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# HW1 ECE 332 Feedback Control

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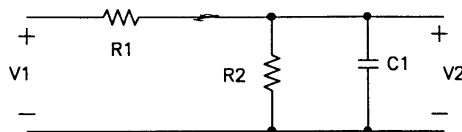
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## 0.1 Problems

### ECE 332 Homework 1

1. For the circuit below, find the transfer function  $\frac{V_2(s)}{V_1(s)}$ .



2. Find the inverse Laplace transforms of the following:

$$(a) F(s) = \frac{1}{(s+2)^4}$$

$$(b) F(s) = \frac{1}{s^3+5s^2+8s+4}$$

$$(c) F(s) = \frac{7s^2+48s+62}{s^3+7s^2+10s}$$

$$(d) F(s) = \frac{10(s+2)}{s(s^2+4)(s+1)}e^{-s}$$

3. A system is described by the differential equation

$$y^{(4)} + y^{(3)} + 5y^{(2)} + 7y^{(1)} + y = u^{(3)} + 2u^{(2)} + 3u^{(1)} + 7u$$

Find the transfer function  $\frac{Y(s)}{U(s)}$ .

4. For the system in Problem 3, find the step response using the *Matlab* toolbox *Simulink* (see handout).
- Plot your output for 30 seconds and comment. Please include descriptive titles and axis labels.
  - In view of the result in (a), a feedback gain  $K$  is introduced into the system. This leads to a system with  $7y^{(1)}$  replaced by  $(7 + K)y^{(1)}$ . Using *Simulink*, find a suitable value of  $K$  so that the unstable oscillations in (a) are eliminated. Plot 30 seconds of the step response for the stabilized system. (Note: Try not to use an excessively large gain if avoidable.)

For this problem submit the two plots as well as a copy of the *Simulink* schematic file used to generate the plots (only one schematic is needed). Please enlarge the necessary blocks so that all of the numbers inside are visible, e.g. the transfer function block.

## 0.2 Problem 1 solution

Let the input impedance be  $Z_{in}$  and the output impedance be  $Z_{out}$ , then

$$Z_{out} = R_2 || Z_{C_1}$$

Where  $Z_{C_1}$  is the capacitor impedance given by  $Z_{C_1} = \frac{1}{C_1 s}$ . To find  $Z_{out}$ , we use the parallel formula

$$\begin{aligned} \frac{1}{Z_{out}} &= \frac{1}{R_2} + \frac{1}{Z_{C_1}} \\ &= \frac{Z_{C_1} + R_2}{R_2 Z_{C_1}} \end{aligned}$$

Therefore

$$Z_{out} = \frac{R_2 Z_{c1}}{Z_{c1} + R_2}$$

Now we find the input impedance, which is

$$\begin{aligned} Z_{in} &= R_1 + R_2 \parallel Z_{c1} \\ &= R_1 + \frac{R_2 Z_{c1}}{Z_{c1} + R_2} \end{aligned}$$

Simplifying gives

$$Z_{in} = \frac{R_1 (Z_{c1} + R_2) + R_2 Z_{c1}}{Z_{c1} + R_2}$$

Now voltage divider is used, which gives

$$\begin{aligned} \frac{V_2(s)}{V_1(s)} &= \frac{Z_{out}}{Z_{in}} \\ &= \frac{\frac{R_2 Z_{c1}}{Z_{c1} + R_2}}{\frac{R_1 (Z_{c1} + R_2) + R_2 Z_{c1}}{Z_{c1} + R_2}} \\ &= \frac{R_2 Z_{c1}}{R_1 (Z_{c1} + R_2) + R_2 Z_{c1}} \\ &= \frac{R_2 Z_{c1}}{R_1 Z_{c1} + R_1 R_2 + R_2 Z_{c1}} \\ &= \frac{1}{\frac{R_1}{R_2} + \frac{R_1}{Z_{c1}} + 1} \\ &= \frac{1}{\frac{R_1}{R_2} + R_1 C s + 1} \\ &= \frac{R_2}{(R_1 + R_2) + R_2 R_1 C s} \end{aligned}$$

Hence the transfer function is

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2}{(R_1 + R_2) + R_2 R_1 C s}$$

The above is a low pass filter since making  $s = j\omega$  gives

$$G(\omega) = \frac{R_2}{(R_1 + R_2) + R_2 R_1 C j\omega}$$

As the frequency  $\omega \rightarrow \infty$  then  $|G(\omega)| \rightarrow 0$  and as  $\omega \rightarrow 0$ ,  $|G(\omega)| \rightarrow \frac{R_2}{R_1 + R_2}$ . So the above transfer function attenuate large frequencies.

### 0.3 Problem 2 solution

#### 0.3.1 Part (a)

$F(s) = \frac{1}{(s+2)^4}$ . Using the Laplace transform property

$$F(s-a) \iff e^{at}f(t) \quad (1)$$

Where  $f(t)$  is the inverse Laplace transform of  $F(s)$ . And also using the property

$$\frac{1}{s^n} \iff \frac{t^{n-1}}{(n-1)!} \quad (2)$$

Then, combining (1) and (2) gives

$$\frac{1}{(s-a)^n} \iff e^{at} \frac{t^{n-1}}{(n-1)!}$$

For  $a = -2$  and  $n = 4$  the above becomes

$$\frac{1}{(s+2)^4} \iff e^{-2t} \frac{t^3}{3!} = e^{-2t} \frac{t^3}{6}$$

#### 0.3.2 Part (b)

$F(s) = \frac{1}{s^3+5s^2+8s+4}$ . Using the computer, the factors are first found

$$s^3 + 5s^2 + 8s + 4 = (s+1)(s+2)^2$$

Applying partial fractions decomposition on  $\frac{1}{(s+1)(s+2)^2}$  gives

$$\frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \quad (1)$$

Multiplying both sides by  $(s+1)$  and evaluating at  $s = -1$  gives

$$A = \frac{1}{(s+2)_{s=-1}^2}$$

$$A = 1$$

Multiplying both sides by  $(s+2)^2$  and evaluating at  $s = -2$  gives

$$C = \frac{1}{(s+1)_{s=-2}}$$

$$C = -1$$

Therefore (1) now becomes

$$\begin{aligned} \frac{1}{(s+1)(s+2)^2} &= \frac{1}{(s+1)} + \frac{B}{(s+2)} - \frac{1}{(s+2)^2} \\ &= \frac{(s+2)^2 + B(s+2)(s+1) - (s+1)}{(s+1)(s+2)^2} \\ &= \frac{2B + 3s + Bs^2 + 3Bs + s^2 + 3}{(s+1)(s+2)^2} \end{aligned}$$

Comparing numerator of RHS and LHS gives

$$1 = (2B + 3) + (3 + 3B)s + (B + 1)s^2$$

Comparing coefficients (powers of  $s$ ) gives  $1 = 2B + 3$  or  $B = -1$ . Therefore the partial fraction decomposition of (1) becomes

$$\boxed{\frac{1}{(s+1)(s+2)^2} = \frac{1}{(s+1)} - \frac{1}{(s+2)} - \frac{1}{(s+2)^2}}$$

The inverse Laplace transform is now applied to each term in the RHS and using  $F(s - a) \Leftrightarrow e^{at}f(t)$  which results in

$$\begin{aligned} \frac{1}{(s+1)} - \frac{1}{(s+2)} - \frac{1}{(s+2)^2} &\Leftrightarrow e^{-t} - e^{-2t} - te^{-2t} \\ &= \boxed{e^{-t}(1 - e^{-t} - te^{-t})} \end{aligned}$$

### 0.3.3 Part (c)

$F(s) = \frac{7s^2 + 48s + 62}{s^3 + 7s^2 + 10s}$ . The denominator can be written as  $s(s^2 + 7s + 10)$  which is now factored to  $s(s+2)(s+5)$ . Applying partial fractions on this gives

$$\begin{aligned} F(s) &= \frac{7s^2 + 48s + 62}{s(s+2)(s+5)} \\ &= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5} \end{aligned}$$

Hence

$$\begin{aligned} A &= \lim_{s \rightarrow 0} \frac{7s^2 + 48s + 62}{(s+2)(s+5)} = \frac{62}{10} \\ B &= \lim_{s \rightarrow (-2)} \frac{7s^2 + 48s + 62}{s(s+5)} = \frac{7(4) + 48(-2) + 62}{-2(-2+5)} = 1 \\ C &= \lim_{s \rightarrow (-5)} \frac{7s^2 + 48s + 62}{s(s+2)} = \frac{7(25) + 48(-5) + 62}{-5(-5+2)} = -\frac{1}{5} \end{aligned}$$

Therefore

$$F(s) = \frac{621}{10s} + \frac{1}{s+2} - \frac{1}{5s+5}$$

The inverse Laplace transform is

$$f(t) = \frac{62}{10} + e^{-2t} - \frac{1}{5}e^{-5t}$$

For  $t \geq 0$ .

### 0.3.4 Part (d)

$F(s) = \frac{10(s+2)}{s(s^2+4)(s+1)}e^{-s}$ . This is solved using time shifting property of Laplace transform, which says that

$$\mathcal{L}\{f(t-a)\} = e^{-as}F(s)$$

Where  $F(s)$  is the Laplace transform of  $f(t)$ . We see that  $a = 1$  in the above. So we just need to find  $\mathcal{L}^{-1}\left(\frac{10(s+2)}{s(s^2+4)(s+1)}\right)$  and then time shift (delay) the result by the amount  $a$  which is 1 in this case. Applying partial fractions on  $\frac{10(s+2)}{s(s^2+4)(s+1)}$  gives

$$\frac{10(s+2)}{s(s^2+4)(s+1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4} \quad (1)$$

Hence

$$A = \lim_{s \rightarrow 0} \frac{10(s+2)}{(s^2+4)(s+1)} = 5$$

$$B = \lim_{s \rightarrow -1} \frac{10(s+2)}{s(s^2+4)} = \frac{10}{-5} = -2$$

Therefore (1) becomes

$$\begin{aligned} \frac{10(s+2)}{s(s^2+4)(s+1)} &= \frac{5}{s} - \frac{2}{s+1} + \frac{Cs+D}{s^2+4} \\ &= \frac{5(s+1)(s^2+4) - 2(s)((s^2+4)) + (Cs+D)(s)(s+1)}{s(s^2+4)(s+1)} \end{aligned}$$

Or

$$\begin{aligned} 10(s+2) &= 5(s+1)(s^2+4) - 2(s)((s^2+4)) + (Cs+D)(s)(s+1) \\ 10s+20 &= 12s+sD+Cs^2+Cs^3+s^2D+5s^2+3s^3+20 \\ &= s(12+D)+20+s^2(C+D+5)+(C+3)s^3 \end{aligned}$$

Comparing coefficients gives

$$\begin{aligned} 12+D &= 10 \\ D &= -2 \end{aligned}$$

And

$$\begin{aligned} 0 &= C + D + 5 \\ C &= -D - 5 \\ &= -3 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{10(s+2)}{s(s^2+4)(s+1)} &= \frac{A}{s} + \frac{B}{(s+1)} + \frac{Cs+D}{(s^2+4)} \\ &= \frac{5}{s} - \frac{2}{(s+1)} - \frac{3s+2}{(s^2+4)} \\ &= \frac{5}{s} - \frac{2}{(s+1)} - 3\frac{s}{(s^2+4)} - \frac{2}{(s^2+4)} \end{aligned} \quad (3)$$

Using tables,  $\mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos(at)$ , hence  $\mathcal{L}^{-1}\left(\frac{s}{(s^2+4)}\right) = \cos 2t$ . Also, from tables  $\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin(at)$ , hence  $\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \sin 2t$ . The complete the inverse Laplace transform of  $\frac{10(s+2)}{s(s^2+4)(s+1)}$  is now found to be

$$\begin{aligned} g(t) &= \mathcal{L}^{-1}\left(\frac{10(s+2)}{s(s^2+4)(s+1)}\right) \\ &= 5 - 2e^{-t} - 3\cos 2t - \sin 2t \end{aligned}$$

Therefore the final answer is just a time shifted version of the above, which is

$$\begin{aligned} f(t) &= g(t-1) \\ &= \boxed{5 - 2e^{-(t-1)} - 3\cos 2(t-1) - \sin 2(t-1)} \end{aligned}$$

For  $t \geq 1$  and zero otherwise.

## 0.4 Problem 3 solution

Taking Laplace transform of both sides, and assuming zero for all initial conditions gives

$$\begin{aligned} s^4Y(s) + s^3Y(s) + 5s^2Y(s) + 7sY(s) + Y(s) &= s^3U(s) + 2s^2U(s) + 3sU(s) + 7U(s) \\ Y(s)(s^4 + s^3 + 5s^2 + 7s + 1) &= U(s)(s^3 + 2s^2 + 3s + 7) \\ \frac{Y(s)}{U(s)} &= \frac{s^3 + 2s^2 + 3s + 7}{s^4 + s^3 + 5s^2 + 7s + 1} \end{aligned}$$

## 0.5 Problem 4 solution

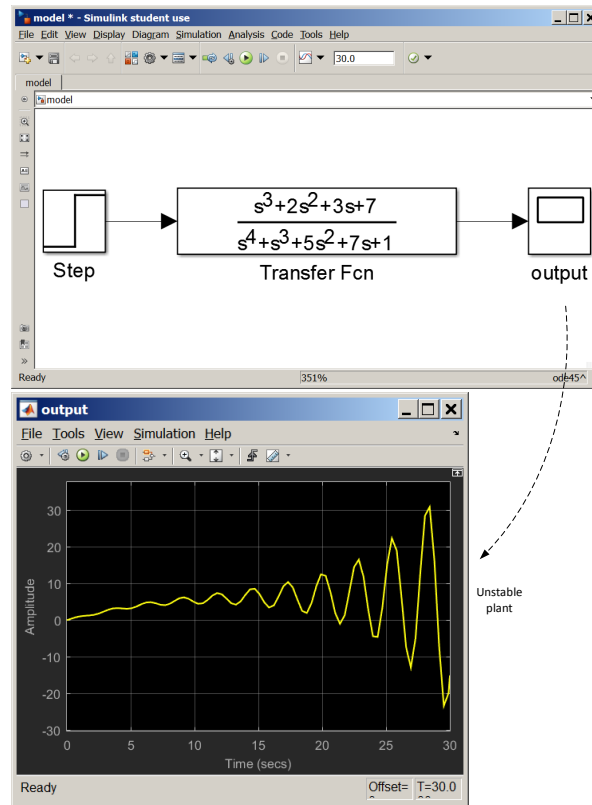
### 0.5.1 Part (a)

Simulink model was setup and run for 30 seconds. The step response shows that the plant is not stable. This is due to the numerator having roots with positive real parts



```
>> double(solve(s^4+s^3+5*s^2+7*s+1==0,s))
-1.1803 + 0.0000i
-0.1608 + 0.0000i
0.1706 - 2.2888i
0.1706 + 2.2888i
```

The following diagram show the simulink model and the output



### 0.5.2 Part (b)

Replacing  $7y^{(1)}$  with  $(7+k)y^{(1)}$  in the given differential equation leads to the following transfer function

$$s^4Y(s) + s^3Y(s) + 5s^2Y(s) + (7+k)sY(s) + Y(s) = s^3U(s) + 2s^2U(s) + 3sU(s) + 7U(s)$$

$$Y(s) [s^4 + s^3 + 5s^2 + (7+k)s + 1] = U(s) (s^3 + 2s^2 + 3s + 7)$$

Hence

$$\frac{Y(s)}{U(s)} = \frac{s^3 + 2s^2 + 3s + 7}{s^4 + s^3 + 5s^2 + (7+k)s + 1}$$

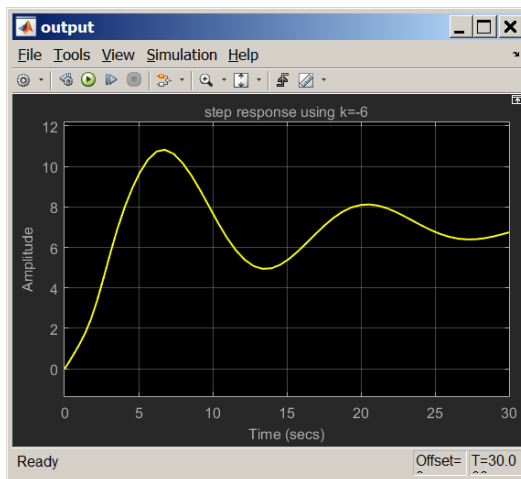
To determine what range of values for  $k$  to use, Routh stability table is first applied to the denominator polynomial

1	5	1
1	$7+k$	0
$-(2+k)$	1	0
$\frac{-(2+k)(7+k)-1}{-(2+k)}$	0	

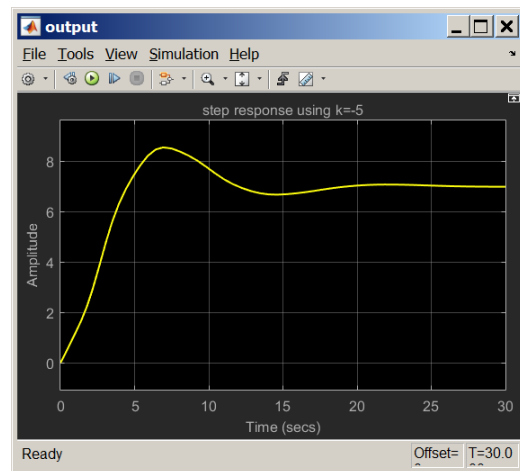
Therefore, for no sign change in the first column, we need  $-(2+k) > 0$  or  $2+k < 0$  or  $k < -2$  and we also need  $\frac{-(2+k)(7+k)-1}{-(2+k)} > 0$  or  $\frac{k^2+9k+15}{2+k} > 0$  which means  $k^2 + 9k + 15 > 0$ . The roots of this quadratic are  $k = -2.2087, k = -6.7913$ . Putting all these conditions together gives the range of value on  $k$  for a stable system as

$$-6.79 < k < -2.208$$

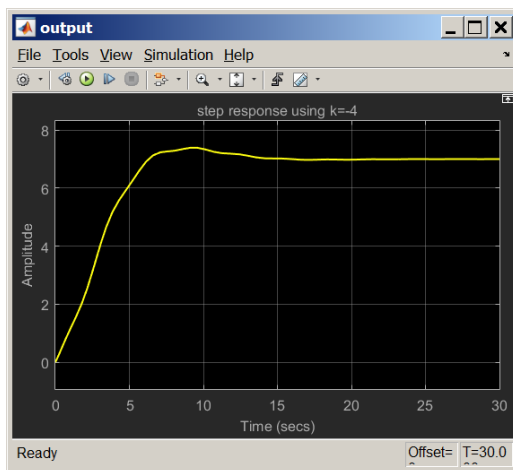
So any  $k$  in this range can be used. Here is the response using simulink using 4 different values of  $k$ , all within the above range to keep the system stable, showing that all the responses are stable, using the same 30 seconds duration.



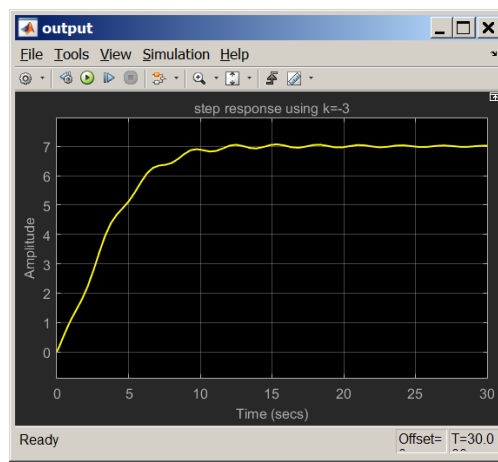
K=-6



K=-5



K=-4



K=-3