

HW 5

Math 703 methods of applied mathematics I

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Contents

1	Problem 4.1.1(d)	3
2	Problem 4.1.2	5
	2.1 Part (a)	6
	2.2 Part(b)	6
3	Problem 4.1.3	8
	3.1 Part(a)	8
	3.2 Part (b)	9
4	Problem 4.1.4	10
5	Problem 4.1.5	11
6	Problem 4.1.8	12
7	Problem 4.1.10	14
8	Problem 4.1.11	15
9	Problem 4.1.16	17
10	Problem 4.1.19	18

11 Problem 4.1.20	20
12 Problem 4.1.26	21
13 Problem 4.3.3	22
13.1 Part(a)	22
13.2 Part(b)	22
14 Problem 4.3.5	23
14.1 Part(a)	23
14.2 Part(b)	24
15 Problem 4.3.6	24
16 Problem 4.3.7	25
16.1 Part(a)	26
16.2 Part(b)	26
17 Problem 4.3.10	26
18 Problem 4.3.21	27
19 Problem 4.3.27	29

1 Problem 4.1.1(d)

4.1.1 Find the Fourier series on $-\pi < x < \pi$ for

- (a) $f(x) = \sin^3 x$, an odd function
- (b) $f(x) = |\sin x|$, an even function
- (c) $f(x) = x^2$, integrating either $x^2 \cos kx$ or the sine series for $f = x$
- (d) $f(x) = e^x$, using the complex form of the series.

What are the even and odd parts of $f(x) = e^x$ and $f(x) = e^{ix}$?

Figure 1: the Problem statement

$$f(x) = e^x = \sum_{k=-\infty}^{\infty} c_k e^{ikx} \quad (1)$$

Where

$$\begin{aligned} c_k &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(1-ik)x} dx \\ &= \frac{1}{2\pi} \left[\frac{e^{(1-ik)x}}{1-ik} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi(1-ik)} \left[\frac{e^{\pi(1-ik)} - e^{-\pi(1-ik)}}{2} \right] \end{aligned}$$

But $\frac{e^z}{2} - \frac{e^{-z}}{2} = \sinh(z)$, hence the above reduces to

$$c_k = \frac{1}{\pi(1-ik)} \sinh(\pi(1-ik)) \quad (2)$$

Substituting (2) into (1) gives

$$e^x = \sum_{k=-\infty}^{\infty} \frac{1}{\pi(1-ik)} \sinh(\pi(1-ik)) e^{ikx}$$

Here are few terms in the series generated using symbolic software:

```
ClearAll[x, k, n, f, ck]
ck[k_, x_] := 1/(2 Pi) Integrate[Exp[x] Exp[-I k x], {x, -Pi, Pi}]
f[k_, x_] := ck[k, x]*Exp[I k x];
term[n_] := If[n == 0, N@f[0, x], N@Simplify@ComplexExpand[f[-n, x] + f[n, x]]]
tbl = Table[{k, Simplify@TrigToExp@ck[k, x]}, {k, -5, 5, 1}];
Grid[Join[{"k", "C_k"}, tbl], Frame -> All]
```

$$\begin{array}{r}
k \\
-5 \\
-4 \\
-3 \\
-2 \\
-1 \\
0 \\
1 \\
2 \\
3 \\
4 \\
5
\end{array}
\begin{array}{l}
C_k \\
\frac{(1-5i)e^{-\pi} - (1-5i)e^{\pi}}{52\pi} \\
\left(\frac{1}{34} - \frac{2i}{17}\right) e^{-\pi} (e^{2\pi} - 1) \\
\frac{(1-3i)e^{-\pi} - (1-3i)e^{\pi}}{20\pi} \\
\left(\frac{1}{10} - \frac{i}{5}\right) e^{-\pi} (e^{2\pi} - 1) \\
-\frac{\left(\frac{1}{4} - \frac{i}{4}\right) e^{-\pi} (e^{2\pi} - 1)}{e^{-\pi} - e^{\pi}} \\
\frac{(1+i)e^{-\pi} - (1+i)e^{\pi}}{2\pi} \\
\left(\frac{1}{10} + \frac{i}{5}\right) e^{-\pi} (e^{2\pi} - 1) \\
\frac{(1+3i)e^{-\pi} - (1+3i)e^{\pi}}{20\pi} \\
\left(\frac{1}{34} + \frac{2i}{17}\right) e^{-\pi} (e^{2\pi} - 1) \\
\frac{(1+5i)e^{-\pi} - (1+5i)e^{\pi}}{52\pi}
\end{array}$$

Here is a plot of Fourier series of e^x for k increasing range to compare with e^x . To generate this plot the terms with $c_{-k} + c_k$ were added in order together to obtain a real valued function before plotting. Plotting was done from $x = -\pi \cdots \pi$. We see as more terms are added, the approximation improves. At 20 terms, the approximations became very good. Here is the plot

```

ck = 1/(2 Pi) Integrate[Exp[x] Exp[-I k1 x], {x, -Pi, Pi}]
f[k_] := (ck /. k1 -> k)*Exp[I k x];
fs[n_] := Sum[Simplify[f[-k] + f[k]], {k, 1, n}] + f[0];
tbl = Table[Plot[{fs[n], Exp[x]}, {x, -Pi, Pi}, Frame -> True, Axes -> False,
FrameLabel -> {"f(x)", None},
{"x", Row[{"Using " <> ToString[n] <> " terms"}]}],
PlotStyle -> {Dashed, Red}], {n, 1, 20, 1}];
Grid[Partition[tbl, 4]]

```

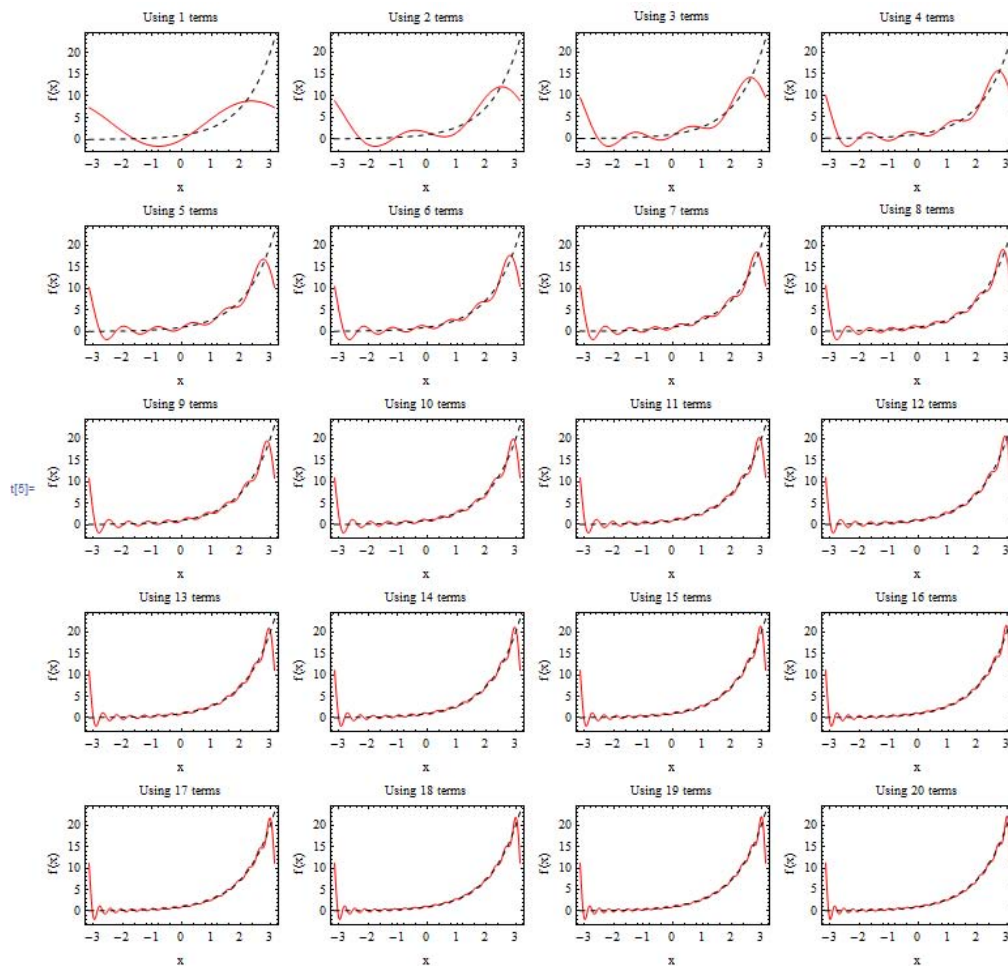


Figure 2: Plot for problem 4.1.1

The even part of e^x are given by $\frac{e^x + e^{-x}}{2} = \cosh x$ and the odd part is $\frac{e^x - e^{-x}}{2} = \sinh x$. For e^{ix} , the even part is $\frac{e^{ix} + e^{-ix}}{2} = \cos x$ and the odd part is $\frac{e^{ix} - e^{-ix}}{2} = i \sin x$

2 Problem 4.1.2

4.1.2 A square wave has $f(x) = -1$ on the left side $-\pi < x < 0$ and $f(x) = +1$ on the right side $0 < x < \pi$.

- (1) Why are all the cosine coefficients $a_k = 0$?
- (2) Find the sine series $\sum b_k \sin kx$ from equation (6).

Figure 3: the Problem statement

2.1 Part (a)

Since $f(-\pi) = -f(-\pi)$ then $f(x)$ is an odd function. For an odd function all the $a_k = 0$ since these go with the even part.

2.2 Part(b)

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \\ &= \frac{1}{\pi} \left(\int_{-\pi}^0 f(x) \sin(kx) dx + \int_0^{\pi} f(x) \sin(kx) dx \right) \\ &= \frac{1}{\pi} \left(\int_{-\pi}^0 -\sin(kx) dx + \int_0^{\pi} \sin(kx) dx \right) \end{aligned}$$

Changing the limits of integration changes the sign, hence the above can be written as

$$\begin{aligned} b_k &= \frac{1}{\pi} \left(\int_0^{\pi} \sin(kx) dx + \int_0^{\pi} \sin(kx) dx \right) \\ &= \frac{2}{\pi} \int_0^{\pi} \sin(kx) dx \\ &= \frac{2}{\pi} \left[\frac{-\cos kx}{k} \right]_0^{\pi} \\ &= \frac{-2}{\pi k} [\cos kx]_0^{\pi} \\ &= \frac{-2}{\pi k} [\cos k\pi - \cos 0] \\ &= \frac{2}{\pi k} (1 - \cos k\pi) \quad k = 1, 2, 3, \dots \end{aligned}$$

Hence

$$b_k = \begin{cases} \frac{4}{\pi k} & k = 1, 3, 5, \dots \\ 0 & k = 2, 4, 6, \dots \end{cases}$$

Hence using $f(x) = \sum_{k=1}^{\infty} b_k \sin kx$, we can write the Fourier series of $f(x)$ as

$$\begin{aligned} f(x) &= \sum_{k=1,3,\dots}^{\infty} \frac{4}{\pi k} \sin kx \\ &= \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \dots \\ &= \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right) \end{aligned}$$

Here is a plot showing the Fourier series approximation to the square wave from $x = -\pi \dots \pi$ as more terms are added

```

Clear[f, k, x];
f[x_, k_] := Sum[2/(Pi n) (1 - Cos[n Pi]) Sin[n x], {n, 1, k}];
tbl = Partition[Table[
Plot[{Sign[x], f[x, k]}, {x, -Pi, Pi},
Exclusions -> None, PlotLabel -> Row[{"k=", k}],
PlotStyle -> {Thin, Red}], {k, 1, 20, 2}], 3];
Grid[tbl, Frame -> All]

```

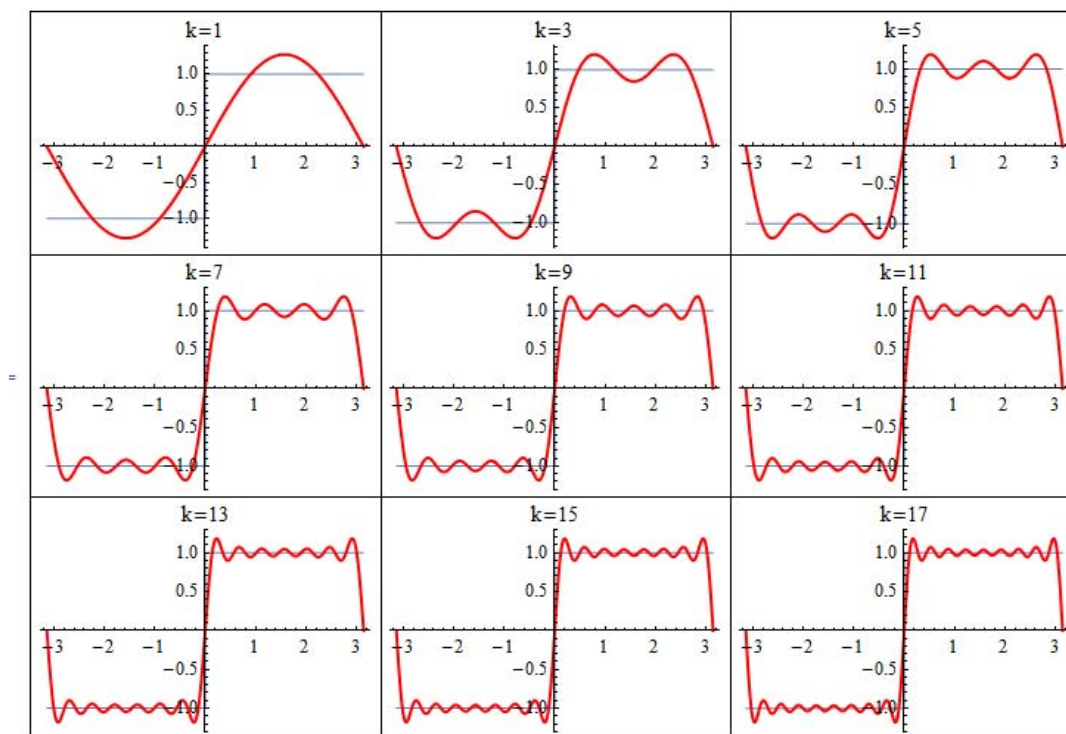


Figure 4: Plot for problem 4.1.2

3 Problem 4.1.3

4.1.3 Find this sine series for the square wave f in another way, by showing that

(a) $df/dx = 2\delta(x) - 2\delta(x + \pi)$ extended periodically

(b) $2\delta(x) - 2\delta(x + \pi) = \frac{4}{\pi} (\cos x + \cos 3x + \dots)$ from (10)

Integrate each term to find the square wave f .

Figure 5: the Problem statement

3.1 Part(a)

We first need to determine the Fourier series for $\delta(x)$ and $\delta(x + \pi)$. For $\delta(x)$ we find

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x) dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x) \cos kx dx = \frac{1}{\pi} \quad (\text{since } \cos 0 = 1)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x) \sin kx dx = 0 \quad (\text{since } \sin 0 = 0)$$

Hence

$$\begin{aligned} \delta(x) &= \frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \cos kx \\ &= \frac{1}{2\pi} + \frac{1}{\pi} (\cos x + \cos 2x + \cos 3x + \dots) \end{aligned}$$

Now to determine Fourier series for $\delta(x + \pi)$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x + \pi) dx = \frac{1}{2\pi}$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x + \pi) \cos kx dx = \frac{(-1)^k}{\pi} \quad (\text{since } \cos(-k\pi) = \cos k\pi = (-1)^k)$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \delta(x) \sin kx dx = 0 \quad (\text{since } \sin(-k\pi) = 0)$$

Hence

$$\begin{aligned}\delta(x + \pi) &= \frac{1}{2\pi} + \sum_{k=1}^{\infty} a_k \cos kx \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \cos kx \\ &= \frac{1}{2\pi} + \frac{1}{\pi} (-\cos x + \cos 2x - \cos 3x + \dots)\end{aligned}$$

Therefore

$$\begin{aligned}2\delta(x) - 2\delta(x + \pi) &= 2 \left[\frac{1}{2\pi} + \frac{1}{\pi} (\cos x + \cos 2x + \cos 3x + \dots) \right] - 2 \left[\frac{1}{2\pi} + \frac{1}{\pi} (-\cos x + \cos 2x - \cos 3x + \dots) \right] \\ &= \frac{1}{\pi} + \frac{2}{\pi} (\cos x + \cos 2x + \cos 3x + \dots) - \frac{1}{\pi} + \frac{2}{\pi} (\cos x - \cos 2x + \cos 3x - \cos 5x + \dots) \\ &= \frac{2}{\pi} (2 \cos x + 2 \cos 3x + 2 \cos 5x + \dots) \\ &= \frac{4}{\pi} (\cos x + \cos 3x + \cos 5x + \dots)\end{aligned}$$

Hence

$$\frac{df}{dx} = \frac{4}{\pi} (\cos x + \cos 3x + \cos 5x + \dots)$$

Hence

$$f(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$$

3.2 Part (b)

We first need to determine the Fourier series for $\delta(x)$ and $\delta(x + \pi)$. For $\delta(x)$ we find

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x) e^{-ikx} dx = \frac{1}{2\pi}$$

Hence

$$\begin{aligned}\delta(x) &= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi} e^{ikx} \\ &= \frac{1}{2\pi} (1 + e^{-ikx} + e^{ikx} + e^{-2ik} + e^{2ik} + \dots) \\ &= \frac{1}{2\pi} (1 + 2 \cos kx + 2 \cos 2kx + 2 \cos 3kx + \dots)\end{aligned}$$

Now to determine Fourier series for $\delta(x + \pi)$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(x + \pi) e^{-ikx} dx = \frac{1}{2\pi} e^{ik\pi} = \frac{1}{2\pi} \cos k\pi = \frac{(-1)^k}{2\pi}$$

Hence

$$\begin{aligned}
 \delta(x + \pi) &= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{2\pi} e^{ikx} \\
 &= \frac{1}{2\pi} (1 - e^{-ix} - e^{ix} + e^{-2ix} + e^{2ix} - e^{-3ix} - e^{3ix} + \dots) \\
 &= \frac{1}{2\pi} (1 - (e^{-ix} + e^{ix}) + e^{-2ix} + e^{2ix} - (e^{-3ix} + e^{3ix}) + \dots) \\
 &= \frac{1}{2\pi} (1 - 2 \cos x + 2 \cos 2x - 2 \cos 3x + \dots)
 \end{aligned}$$

Therefore

$$\begin{aligned}
 2\delta(x) - 2\delta(x + \pi) &= 2 \left[\frac{1}{2\pi} (1 + 2 \cos x + 2 \cos 2x + 2 \cos 3x + \dots) \right] - 2 \left[\frac{1}{2\pi} (1 - 2 \cos x + 2 \cos 2x - 2 \cos 3x + \dots) \right] \\
 &= \frac{1}{\pi} (1 + 2 \cos x + 2 \cos 2x + 2 \cos 3x + \dots) - \frac{1}{\pi} (1 - 2 \cos x + 2 \cos 2x - 2 \cos 3x + \dots) \\
 &= \frac{1}{\pi} (4 \cos x + 4 \cos 3x + 4 \cos 5x + \dots) \\
 &= \frac{4}{\pi} (\cos x + \cos 3x + \cos 5x + \dots)
 \end{aligned}$$

Hence

$$\frac{df}{dx} = \frac{4}{\pi} (\cos x + \cos 3x + \cos 5x + \dots)$$

Therefore

$$f(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$$

Which is the same as above using the a_k, b_k method.

4 Problem 4.1.4

4.1.4 At $x = \pi/2$ the square wave equals 1. From the Fourier series at this point find the alternating sum that equals π :

$$\pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \dots \right).$$

Figure 6: the Problem statement

From above we found that the Fourier series for square wave is

$$f(x) = \frac{4}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$$

Therefore at $x = \frac{\pi}{2}$, the above becomes

$$1 = \frac{4}{\pi} \left(\sin \frac{\pi}{2} + \frac{1}{3} \sin 3 \frac{\pi}{2} + \frac{1}{5} \sin 5 \frac{\pi}{2} + \dots \right)$$

Hence

$$\begin{aligned}\pi &= 4 \left(\sin \frac{\pi}{2} + \frac{1}{3} \sin 3 \frac{\pi}{2} + \frac{1}{5} \sin 5 \frac{\pi}{2} + \dots \right) \\ &= 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right)\end{aligned}$$

5 Problem 4.1.5

4.1.5 From Parseval's formula the square wave sine coefficients satisfy

$$\pi(b_1^2 + b_2^2 + \dots) = \int_{-\pi}^{\pi} |f(x)|^2 dx = \int_{-\pi}^{\pi} 1 dx = 2\pi.$$

Derive another remarkable sum $\pi^2 = 8(1 + \frac{1}{9} + \frac{1}{25} + \dots)$.

Figure 7: the Problem statement

We found that only the b_k survive for the Fourier series of the wave function. They are

$$b_k = \begin{cases} \frac{4}{\pi k} & k = 1, 3, 5, \dots \\ 0 & k = 2, 4, 6, \dots \end{cases}$$

Applying Parseval's formula leads to

$$\pi(b_1^2 + b_3^2 + b_5^2 + \dots) = \int_{-\pi}^{\pi} |f(x)|^2 dx = 2\pi$$

Where we used only the odd b_k terms since all others are zero. The above becomes

$$\begin{aligned}\pi \left(\left(\frac{4}{\pi} \right)^2 + \left(\frac{4}{3\pi} \right)^2 + \left(\frac{4}{5\pi} \right)^2 + \dots \right) &= 2\pi \\ \pi \left(\frac{1}{\pi^2} 4^2 + \frac{1}{\pi^2} \left(\frac{4}{3} \right)^2 + \frac{1}{\pi^2} \left(\frac{4}{5} \right)^2 + \dots \right) &= 2\pi \\ \left(4^2 + \left(\frac{4}{3} \right)^2 + \left(\frac{4}{5} \right)^2 + \dots \right) &= 2\pi^2 \\ \pi^2 &= 8 \left(1 + \left(\frac{1}{3} \right)^2 + \left(\frac{1}{5} \right)^2 + \dots \right)\end{aligned}$$

Hence

$$\pi^2 = 8 \left(1 + \frac{1}{9} + \frac{1}{25} + \dots \right)$$

6 Problem 4.1.8

4.1.8 Suppose f has period T instead of 2π , so that $f(x) = f(x + T)$. Its graph from $-T/2$ to $T/2$ is repeated on each successive interval and its real and complex Fourier series are

$$f(x) = a_0 + a_1 \cos \frac{2\pi x}{T} + b_1 \sin \frac{2\pi x}{T} + \dots = \sum_{-\infty}^{\infty} c_j e^{ij2\pi x/T}.$$

Multiplying by the right functions and integrating from $-T/2$ to $T/2$, find a_k , b_k , and c_k .

Figure 8: the Problem statement

ps. In the solution below, I was using T when I should be using $\frac{T}{2}$ in all the limits. Need to correct later. Or just let period be $2T$ then the math works ok.

In this problem, the basic idea is to observe that when the period was 2π then

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx \\ f(x) &= \sum_{k=-\infty}^{\infty} c_k e^{ikx} \end{aligned}$$

Now when the period is a general value T we use $\left(\frac{2\pi}{T}k\right)$ in place of just k . So the above becomes

$$f(x) = \sum_{k=0}^{\infty} a_k \cos \left(k \frac{2\pi}{T} x\right) + \sum_{k=1}^{\infty} b_k \sin \left(k \frac{2\pi}{T} x\right) \quad (1)$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\left(\frac{2\pi}{T}k\right)x} \quad (2)$$

We now need to determine a_k, b_k, c_k using (1) and (2) in similar way we did when the period was 2π .

To find a_k we multiply (1) by $\cos\left(m \frac{2\pi}{T} x\right)$ where m is some integer between $1 \dots \infty$, and integrating from $-T$ to T gives

$$\begin{aligned} \int_{-T}^T f(x) \cos \left(m \frac{2\pi}{T} x\right) dx &= \int_{-T}^T \sum_{k=0}^{\infty} a_k \cos \left(k \frac{2\pi}{T} x\right) \cos \left(m \frac{2\pi}{T} x\right) dx + \int_{-T}^T \sum_{k=1}^{\infty} b_k \sin \left(k \frac{2\pi}{T} x\right) \cos \left(m \frac{2\pi}{T} x\right) dx \\ &= \sum_{k=0}^{\infty} \int_{-T}^T a_k \cos \left(k \frac{2\pi}{T} x\right) \cos \left(m \frac{2\pi}{T} x\right) dx + \sum_{k=1}^{\infty} \int_{-T}^T b_k \sin \left(k \frac{2\pi}{T} x\right) \cos \left(m \frac{2\pi}{T} x\right) dx \end{aligned}$$

Due to orthogonality between the sin and cos, all the product of sin cos vanish, and only one term in the product of cos cos remain which is the one when $k = m$, hence the above reduces to

$$\int_{-T}^T f(x) \cos \left(m \frac{2\pi}{T} x\right) dx = \int_{-T}^T a_m \cos \left(m \frac{2\pi}{T} x\right) \cos \left(m \frac{2\pi}{T} x\right) dx$$

Since m is arbitrary, we can rename it back to k to keep the same naming as before.

$$\int_{-T}^T f(x) \cos\left(k \frac{2\pi}{T} x\right) dx = \int_{-T}^T a_k \cos^2\left(k \frac{2\pi}{T} x\right) dx \quad (3)$$

When $k = 0$ we find

$$\begin{aligned} \int_{-T}^T f(x) dx &= \int_{-T}^T a_0 dx \\ &= 2a_0 T \end{aligned}$$

Hence

$$a_0 = \frac{1}{2T} \int_{-T}^T f(x) dx$$

Notice, when $T = \pi$, the above reduces to $a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$. Now to find a_k for $k \geq 1$, then from (3)

$$\begin{aligned} \int_{-T}^T f(x) \cos\left(k \frac{2\pi}{T} x\right) dx &= \int_{-T}^T a_k \cos^2\left(k \frac{2\pi}{T} x\right) dx \\ &= a_k T \end{aligned}$$

Hence

$$a_k = \frac{1}{T} \int_{-T}^T f(x) \cos\left(k \frac{2\pi}{T} x\right) dx$$

Notice that when $T = \pi$ the above reduces to $a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx$ as before.

Now we find b_k similarly. We multiply (1) by $\sin\left(m \frac{2\pi}{T} x\right)$ where m is some integer between $1 \cdots \infty$, and integrating from $-T$ to T gives

$$\int_{-T}^T f(x) \sin\left(m \frac{2\pi}{T} x\right) dx = \sum_{k=0}^{\infty} \int_{-T}^T a_k \cos\left(k \frac{2\pi}{T} x\right) \sin\left(m \frac{2\pi}{T} x\right) dx + \sum_{k=1}^{\infty} \int_{-T}^T b_k \sin\left(k \frac{2\pi}{T} x\right) \sin\left(m \frac{2\pi}{T} x\right) dx$$

Due to orthogonality between the sin and cos, all the products of sin cos vanish, and only one term in the product of sin sin remain which is the one when $k = m$, hence the above reduces to

$$\int_{-T}^T f(x) \sin\left(m \frac{2\pi}{T} x\right) dx = \int_{-T}^T b_m \sin\left(m \frac{2\pi}{T} x\right) \sin\left(m \frac{2\pi}{T} x\right) dx$$

Since m is arbitrary, we can rename it back to k to keep the same naming as before.

$$\begin{aligned} \int_{-T}^T f(x) \sin\left(k \frac{2\pi}{T} x\right) dx &= \int_{-T}^T b_k \sin^2\left(k \frac{2\pi}{T} x\right) dx \\ &= b_k T \end{aligned}$$

Hence

$$b_k = \frac{1}{T} \int_{-T}^T f(x) \sin\left(k \frac{2\pi}{T} x\right) dx$$

Notice that when $T = \pi$ the above reduces to $b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx$ as before. We now find c_k .

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{i\left(k\frac{2\pi}{T}\right)x}$$

Multiplying both side by $e^{-i\left(m\frac{2\pi}{T}\right)x}$ and integrating over the period

$$\int_{-T}^T f(x) e^{-i\left(m\frac{2\pi}{T}\right)x} dx = \sum_{k=-\infty}^{\infty} \int_{-T}^T c_k e^{i\left(k\frac{2\pi}{T}\right)x} e^{-i\left(m\frac{2\pi}{T}\right)x} dx$$

All terms other than ones which $k = m$ remain. Hence the above becomes

$$\begin{aligned} \int_{-T}^T f(x) e^{-i\left(m\frac{2\pi}{T}\right)x} dx &= \int_{-T}^T c_m e^{i\left(m\frac{2\pi}{T}\right)x} e^{-i\left(m\frac{2\pi}{T}\right)x} dx \\ &= \int_{-T}^T c_m dx \end{aligned}$$

Therefore, since m is now arbitrary, we rename it back to k and simplifying

$$\begin{aligned} \int_{-T}^T f(x) e^{-i\left(k\frac{2\pi}{T}\right)x} dx &= 2T c_k \\ c_k &= \frac{1}{2T} \int_{-T}^T f(x) e^{-i\left(k\frac{2\pi}{T}\right)x} dx \end{aligned}$$

7 Problem 4.1.10

4.1.10 What constant function is closest in the least square sense to $f = \cos^2 x$? What multiple of $\cos x$ is closest to $f = \cos^3 x$?

Figure 9: the Problem statement

The a_0 term in the Fourier series of $\cos^2 x$ is the constant term. Hence it is the constant that is closest to $\cos^2 x$ in the square sense. Therefore

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2 x dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^2 x dx \\ &= \frac{1}{2} \end{aligned}$$

To find the multiple of $\cos x$ which is closest to $\cos^3 x$, we find a_1 term in the Fourier series of $\cos^3 x$ since that is the term which has $a_1 \cos x$ in it. Hence

$$\begin{aligned} a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos^3 x \cos x dx \\ &= \frac{1}{\pi} \left(\frac{3\pi}{4} \right) \\ &= \frac{3}{4} \end{aligned}$$

8 Problem 4.1.11

4.1.11 Sketch the graph and find the Fourier series of the even function $f = 1 - |x|/\pi$ (extended periodically) in either of two ways: integrate the square wave or compute (with $a_0 = \frac{1}{2}$)

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{x}{\pi} \right) \cos kx dx.$$

Figure 10: the Problem statement

The function we are approximating using Fourier series is

```
f[x_] := Piecewise[{{1 + x/Pi, x < 0}, {1 - x/Pi, x >= 0}}];
Plot[f[x], {x, -Pi, Pi}]
```

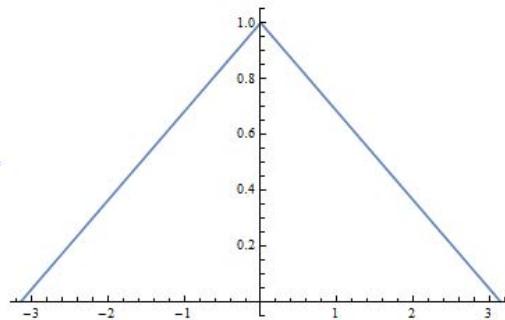


Figure 11: Plot for problem 4.1.11

Since it is even, we only need to determine a_k

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{x}{\pi}\right) \cos kx dx \\ &= \frac{2}{\pi} \left(\frac{1 - \cos k\pi}{k^2 \pi} \right) \end{aligned}$$

Hence

$$\begin{aligned} f(x) &= a_0 + \sum_{k=1}^{\infty} a_k \cos kx \\ &= \frac{1}{2} + \frac{2}{\pi} \left(\frac{1 - \cos \pi}{\pi} \right) \cos x + \frac{2}{\pi} \left(\frac{1 - \cos 2\pi}{4\pi} \right) \cos 2x + \frac{2}{\pi} \left(\frac{1 - \cos 3\pi}{9\pi} \right) \cos 3x + \dots \\ &= \frac{1}{2} + \frac{2}{\pi} \left(\frac{2}{\pi} \right) \cos x + \frac{2}{\pi} \left(\frac{2}{9\pi} \right) \cos 3x + \frac{2}{\pi} \left(\frac{2}{25\pi} \right) \cos 5x + \dots \\ &= \frac{1}{2} + \frac{4}{\pi^2} \cos x + \frac{4}{9\pi^2} \cos 3x + \frac{4}{25\pi^2} \cos 5x + \dots \end{aligned}$$

Here is a plot showing the approximation as more terms are added. The label of each plot show the number of terms used. The more terms we use, the better the approximation

```
ck = (2/Pi) Integrate[(1 - x/Pi) Cos[k x], {x, 0, Pi}];
upTo[n_, x_] := (1/2) + Sum[(ck /. k -> m)* Cos[m x], {m, 1, n}];
tbl = Table[Plot[upTo[m, x], {x, -Pi, Pi},
PlotLabel -> Row[{"terms used =", m}]], {m, 0, 18, 2}];
Grid[Partition[tbl, 3], Frame -> All]
```

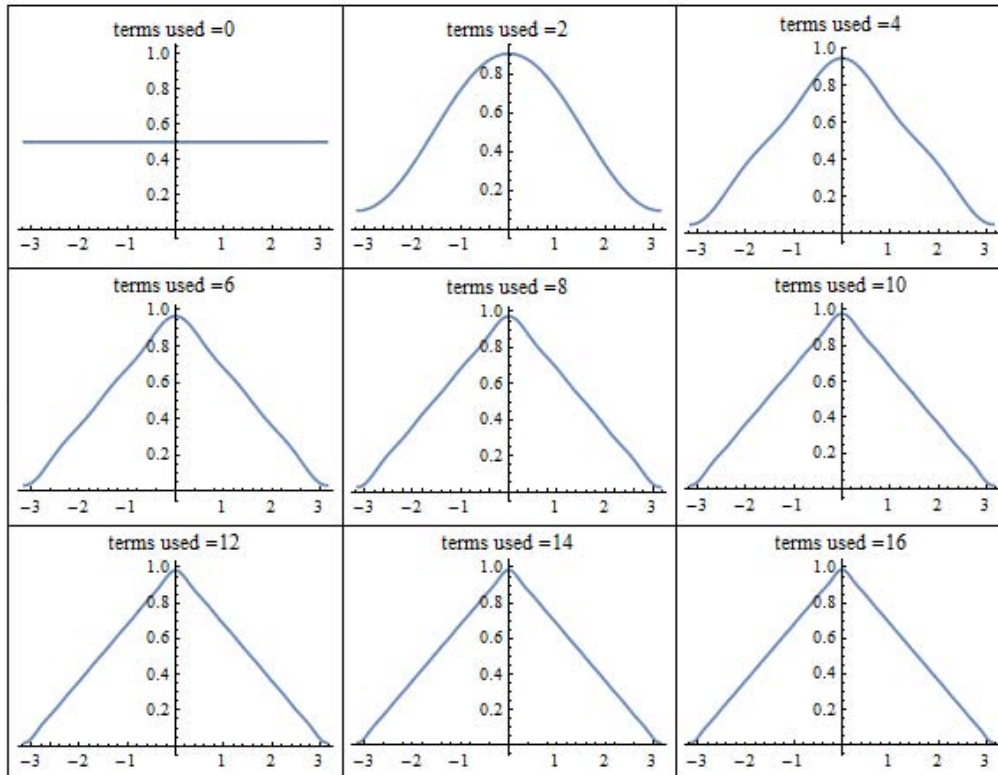



Figure 12: Plot for problem 4.1.11 part 2

9 Problem 4.1.16

4.1.16 If the boundary condition for Laplace's equation is $u_0 = 1$ for $0 < \theta < \pi$ and $u_0 = 0$ for $-\pi < \theta < 0$, find the Fourier series solution $u(r, \theta)$ inside the unit circle. What is u at the origin?

Figure 13: the Problem statement

The first step is to obtain the a_k, b_k coefficients by expanding the boundary value of the solution using Fourier series. On the boundary

$$u_0 = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & -\pi < \theta < 0 \end{cases}$$

Hence

$$a_0 = \frac{1}{2\pi} \int_0^{\pi} d\theta = \frac{1}{2}$$

And

$$a_k = \frac{1}{\pi} \int_0^{\pi} \cos k\theta d\theta = \frac{1}{k\pi} [\sin k\theta]_0^{\pi} = 0$$

And

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_0^{\pi} \sin k\theta d\theta = \frac{1}{k\pi} [-\cos k\theta]_0^{\pi} = 0 = \frac{-1}{k\pi} [\cos k\pi - \cos 0] \\ &= \left\{ \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \dots \right\} \end{aligned}$$

Only odd values of k survive. Now that we found the Fourier coefficient, we use them in the solution given in equation (22), page 276 on the book

$$\begin{aligned} u(r, \theta) &= a_0 + b_1 r \sin \theta + b_3 r^3 \sin 3\theta + b_5 r^5 \sin^5 \theta + \dots \\ &= \frac{1}{2} + \frac{2}{\pi} \left(r \sin \theta + \frac{1}{3} r^3 \sin 3\theta + \frac{1}{5} r^5 \sin^5 \theta + \dots \right) \end{aligned}$$

At the origin, let $r = 0$

$$u(0, \theta) = \frac{1}{2}$$

10 Problem 4.1.19

4.1.19 A plucked string goes linearly from $f(0) = 0$ to $f(p) = 1$ and back to $f(\pi) = 0$. The linear part $f = x/p$ reaches to $x = p$, followed by $f = (\pi - x)/(\pi - p)$ to $x = \pi$. Sketch f as an

odd function and find a plucking point p for which the second harmonic $\sin 2x$ will not be heard ($b_2 = 0$).

Figure 14: the Problem statement

A sketch of the function (string) is below.

```
Clear[x, f, p];
f[x_, p_] := Piecewise[{{(-x - Pi)/(Pi - p), x < -p},
{(x + p)/p - 1, -p < x < 0}, {x/p, 0 < x < p},
{(x - Pi)/(p - Pi), p < x < Pi}}]
Plot[f[x, .8 Pi], {x, -Pi, Pi}, Frame -> True,
FrameLabel -> {"f(x)", None}, {x, "problem 4.1.19"}]
```

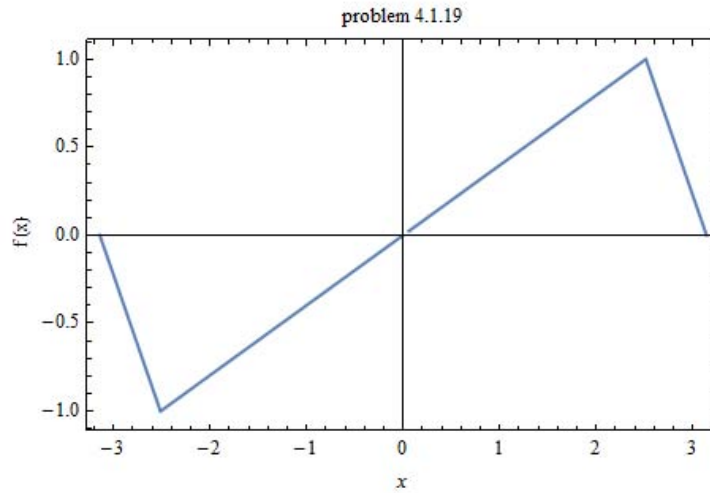


Figure 15: Plot for problem 4.1.19

Since $f(x)$ is odd, we only need to determine b_k

$$\begin{aligned}
 b_k &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin kx dx \\
 &= \frac{2}{\pi} \left(\int_0^p \frac{x}{p} \sin kx dx + \int_p^{\pi} \frac{x-\pi}{p-\pi} \sin kx dx \right) \\
 &= \frac{2}{\pi} \left(\frac{\sin kp - kp \cos kp}{k^2 p} + \frac{k(\pi-p) \cos kp + \sin kp - \sin k\pi}{k^2 (\pi-p)} \right) \\
 &= \frac{2(\pi \sin kp - p \sin k\pi)}{k^2 p \pi (\pi-p)}
 \end{aligned}$$

For $k = 2$

$$\begin{aligned}
 b_2 &= \frac{(\pi \sin 2p - p \sin 2\pi)}{2p\pi(\pi-p)} \\
 &= \frac{\pi \sin 2p}{2p\pi(\pi-p)}
 \end{aligned}$$

For zero, we need

$$\begin{aligned}
 0 &= \pi \sin 2p \\
 \sin 2p &= 0
 \end{aligned}$$

Hence

$$p = \frac{\pi}{2}$$

11 Problem 4.1.20

4.1.20 Show that $P_2 = x^2 - \frac{1}{3}$ is orthogonal to $P_0 = 1$ and $P_1 = x$ over the interval $-1 \leq x \leq 1$. Can you find the next Legendre polynomial by choosing c to make $x^3 - cx$ orthogonal to P_0, P_1 , and P_2 ?

Figure 16: the Problem statement

Two functions f, g are if the inner product is zero $\int_{-1}^1 f(x)g(x) dx = 0$. Hence

$$\int_{-1}^1 P_2 P_0 dx = \int_{-1}^1 (x^2 - 1) dx = \left(\frac{x^3}{3} - x \right)_{-1}^1 = 0$$

And

$$\int_{-1}^1 P_2 P_1 dx = \int_{-1}^1 (x^2 - 1) x dx = \left(\frac{x^4}{4} - \frac{x^2}{2} \right)_{-1}^1 = 0$$

Now let $P_3 = x^3 - cx$, we want this to be orthogonal to P_0, P_1, P_2 . Hence

$$\begin{aligned} \int_{-1}^1 P_3 P_0 dx &= \int_{-1}^1 x^3 - cx dx = \left(\frac{x^4}{4} - c \frac{x^2}{2} \right)_{-1}^1 = \left(\frac{1}{4} - c \frac{1}{2} \right) - \left(\frac{1}{4} - c \frac{1}{2} \right) \\ &= 0 \end{aligned}$$

This equation did not help us find c . We try the next one

$$\begin{aligned} \int_{-1}^1 P_3 P_1 dx &= \int_{-1}^1 (x^3 - cx) x dx = \left(\frac{x^5}{5} - c \frac{x^3}{3} \right)_{-1}^1 = \left(\frac{1}{5} - c \frac{1}{3} \right) - \left(-\frac{1}{5} + c \frac{1}{3} \right) = \frac{2}{5} - \frac{2}{3}c \\ \frac{2}{5} - \frac{2}{3}c &= 0 \\ c &= \frac{2/5}{2/3} \\ &= \frac{3}{5} \end{aligned}$$

Hence

$$P_3 = x^3 - \frac{3}{5}x$$

12 Problem 4.1.26

4.1.26 If f has the double sine series $\sum \sum b_{kl} \sin kx \sin ly$, show that Poisson's equation $-u_{xx} - u_{yy} = f$ is solved by the double sine series $u = \sum \sum b_{kl} \sin kx \sin ly / (k^2 + l^2)$. This is the solution with $u = 0$ on the boundary of the square $-\pi < x, y < \pi$.

Figure 17: the Problem statement

The proposed solution is

$$u(x, y) = \sum \sum \frac{b_{kl} \sin kx \sin ly}{(k^2 + l^2)} \quad (1)$$

To see if this solves

$$-u_{xx} - u_{yy} = f = \sum \sum b_{kl} \sin kx \sin ly \quad (1A)$$

we will take (1) and substitute in the LHS of Poisson equation (1A) and see if we get the RHS of (1A) which is f .

$$\begin{aligned} \frac{\partial u}{\partial x} &= \sum \sum \frac{b_{kl} k \cos kx \sin ly}{(k^2 + l^2)} \\ \frac{\partial^2 u}{\partial x^2} &= \sum \sum \frac{-b_{kl} k^2 \sin kx \sin ly}{(k^2 + l^2)} \end{aligned} \quad (2)$$

And

$$\begin{aligned} \frac{\partial u}{\partial y} &= \sum \sum \frac{b_{kl} \sin(kx) l \cos ly}{(k^2 + l^2)} \\ \frac{\partial^2 u}{\partial y^2} &= \sum \sum \frac{-b_{kl} \sin(kx) l^2 \sin ly}{(k^2 + l^2)} \end{aligned} \quad (3)$$

Substituting (2) and (3) in the LHS of (1A) gives

$$\begin{aligned} -u_{xx} - u_{yy} &= \sum \sum \frac{b_{kl} k^2 \sin kx \sin ly}{(k^2 + l^2)} + \sum \sum \frac{b_{kl} \sin(kx) l^2 \sin ly}{(k^2 + l^2)} \\ &= \sum \sum \frac{b_{kl} k^2 \sin kx \sin ly + b_{kl} \sin(kx) l^2 \sin ly}{(k^2 + l^2)} \\ &= \sum \sum \frac{(b_{kl} \sin kx \sin ly) (k^2 + l^2)}{(k^2 + l^2)} \\ &= \sum \sum b_{kl} \sin kx \sin ly \end{aligned}$$

Which is f . Hence $u(x, y) = \sum \sum \frac{b_{kl} \sin kx \sin ly}{(k^2 + l^2)}$ is the solution verified.

13 Problem 4.3.3

4.3.3 Find the inverse transforms of

(a) $\hat{f}(k) = \delta(k)$ (b) $\hat{f}(k) = e^{-|k|}$ (separate $k < 0$ from $k > 0$).

Figure 18: the Problem statement

13.1 Part(a)

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{k=-\infty}^{\infty} \delta(k) e^{ikx} dk \\ &= \frac{1}{2\pi} [e^{ikx}]_{k=0} = \frac{1}{2\pi} \end{aligned}$$

13.2 Part(b)

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{k=-\infty}^{\infty} e^{-|k|} e^{ikx} dk \\ &= \frac{1}{2\pi} \left(\int_{k=-\infty}^0 e^k e^{ikx} dk + \int_0^{\infty} e^{-k} e^{ikx} dk \right) \\ &= \frac{1}{2\pi} \left(\int_{k=-\infty}^0 e^{k(1+ix)} dk + \int_0^{\infty} e^{k(-1+ix)} dk \right) \\ &= \frac{1}{2\pi} \left(\left[\frac{e^{k(1+ix)}}{1+ix} \right]_{-\infty}^0 + \left[\frac{e^{k(-1+ix)}}{-1+ix} \right]_0^{\infty} \right) \end{aligned} \tag{1}$$

Looking at the first integral result

$$\left[\frac{e^{k(1+ix)}}{1+ix} \right]_{-\infty}^0 = \frac{1}{1+ix} - \frac{e^{-\infty(1+ix)}}{1+ix} = \frac{1}{1+ix}$$

Where we looked at real part of $e^{-\infty(1+ix)} = 0$ so that we can make $e^{-\infty(1+ix)}$ to be zero.

Looking at the second integral result

$$\left[\frac{e^{k(-1+ix)}}{-1+ix} \right]_0^{\infty} = \frac{e^{\infty(-1+ix)}}{-1+ix} - \frac{1}{-1+ix} = -\frac{1}{-1+ix}$$

Where we looked at real part of $e^{\infty(-1+ix)} = 0$ so that we can make $e^{\infty(-1+ix)}$ to be zero. Hence, using

the above two results in (1) gives

$$\begin{aligned}
 f(x) &= \frac{1}{2\pi} \left(\frac{1}{1+ix} - \frac{1}{-1+ix} \right) \\
 &= \frac{1}{2\pi} \left(\frac{1}{1+ix} + \frac{1}{1-ix} \right) \\
 &= \frac{1}{2\pi} \left(\frac{(1-ix) + (1+ix)}{(1+ix)(1-ix)} \right) \\
 &= \frac{1}{2\pi} \left(\frac{2}{1+x^2} \right) \\
 &= \frac{1}{\pi} \frac{1}{1+x^2}
 \end{aligned}$$

14 Problem 4.3.5

4.3.5 Verify Plancherel's energy equation for $f = \delta$ and $f = e^{-x^2/2}$. Infinite energy is allowed.

Figure 19: the Problem statement

14.1 Part(a)

For $f(x) = \delta(x)$

$$2\pi \int_{-\infty}^{\infty} \delta^2(x) dx = 2\pi \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta(x) g_n(x) dx$$

Where $g_n(x)$ is sequence of Gaussian functions. The RHS above becomes

$$2\pi \int_{-\infty}^{\infty} \delta^2(x) dx = 2\pi \lim_{n \rightarrow \infty} g_n(0)$$

But $\lim_{n \rightarrow \infty} g_n(0) = \infty$ hence

$$2\pi \int_{-\infty}^{\infty} \delta^2(x) dx = \infty$$

Now $\hat{f}(k) = 1$ for the Dirac delta. Hence

$$\begin{aligned}
 \hat{f}(k) &= \int_{-\infty}^{\infty} (1) e^{-ikx} dx \\
 &= \int_{-\infty}^{\infty} e^{-ikx} dx \\
 &= \left[\frac{e^{-ikx}}{-ik} \right]_{-\infty}^{\infty} = \frac{1}{-ik} (e^{-ik\infty} - e^{+ik\infty}) = \frac{1}{-ik} (0 - \infty) = \infty
 \end{aligned}$$

Hence verified for δ OK.

14.2 Part(b)

For $f(x) = e^{-\frac{x^2}{2}}$ then

$$\begin{aligned} 2\pi \int_{-\infty}^{\infty} |f(x)|^2 dx &= 2\pi \int_{-\infty}^{\infty} \left| e^{-\frac{x^2}{2}} \right|^2 dx \\ &= 2\pi \int_0^{\infty} e^{-x^2} dx \\ &= 2\pi \left(\frac{\sqrt{\pi}}{2} \right) \\ &= \pi^{\frac{3}{2}} \end{aligned}$$

Now $\hat{f}(k)$ for the above function is

$$\begin{aligned} \hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx \\ &= e^{-\frac{k^2}{2}} \sqrt{2\pi} \end{aligned}$$

Hence

$$\begin{aligned} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 dk &= \int_{-\infty}^{\infty} \left| e^{-\frac{k^2}{2}} \sqrt{2\pi} \right|^2 dk \\ &= 2\pi \int_{-\infty}^{\infty} \left| e^{-\frac{k^2}{2}} \right|^2 dk \\ &= 2\pi \int_0^{\infty} e^{-k^2} dk \\ &= 2\pi \left(\frac{\sqrt{\pi}}{2} \right) \\ &= \pi^{\frac{3}{2}} \end{aligned}$$

Which is the same as before. Hence verified.

15 Problem 4.3.6

4.3.6 What are the half-widths W_x and W_k of the bell-shaped function $f = e^{-x^2/2}$ and its transform? Show that equality holds in the uncertainty principle.

Figure 20: the Problem statement

For $f(x) = e^{-x^2/2}$

$$\begin{aligned} W_x^2 &= \frac{\int_{-\infty}^{\infty} x^2 |f(x)|^2 dx}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \\ &= \frac{\int_{-\infty}^{\infty} x^2 \left| e^{-x^2/2} \right|^2 dx}{\int_{-\infty}^{\infty} \left| e^{-x^2/2} \right|^2 dx} \\ &= \frac{\int_0^{\infty} x^2 e^{-x^2} dx}{\int_0^{\infty} e^{-x^2} dx} \\ &= \frac{\frac{\sqrt{\pi}}{4}}{\frac{\sqrt{\pi}}{2}} = \frac{1}{2} \end{aligned}$$

Now $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-ikx} dx = e^{-k^2/2} \sqrt{2\pi}$, hence

$$\begin{aligned} W_k^2 &= \frac{\int_{-\infty}^{\infty} k^2 |\hat{f}(k)|^2 dx}{\int_{-\infty}^{\infty} |\hat{f}(k)|^2 dx} \\ &= \frac{\int_{-\infty}^{\infty} k^2 \left| e^{-k^2/2} \sqrt{2\pi} \right|^2 dx}{\int_{-\infty}^{\infty} \left| e^{-k^2/2} \sqrt{2\pi} \right|^2 dx} \\ &= \frac{2\pi \int_0^{\infty} k^2 e^{-k^2} dx}{2\pi \int_0^{\infty} e^{-k^2} dx} \\ &= \frac{\frac{\sqrt{\pi}}{4}}{\frac{\sqrt{\pi}}{2}} = \frac{1}{2} \end{aligned}$$

Hence

$$W_x W_k = \sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} = \frac{1}{2}$$

But uncertainty principle says that $W_x W_k \geq \frac{1}{2}$. Hence verified OK.

16 Problem 4.3.7

4.3.7 What is the transform of $x e^{-x^2/2}$? What about $x^2 e^{-x^2/2}$, using 4L?

Figure 21: the Problem statement

16.1 Part(a)

Using 4L(1), let $f(x) = e^{-\frac{x^2}{2}}$, which has $\hat{f}(k) = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{-ikx} dx = \sqrt{2\pi} e^{-\frac{k^2}{2}}$, hence $\frac{d}{dx}f(x)$ will have the transform $ik\hat{f}(k)$, therefore,

$$\mathcal{F}\left(\frac{d}{dx}f(x)\right) = \mathcal{F}\left(-xe^{-\frac{x^2}{2}}\right) = ik\sqrt{2\pi}e^{-\frac{k^2}{2}}$$

Therefore $xe^{-\frac{x^2}{2}}$ has the transform $-ik\sqrt{2\pi}e^{-\frac{k^2}{2}}$

16.2 Part(b)

Let $f(x) = xe^{-\frac{x^2}{2}}$, which has $\hat{f}(k) = -ik\sqrt{2\pi}e^{-\frac{k^2}{2}}$ from part(a). But $\frac{d}{dx}f(x) = e^{-\frac{x^2}{2}} - x^2e^{-\frac{x^2}{2}}$. Hence the transform of $\frac{d}{dx}f(x) = ik\hat{f}(k)$. Therefore

$$\begin{aligned}\mathcal{F}\left(e^{-\frac{x^2}{2}} - x^2e^{-\frac{x^2}{2}}\right) &= ik\left(-ik\sqrt{2\pi}e^{-\frac{k^2}{2}}\right) \\ \mathcal{F}\left(e^{-\frac{x^2}{2}}\right) - \mathcal{F}\left(x^2e^{-\frac{x^2}{2}}\right) &= k^2\sqrt{2\pi}e^{-\frac{k^2}{2}}\end{aligned}$$

But $\mathcal{F}\left(e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi}e^{-\frac{k^2}{2}}$, hence

$$\begin{aligned}\mathcal{F}\left(x^2e^{-\frac{x^2}{2}}\right) &= \sqrt{2\pi}e^{-\frac{k^2}{2}} - k^2\sqrt{2\pi}e^{-\frac{k^2}{2}} \\ \mathcal{F}\left(x^2e^{-\frac{x^2}{2}}\right) &= \sqrt{2\pi}e^{-\frac{k^2}{2}}(1 - k^2)\end{aligned}$$

Therefore

$$\mathcal{F}\left(x^2e^{-\frac{x^2}{2}}\right) = \sqrt{2\pi}e^{-\frac{k^2}{2}}(1 - k^2)$$

17 Problem 4.3.10

4.3.10 Solve the differential equation

$$\frac{du}{dx} + au = \delta(x)$$

by taking Fourier transforms to find $\hat{u}(k)$. What is the solution u (the Green's function for this equation)?

Figure 22: the Problem statement

Let $\hat{u}(k)$ be the Fourier transform of $u(x)$. Using $\mathcal{F}\left(\frac{du}{dx}\right) = ik\hat{u}(k)$ and $\mathcal{F}(\delta) = 1$, then applying Fourier transform on the ODE gives

$$ik\hat{u}(k) + a\hat{u}(k) = 1$$

Solving for $\hat{u}(k)$

$$\begin{aligned}\hat{u}(k)(a + ik) &= 1 \\ \hat{u}(k) &= \frac{1}{a + ik}\end{aligned}$$

Hence, from page 310 in text book, it gives the inverse Fourier transform for the above as

$$u(x) = \begin{cases} e^{-ax} & x > 0 \\ 0 & x < 0 \end{cases}$$

18 Problem 4.3.21

4.3.21 Apply Fourier transforms to $\int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy - 2u(x) = f(x)$ to show that the solution is $u = -\frac{1}{2}f + \frac{1}{2}g$, where g comes from integrating f twice. (Its transform is $\hat{g} = \hat{f}/(i\omega)^2$.) If $f = e^{-|x|}$ find u and verify that it solves the integral equation.

Figure 23: the Problem statement

Comparing the integral equation

$$\int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy - 2u(x) = f(x) \tag{1}$$

with the one in the textbook, page 322 in example one, where the Fourier transform of

$$\int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy = f(x)$$

Is given as

$$\frac{2}{1 + \omega^2}\hat{u}(\omega) = \hat{f}(\omega)$$

The only difference is that in this problem we have an extra $-2u(x)$ term, whose Fourier transform is $-2\hat{u}(\omega)$. Hence the Fourier transform for (1) becomes

$$\frac{2}{1 + \omega^2}\hat{u}(\omega) - 2\hat{u}(\omega) = \hat{f}(\omega)$$

Solving for $\hat{u}(\omega)$

$$\begin{aligned}\hat{u}(\omega) \left(\frac{2}{1+\omega^2} - 2 \right) &= \hat{f}(\omega) \\ \hat{u}(\omega) \left(\frac{2 - 2(1+\omega^2)}{1+\omega^2} \right) &= \hat{f}(\omega) \\ \hat{u}(\omega) &= \frac{1+\omega^2}{-2\omega^2} \hat{f}(\omega)\end{aligned}$$

We need to write the above as $\hat{u}(\omega) = \frac{-1}{2}f + \frac{1}{2}g$. Hence

$$\hat{u}(\omega) = \frac{-1}{2}\hat{f}(\omega) + \frac{1}{-2\omega^2}\hat{f}(\omega) \quad (2)$$

Let $f(x) = e^{-|x|}$, then

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \\ &= \int_{-\infty}^{\infty} e^{-|x|} e^{-i\omega x} dx \\ &= \int_{-\infty}^0 e^x e^{-i\omega x} dx + \int_0^{\infty} e^{-x} e^{-i\omega x} dx \\ &= \left[\frac{e^{x(1-i\omega)}}{1-i\omega} \right]_{-\infty}^0 + \left[\frac{e^{-x(1+i\omega)}}{1+i\omega} \right]_0^{\infty} \\ &= \frac{1}{1-i\omega} - \frac{1}{1+i\omega} \\ &= \frac{(1+i\omega) - (1-i\omega)}{(1-i\omega)(1+i\omega)} \\ &= \frac{2}{1+\omega^2}\end{aligned}$$

Hence using (2)

$$\begin{aligned}\hat{u}(\omega) &= \frac{-1}{2}\hat{f}(\omega) + \frac{1}{-2\omega^2}\hat{f}(\omega) \\ &= \frac{-1}{2} \frac{2}{1+\omega^2} + \frac{1}{-2\omega^2} \frac{2}{1+\omega^2} \\ &= -\frac{1}{\omega^2}\end{aligned}$$

Hence

$$u(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2} e^{i\omega x} d\omega$$

Using tables $u(x) = \frac{-1}{2} |x|$.

19 Problem 4.3.27

4.3.27 Take Fourier transforms in the equation $d^4G/dx^4 - 2a^2d^2G/dx^2 + a^4G = \delta$ to find the transform \hat{G} of the fundamental solution. How would it be possible to find G ?

Figure 24: the Problem statement

The equation is

$$\frac{d^4G(x)}{dx^4} - 2a^2\frac{d^2G(x)}{dx^2} + a^4G(x) = \delta$$

Taking Fourier transform, and using $\frac{d^n G}{dx^n} \Rightarrow (ik)^n \hat{g}(k)$, hence $G'(x) \Rightarrow ik\hat{g}(k)$, $G''(x) \Rightarrow -k^2\hat{g}(k)$, $G'''(x) \Rightarrow (ik)^3\hat{g}(k) = k^3\hat{g}(k)$. Therefore the Fourier transform of the above differential equation is

$$k^4\hat{g}(k) + 2a^2k^2\hat{g}(k) + a^4\hat{g}(k) = 1$$

Solving for $\hat{g}(k)$

$$\begin{aligned}\hat{g}(k)(k^4 + 2a^2k^2 + a^4) &= 1 \\ \hat{g}(k) &= \frac{1}{k^4 + 2a^2k^2 + a^4} \\ &= \frac{1}{(k^2 + a^2)^2}\end{aligned}$$

To find $G(x)$ we need to find the inverse Fourier transform.

$$G(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(k^2 + a^2)^2} e^{ikx} dk$$

With the help of computer, I obtained the following result

$$G(x) = \frac{(1 + a|x|)}{4a^3} e^{-a|x|}$$