find $\frac{E(s)}{V_c(s)}$. From the block diagram[2].

$$V_b(s) = E(s) K_a \left(\frac{K_m}{1 + s\tau_m}\right) \tag{1}$$

And

$$E(s) = V_c(s)\left(\frac{K_g}{1 + s\tau_g}\right) - K_t V_b(s) \tag{2}$$

[2]Notice that the problem is saying $E(s)$ is the variable to the left of the amplifier K_a and this solution is based on this and not on using $E(s) = V_b(s) - V_c(s)$

Replacing $V_b(s)$ in (2) with (1) gives

$$E(s) = V_c(s)\left(\frac{K_g}{1+s\tau_g}\right) - K_t E(s) K_a \left(\frac{K_m}{1+s\tau_m}\right)$$

$$E(s)\left(1 + K_t K_a\left(\frac{K_m}{1+s\tau_m}\right)\right) = V_c(s)\left(\frac{K_g}{1+s\tau_g}\right)$$

$$\frac{E(s)}{V_c(s)} = \frac{\left(\frac{K_g}{1+s\tau_g}\right)}{1 + K_t K_a\left(\frac{K_m}{1+s\tau_m}\right)}$$

Hence

$$\boxed{\frac{E(s)}{V_c(s)} = \left(\frac{1+s\tau_m}{1+s\tau_g}\right)\frac{K_g}{1+s\tau_m+K_tK_aK_m}}$$

When $V_c(s) = \frac{25}{s}, K_g = K_t = 1$ then $E(s)$ from above becomes

$$E(s) = \frac{25}{s}\left(\frac{1+s\tau_m}{1+s\tau_g}\right)\frac{1}{(1+K_aK_m)+s\tau_m}$$

The above $E(s)$ has one pole at the origin, and has a pole at $s = \frac{-1}{\tau_g}$ and a pole at $s = -\frac{1+K_aK_m}{\tau_m}$. Hence this is stable (assuming $K_aK_m > -1$). Applying F.V.T. gives

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} 25\left(\frac{1+s\tau_m}{1+s\tau_g}\right)\frac{1}{(1+K_aK_m)+s\tau_m} \end{aligned}$$

Hence

$$e_{ss} = \frac{25}{1+K_aK_m}$$

0.5.2 Part (b)

When e_{ss} is one degree per second, then from the above

$$1 = \frac{25}{1+K_aK_m}$$

Or

$$\boxed{K_aK_m = 24}$$

0.5.3 Part(c)

To find the step response, we find the closed loop $\frac{V_b(s)}{V_c(s)}$ transfer function first. Substituting (2) into (1) found in part (a) above to obtain an expression for $V_b(s)$

$$\begin{aligned} V_b(s) &= \left(V_c(s)\left(\frac{K_g}{1+s\tau_g}\right) - K_t V_b(s)\right) K_a \left(\frac{K_m}{1+s\tau_m}\right) \\ V_b(s)\left(1+\frac{K_m K_t K_a}{1+s\tau_m}\right) &= V_c(s)\frac{K_g K_a K_m}{\left(1+s\tau_g\right)(1+s\tau_m)} \end{aligned}$$

Hence the closed loop transfer function is

$$\begin{aligned} \frac{V_b(s)}{V_c(s)} &= \frac{\frac{K_g K_a K_m}{\left(1+s\tau_g\right)(1+s\tau_m)}}{1+\frac{K_m K_t K_a}{1+s\tau_m}} \\ &= \frac{1}{\left(1+s\tau_g\right)}\frac{K_g K_a K_m}{(1+s\tau_m)+K_m K_t K_a} \end{aligned}$$

Using same assumptions as part (a), and now using that τ_g is negligible so that $\frac{1}{\left(1+s\tau_g\right)} \approx 1$ in the above, and using $V_c(s) = \frac{1}{s}$ since we are told in this part it is a step input (should we have used $\frac{25}{s}$ again here? It is not clear, but it says step input so I think $\frac{1}{s}$ should be used in this part), then the above simplifies to

$$\begin{aligned} V_b(s) &= \left(\frac{1}{s}\right)\frac{K_a K_m}{(1+s\tau_m)+K_m K_a} \\ &= \frac{1}{s}\frac{K_a K_m}{(1+K_m K_a)+s\tau_m} \\ &= \frac{K_a K_m}{\tau_m}\frac{1}{s}\frac{1}{s+\left(\frac{1+K_m K_a}{\tau_m}\right)} \end{aligned}$$

We now need to find the inverse Laplace transform. Using partial fractions

$$\left(\frac{K_a K_m}{\tau_m}\right)\frac{1}{s}\frac{1}{s+\left(\frac{1+K_m K_a}{\tau_m}\right)} = \frac{A}{s}+\frac{B}{s+\left(\frac{1+K_m K_a}{\tau_m}\right)} \tag{3}$$

Hence

$$A = \lim_{s\to 0}\left(\frac{K_a K_m}{\tau_m}\right)\frac{1}{\frac{1+K_m K_a}{\tau_m}+s} = \frac{K_a K_m}{1+K_m K_a}$$

And

$$B = \lim_{s\to \frac{-(1+K_m K_a)}{\tau_m}} \frac{K_a K_m}{\tau_m}\frac{1}{s} = \frac{K_a K_m}{\tau_m}\frac{1}{\frac{-(1+K_m K_a)}{\tau_m}} = -\frac{K_a K_m}{1+K_m K_a}$$

Now that we found A, B using partial fractions, we replace these values in (3) to obtain $V_b(s)$

$$V_b(s) = \frac{K_a K_m}{1 + K_m K_a} \frac{1}{s} - \frac{K_a K_m}{1 + K_m K_a} \frac{1}{\frac{1 + K_m K_a}{\tau_m} + s} \tag{4}$$

Now we can apply inverse Laplace transform. Hence

$$\begin{aligned} v_b(t) &= \left(\frac{K_a K_m}{1 + K_m K_a} - \frac{K_a K_m}{1 + K_m K_a} e^{-\frac{1 + K_m K_a}{\tau_m} t} \right) u(t) \\ &= \frac{K_a K_m}{1 + K_m K_a} \left(1 - e^{-\frac{1 + K_m K_a}{\tau_m} t} \right) u(t) \end{aligned}$$

Let

$$\boxed{q = \frac{1 + K_m K_a}{\tau_m}}$$

and

$$\boxed{k = \frac{K_a K_m}{\tau_m}}$$

Then $v_b(t)$ can be written as required

$$v_b(t) = \frac{k}{q} \left(1 - e^{-qt} \right) u(t)$$

0.5.4 Part(d)

We first need to find the steady state $v_b(t)$. From (4) found above in part (c)

$$V_b(s) = \frac{K_a K_m}{1 + K_m K_a} \frac{1}{s} - \frac{K_a K_m}{1 + K_m K_a} \frac{1}{\frac{1 + K_m K_a}{\tau_m} + s}$$

Then applying F.V.T. assuming stability

$$\begin{aligned} V_b(\infty) &= \lim_{s \to 0} s V_b(s) \\ &= \lim_{s \to 0} s \left(\frac{K_a K_m}{1 + K_m K_a} \frac{1}{s} - \frac{K_a K_m}{1 + K_m K_a} \frac{1}{\frac{1 + K_m K_a}{\tau_m} + s} \right) \\ &= \frac{K_a K_m}{1 + K_m K_a} \\ &= \frac{k}{q} \end{aligned}$$

Let the settling time be t_s, then we want to solve for t_s from

$$\begin{aligned} v_b(t_s) &= 0.98 V_b(\infty) \\ \frac{k}{q} \left(1 - e^{-q t_s} \right) &= 0.98 \frac{k}{q} \\ 1 - e^{-q t_s} &= 0.98 \\ e^{-q t_s} &= 0.02 \end{aligned}$$

Taking natural logs on both sides gives

$$-qt_s = \ln(0.02)$$
$$qt_s = 3.912$$

Hence

$$\boxed{t_s = \frac{3.912}{\left(\frac{1+K_mK_a}{t_m}\right)}}$$

Using $t_m = 0.4$ seconds in the above gives

$$t_s = \frac{1.5648}{1 + K_mK_a}$$

For $t_s \leq 0.04$ then

$$\frac{1.5648}{(1 + K_mK_a)} \leq 0.04$$
$$1 + K_mK_a \geq \frac{1.5648}{0.04}$$
$$1 + K_mK_a \geq 39.12$$

Hence

$$\boxed{K_mK_a \geq 38.12}$$