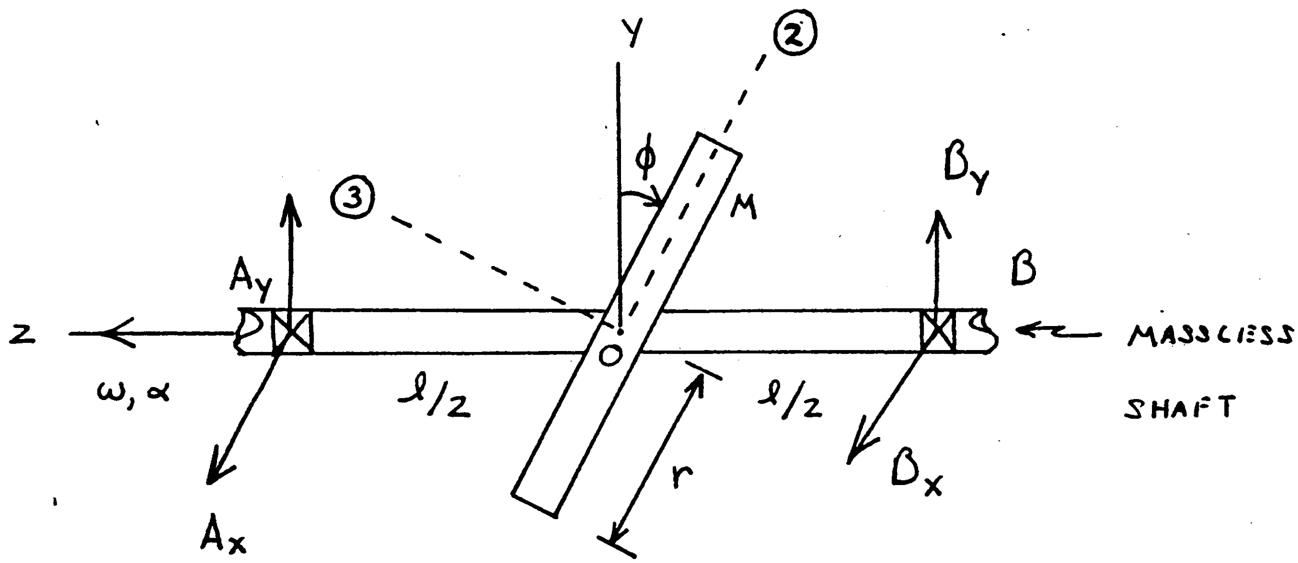
EMA 542

Home Work to be Handed In

10) A thin disk of radius r and mass m is rotating about the z axis with angular velocity ω and angular acceleration α. Use angular momentum methods and direct integration to determine the bearing loads acting on the massless shaft at points A and B.

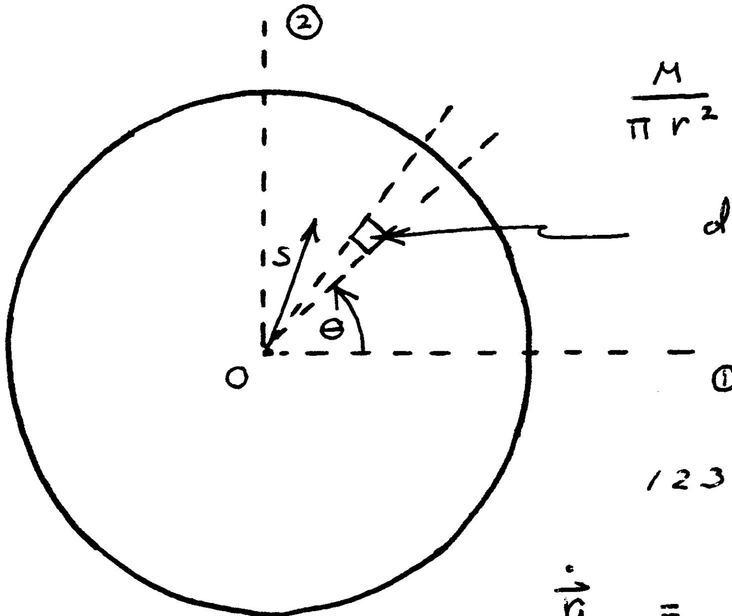


SOLUTION TO 542 HWK (10)

$$\vec{M}_p = \dot{\vec{h}}_p + \vec{\rho}_c \times M\ddot{\vec{r}}_p$$

PICK PT O AS REFERENCE PT $\Rightarrow$ $\ddot{\vec{r}}_o = 0$

$$\therefore \quad \vec{M}_p = \dot{\vec{h}}_p = \dot{\vec{H}}_p = \dot{\vec{H}}_o$$

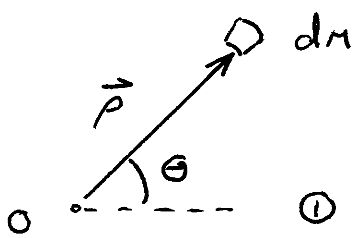


$$\frac{M}{\pi r^2} = \text{MASS / AREA}$$

$$dM = \frac{M}{\pi r^2}\, s\, d\theta\, ds$$

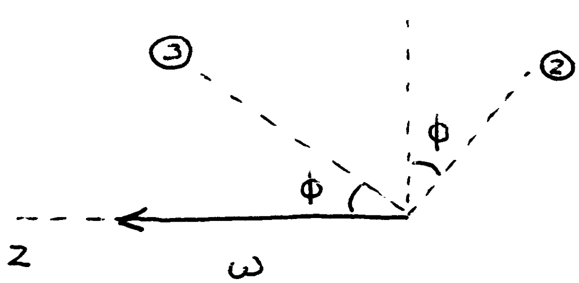
123 AXES FIXED TO DISK

$$\dot{\vec{r}}_{dM} = \dot{\vec{r}}_o + \vec{\omega}_{cs} \times \vec{\rho} + \dot{\vec{\rho}}_r$$



$$\dot{\vec{r}}_o = 0 \qquad \dot{\vec{\rho}}_r = 0$$

$$\vec{\rho} = s\cos\theta\, \vec{e}_1 + s\sin\theta\, \vec{e}_2$$



$$\vec{\omega}_{cs} = -\omega\sin\phi\, \vec{e}_2 + \omega\cos\phi\, \vec{e}_3$$

$$\therefore \quad \dot{\vec{r}}_{dm} = \vec{\omega}_{cs} \times \vec{\rho} = \left[-\omega \sin\phi \, \vec{e}_2 + \omega \cos\phi \, \vec{e}_3 \right]$$

$$\times \left[s \cos\theta \, \vec{e}_1 + s \sin\theta \, \vec{e}_2 \right]$$

$$\dot{\vec{r}}_{dm} = \omega s \sin\phi \cos\theta \, \vec{e}_3 + s \omega \cos\phi \cos\theta \, \vec{e}_2$$

$$- \omega s \cos\phi \sin\theta \, \vec{e}_1$$

ANGULAR MOMENTUM:

$$d\vec{H}_o = \vec{\rho} \times \dot{\vec{r}}_{dm} \; dm$$

$$= \left[s \cos\theta \, \vec{e}_1 + s \sin\theta \, \vec{e}_2 \right]$$

$$\times \left[\sin\phi \cos\theta \, \vec{e}_3 + \cos\phi \cos\theta \, \vec{e}_2 \right.$$

$$\left. - \cos\phi \sin\theta \, \vec{e}_1 \right] \omega s^2 \frac{M}{\pi r^2} \, d\theta \, ds$$

$$\Rightarrow \quad d\vec{H}_o = \frac{M}{\pi r^2} \omega s^2 \left[- s \sin\phi \cos^2\theta \, \vec{e}_2 + s \cos\phi \cos^2\theta \, \vec{e}_3 \right.$$

$$\left. + s \sin\phi \sin\theta \cos\theta \, \vec{e}_1 + s \cos\phi \sin^2\theta \, \vec{e}_3 \right] d\theta \, ds$$

OR

$$d\vec{H}_o = \frac{M}{\pi r^2}\,\omega\, s^3 \Big[-\sin\phi \cos^2\theta\, \vec{e}_2$$

$$+ \sin\phi \sin\theta\cos\theta\, \vec{e}_1 + \cos\phi\, \vec{e}_3 \Big]\, d\theta\, ds$$

$$\vec{H}_o = \int_0^r \int_0^{2\pi} \frac{M}{\pi r^2}\,\omega\, s^3 \Big[-\sin\phi \cos^2\theta\, \vec{e}_2$$

$$+ \sin\phi \sin\theta\cos\theta\, \vec{e}_1 + \cos\phi\, \vec{e}_3 \Big]\, d\theta\, ds$$

INTEGRATE FIRST IN θ:

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta \right]_0^{2\pi} = \pi$$

$$\int_0^{2\pi} \sin\theta\cos\theta\, d\theta = \frac{\sin^2\theta}{2} \Big|_0^{2\pi} = 0$$

$$\therefore \quad \vec{H}_o = \frac{M}{\pi r^2}\,\omega \int_0^r s^3 \Big[-\pi \sin\phi\, \vec{e}_2 + \overset{2\pi}{\cos\phi}\, \vec{e}_3 \Big]\, ds$$

$$\vec{H}_o = \frac{M}{\pi r^2}\,\omega\, \pi\, \frac{1}{4} r^4 \Big[-\sin\phi\, \vec{e}_2 + 2\cos\phi\, \vec{e}_3 \Big]$$

$$\Rightarrow \boxed{\vec{H}_o = -\frac{1}{4} M r^2 \omega \sin\phi \, \vec{e}_2 + \frac{1}{2} M r^2 \omega \cos\phi \, \vec{e}_3}$$

NOW TAKE TIME DERIVATIVE:

$$\dot{\vec{H}}_o = \dot{\vec{H}}_{oR} = \dot{\vec{H}}_{or} + \vec{\omega}_{cs} \times \vec{H}_o$$

$$\underline{\dot{\vec{H}}_{or}} = -\frac{1}{4} M r^2 \alpha \sin\phi \, \vec{e}_2 + \frac{1}{2} M r^2 \alpha \cos\phi \, \vec{e}_3$$

TIME DERIVATIVE IN $\vec{e}_1 \, \vec{e}_2 \, \vec{e}_3$

$$\vec{\omega}_{cs} \times \vec{H}_o = \left[-\omega \sin\phi \, \vec{e}_2 + \omega \cos\phi \, \vec{e}_3 \right]$$

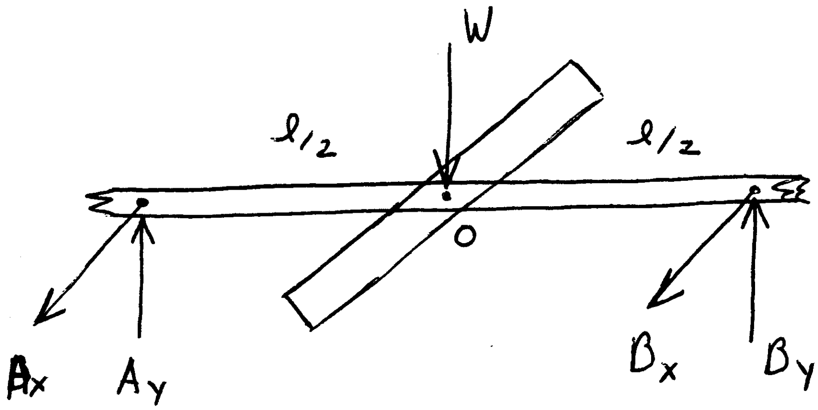
$$\times \left[-\frac{1}{4} M r^2 \omega \sin\phi \, \vec{e}_2 + \frac{1}{2} M r^2 \omega \cos\phi \, \vec{e}_3 \right]$$

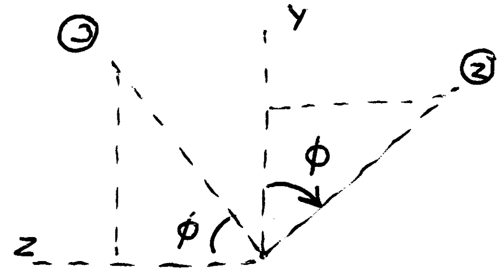
$$= -\frac{1}{2} M r^2 \omega^2 \sin\phi \cos\phi \, \vec{e}_1$$

$$+ \frac{1}{4} M r^2 \omega^2 \sin\phi \cos\phi \, \vec{e}_1$$

$$= -\frac{1}{4} M r^2 \omega^2 \sin\phi \cos\phi \, \vec{e}_1$$

$$\Rightarrow \boxed{\begin{aligned} \vec{M}_o = & -\frac{1}{4} M r^2 \omega^2 \sin\phi \cos\phi \, \vec{e}_1 - \frac{1}{4} M r^2 \alpha \sin\phi \, \vec{e}_2 \\ & + \frac{1}{2} M r^2 \alpha \cos\phi \, \vec{e}_3 \end{aligned}}$$





$$\vec{e}_1 = \bar{i} \qquad \vec{e}_2 = \cos\phi\,\bar{j} - \sin\phi\,\bar{k}$$

$$\vec{e}_3 = \sin\phi\,\bar{j} + \cos\phi\,\bar{k}$$

$$\Rightarrow \vec{M}_o = -\tfrac{1}{4} Mr^2\omega^2 \sin\phi\cos\phi\,\bar{i} - \tfrac{1}{4} Mr^2\alpha \sin\phi\cos\phi\,\bar{j}$$

$$+ \tfrac{1}{4} Mr^2\alpha \sin^2\phi\,\bar{k} + \tfrac{1}{2} Mr^2\alpha \sin\phi\cos\phi\,\bar{j}$$

$$+ \tfrac{1}{2} Mr^2\alpha \cos^2\phi\,\bar{k}$$

$$\Rightarrow \vec{M}_o = -\tfrac{1}{4} Mr^2\omega^2 \sin\phi\cos\phi\,\bar{i} + \tfrac{1}{4} Mr^2\alpha \sin\phi\cos\phi\,\bar{j}$$

$$+ Mr^2\alpha\left[\tfrac{1}{4}\sin^2\phi + \tfrac{1}{2}\cos^2\phi\right]\bar{k}$$

<u>Sum Forces:</u>

$$A_y + B_y - Mg = 0 \qquad (1)$$

$$A_x + B_x = 0 \qquad (2)$$

Sum Moments about O:

$$\Sigma M_x \Rightarrow -A_y \frac{l}{2} + B_y \frac{l}{2} = -\frac{1}{4} M r^2 \omega^2 \sin\phi \cos\phi \quad (3)$$

$$(1) \Rightarrow A_y = Mg - B_y$$

$$(3) \Rightarrow -Mg \frac{l}{2} + B_y \frac{l}{2} + B_y \frac{l}{2} = -\frac{1}{4} M r^2 \omega^2 \sin\phi \cos\phi$$

$$\Rightarrow \boxed{B_y = -\frac{1}{4} \frac{M}{l} r^2 \omega^2 \sin\phi \cos\phi + \frac{1}{2} Mg}$$

$$\Rightarrow \boxed{A_y = \frac{1}{4} \frac{M}{l} r^2 \omega^2 \sin\phi \cos\phi + \frac{1}{2} Mg}$$

$$\Sigma M_y \Rightarrow A_x \frac{l}{2} - B_x \frac{l}{2} = \frac{1}{4} M r^2 \alpha \sin\phi \cos\phi \quad (4)$$

$$(2) \Rightarrow A_x = -B_x$$

$$(4) \Rightarrow \boxed{A_x = \frac{1}{4} \frac{M}{l} r^2 \alpha \sin\phi \cos\phi}$$

$$\boxed{B_x = -\frac{1}{4} \frac{M}{l} r^2 \alpha \sin\phi \cos\phi}$$

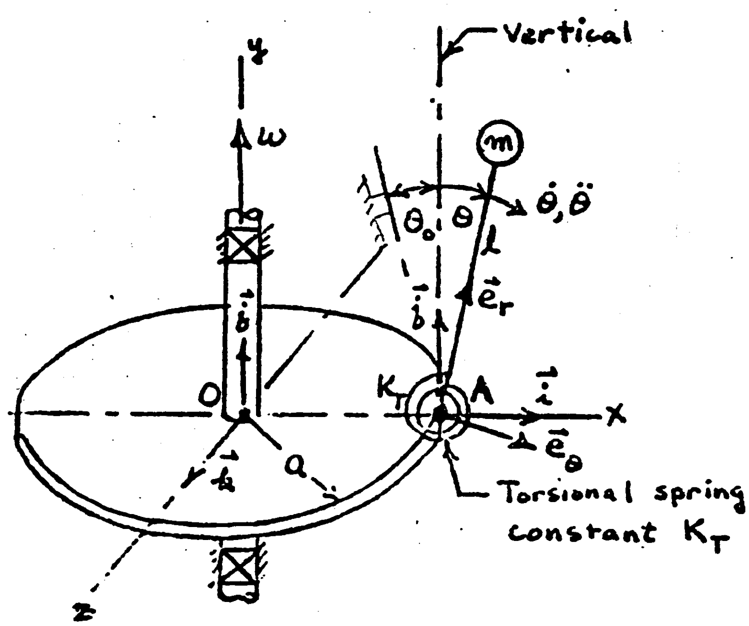
EMA 542

Home Work to be Handed In

9) The circular platform of radius a rotates about a vertical axis at a constant angular velocity ω. The axes x, y, z are body axes attached to the platform. A simple pendulum of mass m and length l is supported at A by a bearing which allows rotation about an axis at A parallel to the z body axis. The pendulum is constrained by a torsional spring at A with spring constant K_T which provides a torsional moment proportional to the angular displacement. The torsional spring is designed such that when $\dot{\theta} = \ddot{\theta} = 0$, the pendulum remains vertical for ω = constant. At position $\theta = -\theta_o$ as shown in the figure, the spring is undeformed. Consider that the pendulum is disturbed so that it vibrates about the vertical position $\theta = 0$.

a) Determine θ_o and the nonlinear equation for rotational motion of the pendulum about the bearing A using the **relative angular momentum method.**

b) For small angles, what is the natural frequency of oscillation?



SOLUTION TO 542 HWK 9

ω = CONSTANT MASS = M SPRING – K_T

@ ω $\Rightarrow$ $\theta = 0 = \dot{\theta} = \ddot{\theta}$ STEADY STATE

USING RELATIVE ANGULAR MOMENTUM:

$$\vec{M}_P = \dot{\vec{h}}_P + \vec{\rho} \times M \ddot{\vec{r}}_P \qquad A = P$$

$$\vec{\rho} = l \sin\theta \, \bar{i} + l \cos\theta \, \bar{j}$$

$$\ddot{\vec{r}}_A = -a\omega^2 \bar{i} \qquad \text{STEADY CIRCULAR MOTION}$$

$$\Rightarrow \underline{\vec{\rho} \times M \ddot{\vec{r}}_P} = \left[l \sin\theta \, \bar{i} + l \cos\theta \, \bar{j} \right] \times (-a\omega^2 \bar{i}) M$$

$$= M l a \omega^2 \cos\theta \, \bar{k} \qquad (1)$$

FORM APPARENT ANGULAR MOMENTUM:

$$\vec{h}_R = \vec{\rho} \times M \dot{\vec{\rho}}$$

$$\vec{V}_M = \vec{V}_A + \vec{\omega}_{c3} \times \vec{\rho} + \dot{\vec{\rho}}_r \qquad (2)$$

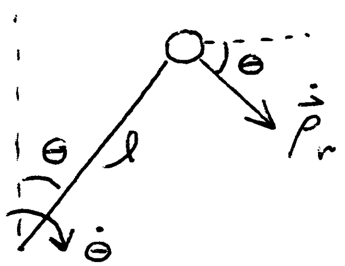
FIX XYZ TO PLATFORM AND PLACE ORIGIN AT A.

$$\dot{\vec{\rho}} = \vec{V}_M - \vec{V}_A = \vec{\omega}_{cs} \times \vec{\rho} + \dot{\vec{\rho}}_r$$ RELATIVE VELOCITY

$$\vec{\omega}_{cs} = \omega \bar{j}$$

$$\vec{\omega}_{cs} \times \vec{\rho} = \omega \bar{j} \times [l \sin\theta \, \bar{i} + l \cos\theta \, \bar{j}]$$

$$= - l \omega \sin\theta \, \bar{k}$$



$$\dot{\vec{\rho}}_r = l \dot{\theta} \cos\theta \, \bar{i} - l \dot{\theta} \sin\theta \, \bar{j}$$

$$\therefore \underline{\dot{\vec{\rho}}} = l \dot{\theta} \cos\theta \, \bar{i} - l \dot{\theta} \sin\theta \, \bar{j} - l \omega \sin\theta \, \bar{k}$$

$$\therefore \vec{h}_A = M [l \sin\theta \, \bar{i} + l \cos\theta \, \bar{j}] \times [l \dot{\theta} \cos\theta \, \bar{i}$$

$$- l \dot{\theta} \sin\theta \, \bar{j} - l \omega \sin\theta \, \bar{k}]$$

$$= M [- l^2 \dot{\theta} \sin^2\theta \, \bar{k} + l^2 \omega \sin^2\theta \, \bar{j}$$

$$- l^2 \dot{\theta} \cos^2\theta \, \bar{k} - l^2 \omega \cos\theta \sin\theta \, \bar{i}]$$

$$\Rightarrow \quad \vec{h}_A = -Ml^2\omega \sin\theta\cos\theta\,\bar{i} + l^2 M\omega \sin^2\theta\,\bar{j} - Ml^2\dot{\theta}\,\bar{k}$$

TAKE TIME DERIVATIVE:

$$\dot{\vec{h}}_{AR} = \dot{\vec{h}}_{Ar} + \vec{\omega}_{cs} \times \vec{h}_A = \dot{\vec{h}}_A \quad \text{TOTAL DERIVATIVE}$$

$$\vec{\omega}_{cs} \times \vec{h}_A = \omega\bar{j} \times \left[-Ml^2\omega\sin\theta\cos\theta\,\bar{i} + Ml^2\omega\sin^2\theta\,\bar{j} - Ml^2\dot{\theta}\,\bar{k}\right]$$

$$= Ml^2\omega^2 \sin\theta\cos\theta\,\bar{k} - Ml^2\dot{\theta}\omega\,\bar{i}$$

$\dot{\vec{h}}_{Ar}$ = TIME DERIVATIVE WITHIN XYZ

$$= \left[-Ml^2\omega\dot{\theta}\cos^2\theta + Ml^2\omega\dot{\theta}\sin^2\theta\right]\bar{i} + 2Ml^2\omega\dot{\theta}\sin\theta\cos\theta\,\bar{j} - Ml^2\ddot{\theta}\,\bar{k}$$

$$\therefore \quad \dot{\vec{h}}_A = \left[-Ml^2\omega\dot{\theta}\cos^2\theta + Ml^2\omega\dot{\theta}\sin^2\theta - Ml^2\dot{\theta}\omega\right]\bar{i} + 2Ml^2\omega\dot{\theta}\sin\theta\cos\theta\,\bar{j} + \left[-Ml^2\ddot{\theta} + Ml^2\omega^2\sin\theta\cos\theta\right]\bar{k}$$

$$\therefore \quad \vec{M}_p = \left[\sin^2\theta - \cos^2\theta - 1\right] M l^2 \omega \dot{\theta}\, \bar{i}$$

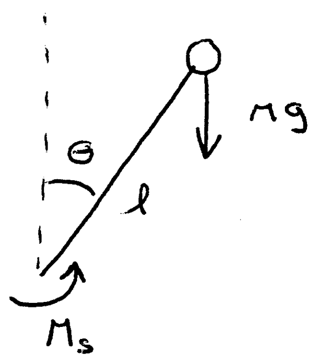
$$+ 2 M l^2 \omega \dot{\theta} \sin\theta \cos\theta\, \bar{j}$$

$$+ \left[- M l^2 \ddot{\theta} + M l^2 \omega^2 \sin\theta \cos\theta \right.$$

$$\left. + M l a \omega^2 \cos\theta \right] \bar{k}$$

LOOK AT Z COMPONENT:

$$\Rightarrow \quad M_z = -Mgl\sin\theta + K_T(\theta_0 + \theta)$$



$$\therefore -Mgl\sin\theta + K_T(\theta_0 + \theta)$$

$$= -M l^2 \ddot{\theta} + M l^2 \omega^2 \sin\theta \cos\theta$$

$$+ M l a \omega^2 \cos\theta$$

$$\Rightarrow M l^2 \ddot{\theta} + K_T(\theta_0 + \theta) - M l^2 \omega^2 \sin\theta\cos\theta$$

$$- M l a \omega^2 \cos\theta - Mgl\sin\theta = 0$$

AT STEADY STATE: $\theta = 0 \quad \ddot{\theta} = 0$

$$\Rightarrow \quad K_T \theta_o - M l a \omega^2 = 0$$

$$\Rightarrow \quad \boxed{\theta_o = \frac{M l a \omega^2}{K_T}}$$

$$\Rightarrow \quad \ddot{\theta} + \frac{K_T}{M l^2}\theta - \omega^2 \sin\theta \cos\theta + \frac{a}{l}\omega^2(1-\cos\theta) - \frac{g}{l}\sin\theta = 0$$

FOR SMALL ANGLES:

$$\ddot{\theta} + \left[\frac{K_T}{M l^2} - \omega^2 - \frac{g}{l}\right]\theta = 0$$

$$\Rightarrow \quad \boxed{\omega_N = \left[\frac{K_T}{M l^2} - \omega^2 - \frac{g}{l}\right]^{1/2}}$$