HW 4

Math 703
methods of applied mathematics I

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Contents

1 Problem 3.1.1

3.1.1 For a bar with constant c but with decreasing $f = 1 - x$, find $w(x)$ and $u(x)$ as in equations (8–10).

Figure 1: the Problem statement

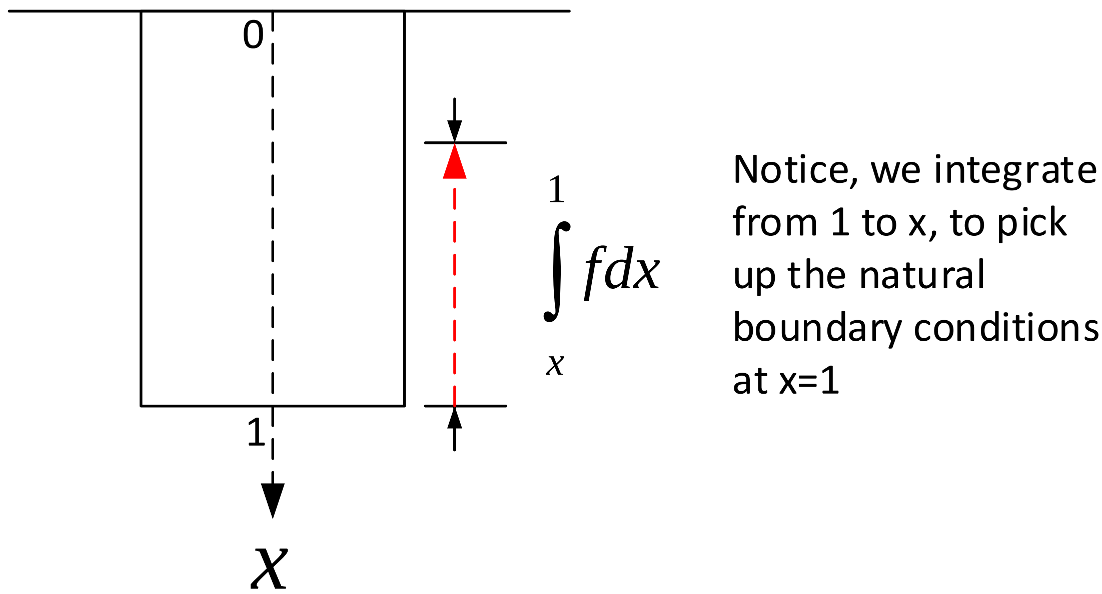

Figure 2: Figure for 3.1.1

Starting with the differential equation for u (which is the longitudinal deformation of the bar along the x axis)

$$-c\frac{d^2u}{dx^2} = f(x)$$

And using $f(x) = 1 - x$ and integrating both sides gives

$$-c\int_x^1 \frac{d^2u}{d\tau^2} d\tau = \int_x^1 (1-\tau)\, d\tau$$

$$-c\left[\frac{du}{d\tau}\right]_x^1 = \left[\tau - \frac{\tau^2}{2}\right]_x^1$$

But $\frac{du}{dx} = w$, and $w(1) = 0$, hence the above becomes

$$-c\left[e(1) - e(x)\right] = \left[\left(1 - \frac{1^2}{2}\right) - \left(x - \frac{x^2}{2}\right)\right]$$

But $ce = w$, hence the above can be written as

$$-\left[w(1) - w(x)\right] = \frac{1}{2} - x + \frac{x^2}{2}$$

But $w(1) = 0$, hence

$$w(x) = \frac{1}{2} - x + \frac{x^2}{2}$$

To find $u(x)$, we use the relation that

$$c\frac{du}{dx} = w(x)$$

This is the same as $ce = w(x)$, since strain $e = \frac{du}{dx}$. So we integrate one more time, but this time, we integrate from 0 to x instead from 1 to x. This is in order to pick up the essential boundary conditions on u at $x = 0$, since $u(1)$ is not known, it would be an error to use the first integration limits used earlier above. Hence

$$\int_0^x c\frac{du}{d\tau}d\tau = \int_0^x w(\tau)\,d\tau$$

$$c\int_0^x \frac{du}{d\tau}d\tau = \int_0^x \frac{1}{2} - \tau + \frac{\tau^2}{2}d\tau$$

$$c\left[u\right]_0^x = \left[\left(\frac{\tau}{2} - \frac{\tau^2}{2} + \frac{\tau^3}{6}\right)\right]_0^x$$

$$c\left(u(x) - u(0)\right) = \left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)$$

But $u(0) = 0$ since fixed there. This is the essential boundary conditions we are give. The above now simplifies to

$$u(x) = \frac{1}{c}\left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6}\right)$$

2 Problem 3.1.2

3.1.2 For a hanging bar with constant f but weakening elasticity $c(x) = 1 - x$, find the displacement $u(x)$. The first step $w = (1 - x)f$ is the same as in (9), but there will be stretching even at $x = 1$ where there is no force. (The condition is $w = c\, du/dx = 0$ at the free end, and $c = 0$ allows $du/dx \neq 0$.)

Figure 3: the Problem statement

Since $ce = w(x)$, then $w(x) = (1 - x)e$ and since $e = \frac{du}{dx}$ then

$$w(x) = (1 - x)\frac{du}{dx}$$

But $-\frac{dw}{dx}=f$, hence integrating both sides gives

$$-\int_x^1 \frac{dw}{d\tau} d\tau = \int_x^1 f d\tau$$
$$-[w]_x^1 = f\int_x^1 d\tau$$
$$-(w(1)-w(x)) = f(1-x)$$

But $w(1)=0$, hence

$$w(x) = f(1-x)$$

We found from above that $w(x)=(1-x)\frac{du}{dx}$, therefore

$$(1-x)\frac{du}{dx} = f(1-x)$$
$$\frac{du}{dx} = f$$

Integrating one more time to find $u(x)$

$$\int_0^x \frac{du}{d\tau} d\tau = \int_0^x f d\tau$$
$$[u]_0^x = fx$$
$$u(x)-u(0) = fx$$

But $u(0)=0$, hence

$$u(x) = fx$$

3 Problem 3.1.4

3.1.4 With the bar still free at both ends, what is the condition on the external force f in order that $-\dfrac{dw}{dx}=f(x)$, $w(0)=w(1)=0$ has a solution? (Integrate both sides of the equation from 0 to 1.) This corresponds in the discrete case to solving $A_0^T y=f$; there is no solution for most f, because the left sides of the equations add to zero.

Figure 4: the Problem statement

Since $-\frac{dw}{dx}=f$, then integrating from 0 to 1, gives

$$-\int_0^1 \frac{dw}{d\tau} d\tau = \int_0^1 f d\tau$$
$$-[w(1)-w(0)] = \int_0^1 f d\tau$$

If $w(1) = 0$ and $w(0) = 0$, then this implies

$$\int_0^1 f d\tau = 0$$

Therefore the only possibility for solution is that $\int_0^1 f d\tau = 0$. For example, a constant none zero f will not work, since this will result in $f = 0$ which is a contradiction.

4 Problem 3.1.5

3.1.5 Find the displacement for an exponential force, $-u'' = e^x$ with $u(0) = u(1) = 0$.

Note that $A + Bx$ is the general solution to $-u'' = 0$; it can be added to any particular solution for the given f, and A and B can be adjusted to fit the boundary conditions.

Figure 5: the Problem statement

The general solution is $u = u_h + u_p$. For the homogeneous solution $u_h = A + Bx$, now we find the particular solution. By inspection we see that $u_p = -e^x$ satisfies the differential equation. Hence

$$u = A + Bx - e^x$$

We now apply the boundary conditions to find A, B. At $x = 0$,

$$\begin{aligned} 0 &= A - e^0 \\ 0 &= A - 1 \\ A &= 1 \end{aligned}$$

Therefore $u = 1 + Bx - e^x$. At $u = 1$ we find

$$\begin{aligned} 0 &= 1 + B - e^1 \\ B &= e - 1 \end{aligned}$$

Hence the solution is

$$u = 1 + (e - 1)x - e^x$$

5 Problem 3.1.6

3.1.6 Suppose the force f is constant but the elastic constant c jumps from $c = 1$ for $x \leq \frac{1}{2}$ to $c = 2$ for $x > \frac{1}{2}$. Solve $-dw/dx = f$ with $w(1) = 0$ as before, and then solve $c\, du/dx = w$ with $u(0) = 0$. Even if c jumps, the combination $w = c\, du/dx$ remains smooth.

Figure 6: the Problem statement

Using $-\frac{dw}{dx}=f$, integrating both sides

$$-\int_x^1 \frac{dw}{d\tau}d\tau = \int_x^1 f d\tau$$
$$-\left[w(\tau)\right]_x^1 = (1-x) f$$
$$-(w(1)-w(x)) = (1-x) f$$
$$w(x) = (1-x) f$$

Since $w(1)=0$. Now we use $ce=w(x)$ to solve for u. Since $e=\frac{du}{dx}$. For $0\leq x\leq\frac{1}{2}$ we solve, using $c=1$

$$c\frac{du}{dx} = (1-x) f$$
$$\int_0^x \frac{du}{d\tau}d\tau = \int_0^x (1-\tau) f d\tau$$
$$\left[u(\tau)\right]_0^x = f\left[\tau-\frac{\tau^2}{2}\right]_0^x$$
$$u(x)-u(0) = f\left(x-\frac{x^2}{2}\right)$$

But $u(0)=0$, hence the solution is

$$u(x) = f\left(x-\frac{x^2}{2}\right) \qquad 0\leq x\leq\frac{1}{2} \tag{1}$$

We now integrate over the second half, where $c=2$

$$c\frac{du}{dx} = (1-x) f$$
$$\int_{\frac{1}{2}}^x 2\frac{du}{d\tau}d\tau = \int_{\frac{1}{2}}^x (1-\tau) f d\tau$$
$$2\left[u(\tau)\right]_{\frac{1}{2}}^x = f\left[\tau-\frac{\tau^2}{2}\right]_{\frac{1}{2}}^x$$
$$2\left(u(x)-u\left(\frac{1}{2}\right)\right) = f\left(\left(x-\frac{x^2}{2}\right)-\left(\frac{1}{2}-\frac{\left(\frac{1}{2}\right)^2}{2}\right)\right)$$
$$2u(x)-2u\left(\frac{1}{2}\right) = f\left(-\frac{1}{2}x^2+x-\frac{3}{8}\right) \tag{2}$$

To find $u\left(\frac{1}{2}\right)$ we use the earlier solution (1) above $u\left(\frac{1}{2}\right) = f\left(\frac{1}{2} - \frac{\left(\frac{1}{2}\right)^2}{2}\right) = \frac{3}{8}f$, hence (2) becomes

$$
\begin{aligned}
2u(x) - \frac{3}{4}f &= \left(-\frac{1}{2}x^2 + x - \frac{3}{8}\right)f \\
2u(x) &= \left(-\frac{1}{2}x^2 + x - \frac{3}{8} + \frac{3}{4}\right)f \\
u(x) &= \left(-\frac{1}{4}x^2 + \frac{1}{2}x + \frac{3}{16}\right)f
\end{aligned}
$$

To verify, let us check that $u(x) = \frac{3}{8}f$ also using the second solution above. Let $x = \frac{1}{2}$ in the above, we find

$$
\begin{aligned}
u\left(\frac{1}{2}\right) &= \left(-\frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\frac{1}{2} + \frac{3}{16}\right)f \\
&= \frac{3}{8}
\end{aligned}
$$

Therefore the solution $u(x)$ is continuous and smooth at $x = \frac{1}{2}$ where the elasticity changes. This is a plot of the solution

```
In[63]:= u[x_] := Piecewise[{{(-1/4 x^2 + 1/2 x + 3/16), 1/2 ≤ x < 1}, {x - x^2/2, 0 < x < 1/2}}]
         Plot[u[x], {x, 0, 1}, PlotTheme → "Detailed", Frame → True,
          FrameLabel → {{"u(x)", None}, {"x", "Solution for problem 3.1.6"}}]
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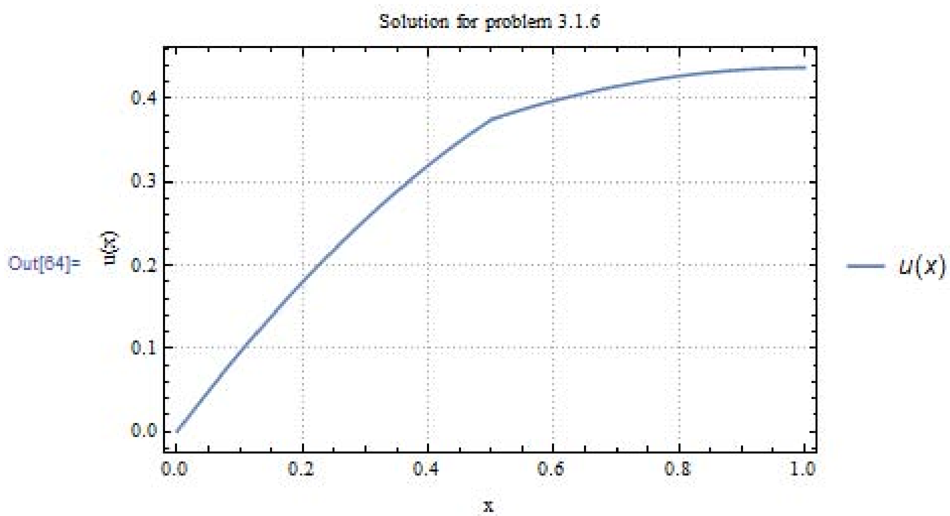

Figure 7: Figure for 3.1.6

6 Problem 3.2.2

3.2.2 What function $u(x)$ with $u(0)=0$ and $u(1)=0$ minimizes

$$P(u)=\int_0^1\left[\tfrac{1}{2}\left(\frac{du}{dx}\right)^2+x\,u(x)\right]dx?$$

Figure 8: the Problem statement

The general form of $P(u(x))$ is

$$P(u(x))=\int_0^1\left[\frac{1}{2}C\left(\frac{du(x)}{dx}\right)^2-f(x)\,u(x)\right]dx \tag{1}$$

We will use theorem proved in class that function $\bar{u}(x)$ minimizes $p(\bar{u})$ iff

$$\int_0^1 C\frac{d\bar{u}}{dx}\frac{dv}{dx}-fvdx=0$$

For any test function $v(x)$. However, this test function must satisfy the essential conditions on $u(x)$. Therefore, since we are told $u(1)=u(0)=0$, then it follows that $v(1)=v(0)=0$. Now we apply Integration by part to (1)

$$\left[C\frac{d\bar{u}}{dx}v\right]_0^1-C\int_0^1\frac{d^2\bar{u}}{dx^2}vdx-\int_0^1 fvdx=0$$

$$C\left[\left.\frac{d\bar{u}}{dx}\right|_{x=1}v(1)-\left.\frac{d\bar{u}}{dx}\right|_{x=0}v(0)\right]-C\int_0^1\frac{d^2\bar{u}}{dx^2}vdx-\int_0^1 fvdx=0$$

Since $v(1)=v(0)=0$ the above reduces to

$$-C\int_0^1\frac{d^2\bar{u}}{dx^2}vdx=\int_0^1 fvdx$$

Since $v(x)$ is arbitrary function (other than having the same essential boundary conditions as $u(x)$) then the above implies

$$-C\frac{d^2\bar{u}}{dx^2}=f \tag{2}$$

Now we can apply this result to the problem at hand, which is to find $\bar{u}$ which minimizes

$$p(u)=\int_0^1\left[\frac{1}{2}\left(\frac{du}{dx}\right)^2+xu\right]dx \tag{3}$$

By comparing (3) and (1), we see that $C=1$ and $f=-x$, hence from (2), we need to solve

$$-\frac{d^2\bar{u}}{dx^2}=-x$$

or

$$\frac{d^2\bar{u}}{dx^2} = x \tag{4}$$

With the boundary conditions $\bar{u}(0) = \bar{u}(1) = 0$. The homogeneous solution to (4) is $\bar{u}_h(x) = Ax + B$. Let the particular solution be $\bar{u}_p(x) = c_1 x^3$, then applying this to (4) gives

$$6c_1 x = x$$

Hence $c_1 = \frac{1}{6}$ and $\bar{u}_p(x) = \frac{1}{6}x^3$. Therefore the general solution is

$$\begin{aligned}\bar{u}(x) &= \bar{u}_h(x) + \bar{u}_p(x)\\ &= Ax + B + \frac{1}{6}x^3\end{aligned}$$

We now apply the essential conditions on the above. Which results in two equations to solve for A, B

$$\begin{aligned}\bar{u}(0) &= 0 = B\\ \bar{u}(1) &= 0 = A + \frac{1}{6}\end{aligned}$$

Hence $B = 0, A = -\frac{1}{6}$, and the solution is

$$\bar{u}(x) = -\frac{1}{6}x + \frac{1}{6}x^3$$

or

$$\bar{u}(x) = -\frac{x}{6}\left(1 - x^2\right)$$

7 Problem 3.2.3

3.2.3 What function $w(x)$ with $dw/dx = x$ (and unknown integration constant) minimizes

$$Q(w) = \int_0^1 \frac{w^2}{2}\,dx?$$

With no boundary condition on w this is dual to Ex. 3.2.2.

Figure 9: the Problem statement

We need to find $\bar{w}(x)$ which minimizes the functional $Q(w(x)) = \int_0^1 \frac{w^2}{2} dx$ with constraint $\frac{dw}{dx} = x$. Since we have a constraint, we need to set up a Lagrangian minimization. Hence we want to minimize

$$L(w, \lambda) = \int_0^1 \frac{w^2}{2} - \lambda\left(\frac{dw}{dx} + x\right) dx$$

Where λ is the Lagrangian. Now we follow the standard method, but work with L instead of Q.

$$L((w + v), \lambda) = L(w, \lambda) + \frac{\delta L(w, \lambda)}{\delta x} v + \cdots$$

Hence

$$
\begin{aligned}
\frac{\delta L(w,\lambda)}{\delta x}v &= L((w+v),\lambda) - L(w,\lambda)\\
&= \int_0^1 \frac{(w+v)^2}{2} - \lambda\left(\frac{d(w+v)}{dx}+x\right)dx - \int_0^1 \frac{w^2}{2} - \lambda\left(\frac{dw}{dx}+x\right)dx\\
&= \int_0^1 \frac{1}{2}\left(w^2+v^2+2vw\right) - \lambda\left(\frac{dw}{dx}+\frac{dv}{dx}+x\right) - \frac{w^2}{2} + \lambda\left(\frac{dw}{dx}+x\right)dx\\
&= \int_0^1 \frac{1}{2}\left(v^2+2vw\right) - \lambda\frac{dv}{dx}dx\\
&= \int_0^1 \frac{1}{2}v^2dx + \int_0^1\left(vw - \lambda\frac{dv}{dx}\right)dx
\end{aligned}
$$

But for small variation v the term $\int_0^1 \frac{1}{2}v^2dx$ is always positive and can be made as small as needed. Hence we ignore it, and what is left is

$$
\frac{\delta L(w,\lambda)}{\delta x}v = \int_0^1\left(vw - \lambda\frac{dv}{dx}\right)dx
$$

Since we want $\frac{\delta L(w,\lambda)}{\delta x} = 0$ for a minimum, and the above must be valid for any non trivial v then

$$
\int_0^1\left(vw - \lambda\frac{dv}{dx}\right)dx = 0
$$

Applying integration by parts to $\int_0^1 \lambda\frac{dv}{dx}dx$ where $\int udv = [uv] - \int vdu$. Let $u = \lambda, dv = \frac{dv}{dx}$, hence the above becomes

$$
\begin{aligned}
0 &= \int_0^1\left(vw - \lambda\frac{dv}{dx}\right)dx\\
&= \int_0^1 vw\ dx - \overbrace{\int_0^1 \lambda\frac{dv}{dx}dx}^{\text{by parts}}\\
&= \int_0^1 vw\ dx - \left[(\lambda v)_0^1 - \int_0^1 \frac{d\lambda}{dx}vdx\right]
\end{aligned}
$$

Assuming $v(0) = v(1) = 0$, then the above reduces to

$$
\int_0^1 vw\ + \frac{d\lambda}{dx}vdx = 0
$$

$$
\int_0^1\left(w\ + \frac{d\lambda}{dx}\right)vdx = 0
$$

Since this is valid for any v, therefore

$$w + \frac{d\lambda}{dx} = 0$$

Hence the $w(x)$ which minimizes $\int_0^1 \frac{w^2}{2} dx$ with constraint $\frac{dw}{dx} = x$ is

$$w(x) = -\frac{d\lambda}{dx}$$

8 Problem 3.2.10

3.2.10 If the ends of a beam are fixed (zero boundary conditions) and the force is $f = 1$ with $c = 1$, solve $d^4u/dx^4 = 1$ and then find M. Why does it have to be done in that order?

Figure 10: the Problem statement

For a beam, the equation of deflection is $u^{(4)} = 1$. The solution is given by integrating 4 times resulting in

$$\begin{aligned} u'''(x) &= x + c_1 \\ u'' &= \frac{x^2}{2} + c_1 x + c_2 \\ u' &= \frac{x^3}{6} + c_1\frac{x^2}{2} + c_2 x + c_3 \\ u &= \frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4 \end{aligned}$$

Since $u(0) = 0$ then $c_4 = 0$ and since $u'(0) = 0$ then $c_3 = 0$, hence

$$u(x) = \frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2}$$

Now, assuming the beam has length 1. Then on the other end, we have also $u(1) = 0$, then

$$u(1) = 0 = \frac{1}{24} + c_1\frac{1}{6} + c_2\frac{1}{2} \tag{1}$$

And since also $u'(1) = 0$, then

$$u'(1) = 0 = \frac{1}{6} + c_1\frac{1}{2} + c_2 \tag{2}$$

From (1) and (2) we can solve for c_2, c_1, giving $c_2 = \frac{1}{12}, c_1 = -\frac{1}{2}$, hence

$$u(x) = \frac{x^4}{24} - \frac{1}{12}x^3 + \frac{1}{24}x^2$$

Now we can find $M(x)$ since $M(x) = c\frac{d^2u}{dx^2}$, hence

$$M(x) = \frac{x^2}{2} - \frac{1}{2}x + \frac{1}{12}$$

If we had used $M = u''$ directly (from page 173 on text, where $c = 1$ now), then the solution would be

$$
\begin{aligned}
Mx + c_1 &= u' \\
\frac{Mx^2}{2} + c_1 x + c_2 &= u
\end{aligned}
$$

At $u(0) = 0$ then $c_2 = 0$, hence $\frac{Mx^2}{2} + c_1 x = u$ and from $u(1) = 0$ we obtain $\frac{M}{2} + c_1 = 0$ or $M = -\frac{c_1}{2}$. But we are now stuck since we can't find c_1.

So to find M, we must first find $u(x)$ and then find $M = cu''$ after solving for u completely.

9 Problem 3.2.12

3.2.12 What is the shape of a uniform beam under zero force, $f = 0$ and $c = 1$, if $u(0) = u(1) = 0$ at the ends but $du/dx(0) = 1$ and $du/dx(1) = -1$? Sketch this shape.

Figure 11: the Problem statement

For a beam, the equation of deflection is $u^{(4)} = 0$. The solution is given by integrating 4 times resulting in

$$
\begin{aligned}
u'''(x) &= c_1 \\
u'' &= c_1 x + c_2 \\
u' &= c_1 \frac{x^2}{2} + c_2 x + c_3 \\
u &= c_1 \frac{x^3}{6} + c_2 \frac{x^2}{2} + c_3 x + c_4
\end{aligned}
$$

For $u(0) = 0$ gives $c_4 = 0$ and $u'(0) = 1$ gives $c_3 = 1$ and $u(1) = 0$ gives $0 = c_1 \frac{1}{6} + c_2 \frac{1}{2} + 1$ and $u'(1) = -1$ gives $-1 = c_1 \frac{1}{2} + c_2 + 1$

Hence we need to solve these

$$
\begin{aligned}
-1 &= c_1 \frac{1}{2} + c_2 + 1 \\
0 &= c_1 \frac{1}{6} + c_2 \frac{1}{2} + 1
\end{aligned}
$$

For c_1, c_2. The solution is: $c_1 = 0, c_2 = -2$. Hence

$$
u(x) = -x^2 + x
$$

A plot is

```
Plot[x - x^2, {x, 0, 1}, Frame → True, AspectRatio → Automatic,
 FrameLabel → {{"u(x)", None}, {"x", "solution to u''''(x)=0"}}]
```

Out[54]=

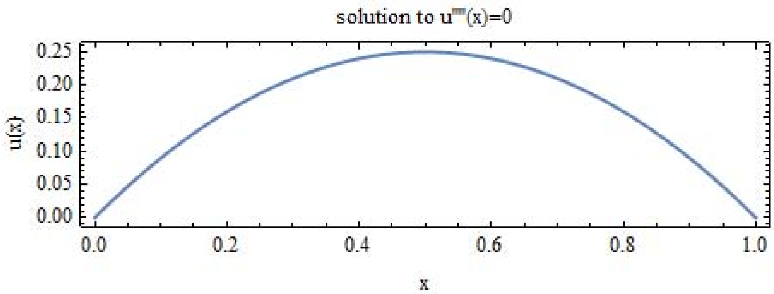

Figure 12: Plot for 3.2.12

10 Problem 3.3.3

3.3.3 *Discrete divergence theorem*: Why is the flow across the "cut" in the figure equal to the sum of the flows from the individual nodes A,B,C,D? *Note*: This is true even if flows like $d_1 - d_6$ from nodes like A are nonzero. If the current law holds and each node has zero net flow, then the exercise says that the flow across every cut is zero.

Figure 13: the Problem statement

11 Problem 3.3.4

3.3.4 *Discrete Stokes theorem*: Why is the voltage drop around the large triangle equal to the sum of the drops around the small triangles? *Note*: This is true even if voltage drops like $d_1 + d_7 + d_6$ around triangles like ABC are nonzero. If the voltage law holds and the drop around each small triangle is zero, then the exercise says that $d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0$.

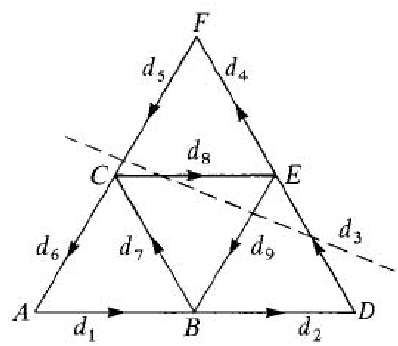

Figure 14: the Problem statement

12 Problem 3.3.5

3.3.5 On a graph the analogue of the gradient is the edge-node incidence matrix A_0. The analogue of the curl is the loop-edge matrix R with a row for each independent loop and a column for each edge. Draw a graph with four nodes and six directed edges, write down A_0 and R, and confirm that $RA_0 = 0$ in analogy with curl grad $= 0$.

Figure 15: the Problem statement