EMA 542

Home Work to be Handed In

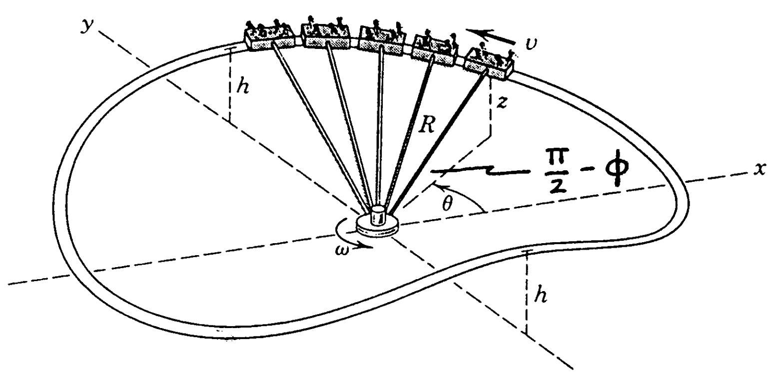
1) The cars of an amusement-park ride are attached to arms of length R which are hinged to a central rotating collar that drives the assembly about the vertical axis with a constant angular rate ω. The cars rise and fall with the track according to the relation $z = \frac{h}{2}(1 - cos 2\theta)$.

Determine for each car as it passes the position $\theta = \frac{\pi}{4}$ rads:

a) The expressions for the r-, θ-, and ϕ-components of velocity $\vec{v}$.

b) The θ-component of the acceleration $\vec{a}$.

Your answers should be in terms of h, R, and ω.



9/15/93

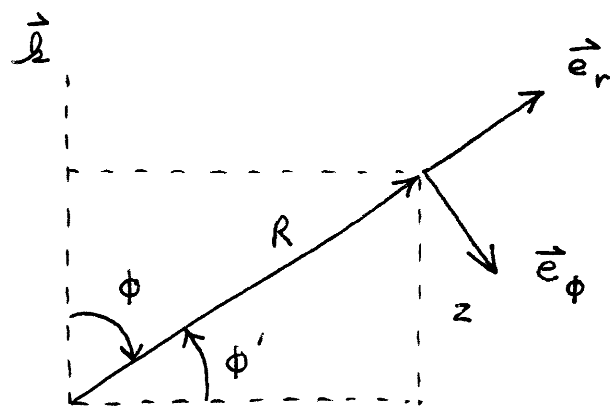
EMA 542 - Spherical Coord. Ex

2/169 $\quad \vec{r} = R\,\vec{e}_r$

$$\therefore \quad \dot{\vec{r}} = \dot{R}\,\vec{e}_r + R\,\dot{\vec{e}}_r = \dot{R}\,\vec{e} + \vec{\omega}_{\vec{r}} \times R\,\vec{e}_r$$

$$= \vec{\omega}_{\vec{r}} \times R\,\vec{e}_r \qquad \dot{R} = 0$$

$$\vec{\omega}_{\vec{r}} = \dot{\theta}\,\vec{k} + \dot{\phi}\,\vec{e}_\theta \qquad \dot{\theta} = \omega = \text{const.}$$



$$z = R\cos\phi = \frac{h}{2}(1 - \cos 2\theta)$$

TAKE TIME DERIVATIVES OF BOTH SIDES

$$\Rightarrow \quad -R\,\dot{\phi}\sin\phi = h\sin 2\theta\,\dot{\theta}$$

$$\Rightarrow \quad \dot{\phi} = -\frac{h}{R}\,\dot{\theta}\,\frac{\sin 2\theta}{\sin\phi} \qquad @\ \theta = \frac{\pi}{4} \quad \cos\phi = \frac{h}{2R}$$

$$\therefore \quad \sin\phi = \sqrt{1 - \left(\frac{h}{2R}\right)^2} \quad \Rightarrow \quad \dot{\phi} = -\frac{h}{R}\,\frac{1}{\sqrt{1 - \left(\frac{h}{2R}\right)^2}}\,\dot{\theta}$$

$$\therefore \quad \vec{\omega}_{\vec{r}} = \dot{\theta}\cos\phi\,\vec{e}_r - \dot{\theta}\sin\phi\,\vec{e}_\phi + \dot{\phi}\,\vec{e}_\theta$$

$$\therefore \quad \vec{\omega}_r \times R\vec{e}_r = \left[\dot{\theta}\cos\phi\,\vec{e}_r - \dot{\theta}\sin\phi\,\vec{e}_\phi + \dot{\phi}\,\vec{e}_\theta\right] \times R\vec{e}_r$$

$$= R\dot{\theta}\sin\phi\,\vec{e}_\theta + R\dot{\phi}\,\vec{e}_\phi$$

$$\therefore \quad \vec{V} = R\omega\sqrt{1-\left(\frac{h}{2R}\right)^2}\,\vec{e}_\theta - \frac{h\omega}{\sqrt{1-\left(\frac{h}{2R}\right)^2}}\,\vec{e}_\phi$$

$$\therefore \quad V_\theta = R\omega\sqrt{1-\left(\frac{h}{2R}\right)^2}$$

$$V_\phi = \frac{-h\omega}{\sqrt{1-\left(\frac{h}{2R}\right)^2}} \quad \text{OR} \quad V_\phi = \frac{h\omega}{\sqrt{1-\left(\frac{h}{2R}\right)^2}}$$

$$V_r = 0$$

USING ϕ IN FIGURE

ACCELERATION

$$\dot{\vec{V}} = \dot{\vec{V}}_\theta + \dot{\vec{V}}_\phi$$

$$\vec{V}_\theta = R\dot{\theta}\sin\phi\,\vec{e}_\theta = V_\theta\,\vec{e}_\theta$$

$$\vec{V}_\phi = R\dot{\phi}\,\vec{e}_\phi = V_\phi\,\vec{e}_\phi$$

$$\therefore \quad \dot{\vec{V}}_\theta = \dot{V}_\theta\,\vec{e}_\theta + \vec{\omega}_\theta \times \vec{V}_\theta$$

NOTE $\quad \vec{\omega}_\theta = \vec{\omega}_r = \vec{\omega}_\phi$

$$\therefore \quad \dot{\vec{V}}_\theta = R\dot{\theta}\dot{\phi}\cos\phi\,\vec{e}_\theta + R\dot{\theta}\sin\phi\,\big[\dot{\theta}\cos\phi\,\vec{e}_r$$

$$- \dot{\theta}\sin\phi\,\vec{e}_\phi + \dot{\phi}\,\vec{e}_\theta\,\big] \times \vec{e}_\theta$$

$$\Rightarrow \quad \dot{\vec{V}}_\theta = R\dot{\theta}\dot{\phi}\cos\phi\,\vec{e}_\theta + R\dot{\theta}\sin\phi\,\big[-\dot{\theta}\cos\phi\,\vec{e}_\phi$$

$$- \dot{\theta}\sin\phi\,\vec{e}_r\,\big]$$

$$\Rightarrow \quad \dot{\vec{V}}_\theta = -R\dot{\theta}^2\sin^2\phi\,\vec{e}_r + \dot{\phi}\cos\phi\,\vec{e}_\theta\,R\dot{\theta}$$

$$- R\dot{\theta}^2\sin\phi\cos\phi\,\vec{e}_\phi$$

$$\dot{\vec{V}}_\phi = \dot{V}_\phi \, \vec{e}_\phi + \vec{\omega}_\phi \times \vec{V}_\phi$$

$$= R\ddot{\phi}\, \vec{e}_\phi + R\dot{\phi} \left[\dot{\theta} \cos\phi \, \vec{e}_r - \dot{\theta} \sin\phi \, \vec{e}_\phi + \dot{\phi}\, \vec{e}_\theta \right] \times \vec{e}_\phi$$

$$= R\ddot{\phi}\, \vec{e}_\phi + R\dot{\phi} \left[\dot{\theta} \cos\phi \, \vec{e}_\theta - \dot{\phi}\, \vec{e}_r \right]$$

$$\therefore \; \dot{\vec{V}}_\phi = -R\dot{\phi}^2 \vec{e}_r + R\ddot{\phi}\, \vec{e}_\phi + R\dot{\phi}\dot{\theta} \cos\phi \, \vec{e}_\theta$$

FROM BEFORE:

$$\ddot{\phi} = -\frac{h}{R} \dot{\theta} \left[2\dot{\theta} \frac{\cos 2\theta}{\sin\phi} - \dot{\phi} \frac{\sin 2\theta \cos\phi}{\sin^2\phi} \right]$$

$$@ \; \theta = \frac{\pi}{4} \qquad \cos 2\theta = 0 \qquad \sin 2\theta = 1$$

$$\cos\phi = \frac{h}{2R} \qquad \sin\phi = \sqrt{1 - \left(\frac{h}{2R}\right)^2}$$

$$\therefore \; \ddot{\phi} = +\frac{h}{R}\, \omega \left(-\frac{h}{R}\, \omega \right) \frac{1}{\sqrt{1 - \left(\frac{h}{2R}\right)^2}} \; \frac{\frac{h}{2R}}{\left(1 - \left[\frac{h}{2R}\right]^2\right)}$$

$$\underline{\ddot{\phi}} = -\frac{1}{2} \left(\frac{h}{R}\right)^3 \omega^2 \left[1 - \left(\frac{h}{2R}\right)^2 \right]^{-3/2}$$

$$\therefore \quad a_r = -R\omega^2\left[1-\left(\frac{h}{2R}\right)^2\right] - R\left(\frac{h}{R}\right)^2 \omega^2 \frac{1}{\left[1-\left(\frac{h}{2R}\right)^2\right]}$$

$$\underline{a_r} = -R\omega^2\left\{1-\left(\frac{h}{2R}\right)^2 - \frac{\left(\frac{h}{R}\right)^2}{\left[1-\left(\frac{h}{2R}\right)^2\right]}\right\}$$

$$a_\theta = R\dot\theta\,\dot\phi\cos\phi + R\dot\phi\,\dot\theta\cos\phi = 2R\dot\theta\dot\phi\cos\phi$$

$$\therefore \quad a_\theta = -2R\omega\left(\frac{h}{2R}\right)\frac{h}{R}\,\omega\frac{1}{\sqrt{1-\left(\frac{h}{2R}\right)^2}}$$

$$\therefore \quad \underline{a_\theta} = -\frac{h^2\omega^2}{R\sqrt{1-\left(\frac{h}{2R}\right)^2}}$$

$$a_\phi = -R\dot\theta^2\sin\phi\cos\phi + R\ddot\phi$$

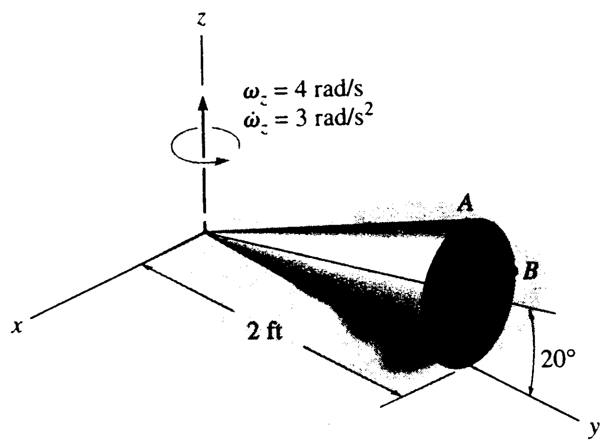
$$= -R\omega^2\frac{h}{2R}\sqrt{1-\left(\frac{h}{2R}\right)^2} - \frac{R}{2}\left(\frac{h}{R}\right)^3\omega^2\left[1-\left(\frac{h}{2R}\right)^2\right]^{-3/2}$$

$$a_\phi = \ \cdots$$

EMA 542

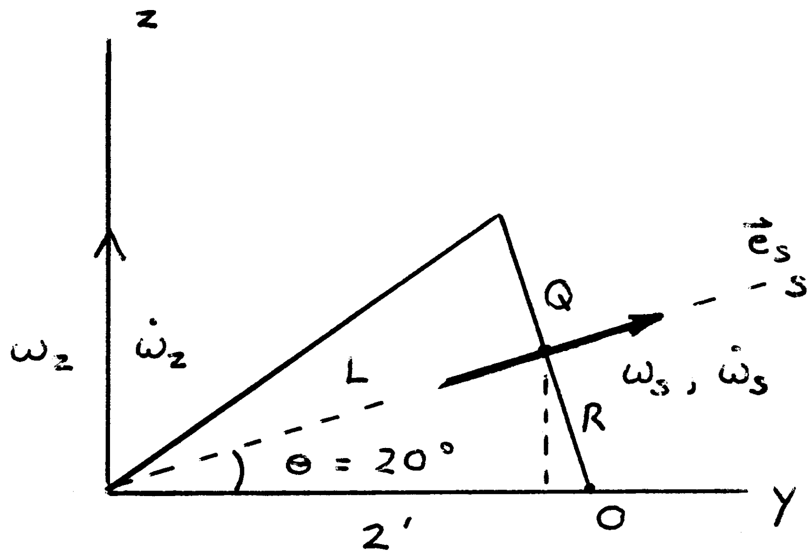
Home Work to be Handed In

1A) The cone rolls without slipping such that at the instant shown, $\omega_z = 4.0$ rad/sec. and $\dot{\omega}_z = 3.0$ rad/sec^2. Determine the total angular velocity and angular acceleration of the cone with respect to the fixed xyz coordinate system. Note that it is easiest to use velocity constraints to fulfill the no slip condition.



9/8/97

Solution to 542 Hwk 1a



No slip

$\omega_z = 4 r/s$

$\dot{\omega}_z = 3 r/s^2$

$V_o = 0$

ω_s = spin angular velocity due to no slip condition

$\vec{\Omega}$ = total angular velocity of cone

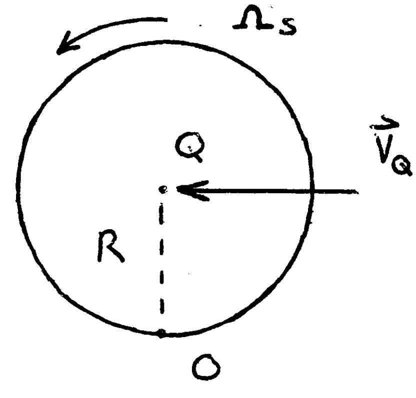
$$\vec{\Omega} = \vec{\omega}_z + \vec{\omega}_s \qquad (A)$$

$$\vec{V}_Q = -\omega_z L \cos 20\, \vec{i} \qquad L = 2\cos 20$$

$$\Rightarrow \quad \vec{V}_Q = -\omega_z L \cos 20\, \vec{i} \qquad (1)$$

Look down s axis

Assume Ω_s in positive direction

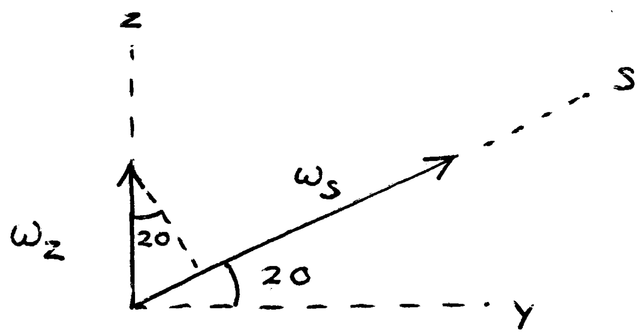


$$\vec{V}_Q = R\Omega_s\, \vec{i}$$

$$R = L \tan 20$$

$$\vec{V}_Q = L\Omega_s \tan 20\, \vec{i} \qquad (2)$$

NOTE THAT Ω_S IS THE TOTAL ANGULAR VELOCITY OF THE CONE ALONG THE S AXIS.



ALSO ASSUME ω_S IN POSITIVE DIRECTION

$$\therefore \text{(A)} \Rightarrow \Omega_S = \omega_z \sin 20 + \omega_S$$

OR

EQUATING (1) + (2):

$$\Rightarrow \quad -\omega_z L \cos 20 = L \Omega_S \tan 20$$

$$\Rightarrow \quad -\omega_z \cos 20 = (\omega_z \sin 20 + \omega_S) \tan 20$$

$$\Rightarrow \quad -\omega_z \cos^2 20 = \omega_z \sin^2 20 + \omega_S \sin 20$$

$$\text{OR} \quad -\frac{\omega_z}{\sin 20} = \omega_S$$

ASSUMED IN WRONG DIRECTION (3)

$$\therefore \quad \vec{\omega}_S = -\frac{\omega_z}{\sin 20} \vec{e}_S = \frac{-\omega_z}{\sin 20}\left[\cos 20 \bar{j} + \sin 20 \bar{k}\right]$$

$$\text{OR} \quad \vec{\omega}_S = -\omega_z \cot 20 \bar{j} - \omega_z \bar{k}$$

$$\therefore \vec{\Omega} = \vec{\omega}_z + \vec{\omega}_s = -\omega_z \cot 20 \bar{j}$$

$$\text{OR} \quad \vec{\Omega} = -10.99 \text{ r/s } \bar{j}$$

Now compute $\dot{\vec{\Omega}}$:

$$\dot{\vec{\Omega}} = \dot{\vec{\omega}}_z + \dot{\vec{\omega}}_s$$

$$\dot{\vec{\omega}}_z = \dot{\omega}_z \bar{k} + \vec{\omega}_{\omega_z} \times \vec{\omega}_z \qquad \vec{\omega}_{\omega_z} = 0$$

$$\therefore \quad \dot{\vec{\omega}}_z = 3\bar{k} \qquad (4)$$

$$\dot{\vec{\omega}}_s = \dot{\omega}_s \vec{e}_s + \vec{\omega}_{\omega_s} \times \vec{\omega}_s \qquad \vec{\omega}_{\omega_s} = \omega_z \bar{k}$$

$$= -\frac{\dot{\omega}_z}{\sin 20} \vec{e}_s + \omega_z \bar{k} \times \omega_z \left[-\cot 20 \bar{j} - \omega_z \bar{k}\right]$$

$$= -\dot{\omega}_z \cot 20 \bar{j} - \dot{\omega}_z \bar{k} + \omega_z^2 \cot 20 \bar{i}$$

$$\dot{\vec{\omega}}_s = 16 \cot 20 \bar{i} - 3 \cot 20 \bar{j} - 3\bar{k} \qquad (5)$$

$$(4) + (5) \Rightarrow \quad \dot{\vec{\Omega}} = 16 \cot 20 \bar{i} - 3 \cot 20 \bar{j}$$

$$= 43.96 \bar{i} - 8.24 \bar{j}$$

EMA 542

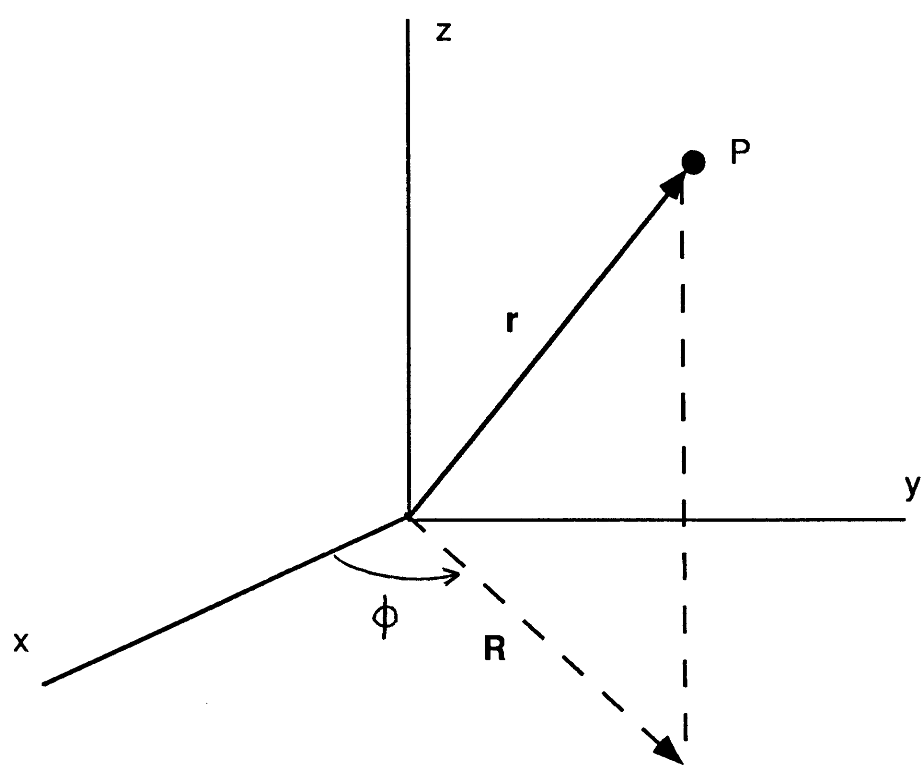
Home Work to be Handed In

2) The motion of a particle P along a fixed path is defined relative to the fixed xyz coordinate system by the parametric equations

$$R = 1.5\,m$$
$$\phi = 2t\ rad$$
$$z = t^2\ m$$

where t is in seconds. At $t = 0.25$ seconds, determine:

a) The binormal unit vector $\vec{e}_b$ in xyz coordinates.
b) The speed v and acceleration $\dot{v}$ along the path.
c) The curvature K.
d) The rate $\dot{\theta}$ at which the normal and tangent vectors rotate within the osculating plane.
e) Why is the binormal unit vector parallel to the vector $\vec{v}_p \times \vec{a}_p$?



EMA 542 - Homework to be Handed In - # 2

$R = 1.5 \qquad \phi = 2t \qquad z = t^2$

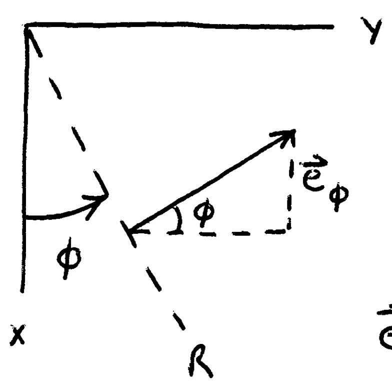
@ $t = 0.25$ sec. $\qquad R = 1.5 \quad \phi = 0.5 \quad z = .0625$

In cylindrical coordinates:

$$\vec{V} = \dot{R}\,\vec{e}_R + R\dot{\phi}\,\vec{e}_\phi + \dot{z}\,\bar{k} \qquad (1)$$

$\dot{R} = 0 \qquad \dot{\phi} = 2 \qquad \dot{z} = 2t$

$$\Rightarrow \quad \vec{V} = R\dot{\phi}\,\vec{e}_\phi + \dot{z}\,\bar{k} = 3\,\vec{e}_\phi + 2t\,\bar{k} \qquad (2)$$



$$\vec{e}_\phi = -\sin\phi\,\bar{i} + \cos\phi\,\bar{j}$$

$$\therefore \quad \vec{V} = -3\sin\phi\,\bar{i} + 3\cos\phi\,\bar{j} + 2t\,\bar{k} \qquad (3)$$

$$\vec{e}_t = \frac{\vec{V}}{|\vec{V}|}$$

$$V = \left[3^2\sin^2\phi + 3^2\cos^2\phi + 4t^2\right]^{1/2}$$

$$= \left[9 + 4t^2\right]^{1/2} \qquad (4)$$

@ $t = 1/4 \qquad V = 3.041$ m/s

$$\therefore \quad \vec{e}_t = \frac{\left[-3\sin\phi\,\bar{i} + 3\cos\phi\,\bar{j} + 2t\,\bar{k}\right]}{\left[9 + 4t^2\right]^{1/2}} \qquad (5)$$

$$@\ t = \tfrac{1}{4} \qquad \vec{e}_t = -.473\,\bar{i} + .866\,\bar{j} + .164\,\bar{k} \qquad (6)$$

$$\text{Eq } (3) \Rightarrow \quad \dot{\vec{V}} = -3\dot{\phi}\cos\phi\,\bar{i} - 3\dot{\phi}\sin\phi\,\bar{j} + 2\bar{k} = \vec{a} \qquad (7)$$

$$@\ t = \tfrac{1}{4} \qquad \vec{a} = -5.265\,\bar{i} - 2.877\,\bar{j} + 2\bar{k} \qquad (8)$$

$$|\vec{a}_t| = \vec{a}\cdot\vec{e}_t = 2.490 \quad - 2.491 \quad + .328$$

$$\Rightarrow \quad a_t = 0.328 \quad m/s^2$$

$$\therefore \quad \vec{a}_t = .328\,\vec{e}_t = -.155\,\bar{i} + .284\,\bar{j} + .054\,\bar{k} \qquad (9)$$

$$\Rightarrow \quad \vec{a}_N = \vec{a} - \vec{a}_t = (-5.265 + .155)\,\bar{i} + (-2.877 - .284)\,\bar{j}$$

$$+ (2 - .054)\,\bar{k}$$

$$\vec{a}_N = -5.110\,\bar{i} - 3.161\,\bar{j} + 1.946\,\bar{k} \qquad (10)$$

$$a_N = \left[(5.110)^2 + (3.161)^2 + (1.946)^2\right]^{1/2}$$

$$a_N = 6.316 \ m/s^2$$

$$a_N = \frac{V^2}{\rho} \Rightarrow 6.316 = \frac{(3.041)^2}{\rho}$$

$$\Rightarrow \rho = 1.464\ m \qquad K = \frac{1}{\rho} = .683\ m^{-1}$$

ALSO: $V = \rho\dot{\theta} \Rightarrow \dot{\theta} = \dfrac{3.041}{1.464}$

$$\therefore \dot{\theta} = 2.077\ r/s$$

ALSO: $\vec{e}_b = \vec{e}_t \times \vec{e}_N$

$$\vec{e}_N = \frac{\vec{a}_N}{a_N} = -.809\bar{i} - .500\bar{j} + .308\bar{k} \qquad (11)$$

$$\therefore \vec{e}_t \times \vec{e}_N = [-.473\bar{i} + .866\bar{j} + .164\bar{k}]$$

$$\times [-.809\bar{i} - .500\bar{j} + .308\bar{k}] = \vec{e}_b$$

$$\therefore \vec{e}_b = .237\bar{k} + .146\bar{j} + .701\bar{k} + .267\bar{i}$$

$$- .133\bar{j} + .082\bar{i}$$

COLLECTING SOLUTIONS:

(a) $\vec{e}_b = .349\,\bar{i} + .013\,\bar{j} + .938\,\bar{k}$

(b) $v = 3.041$ m/s $\qquad \dot{v} = a_t = 0.328$ m/s^2

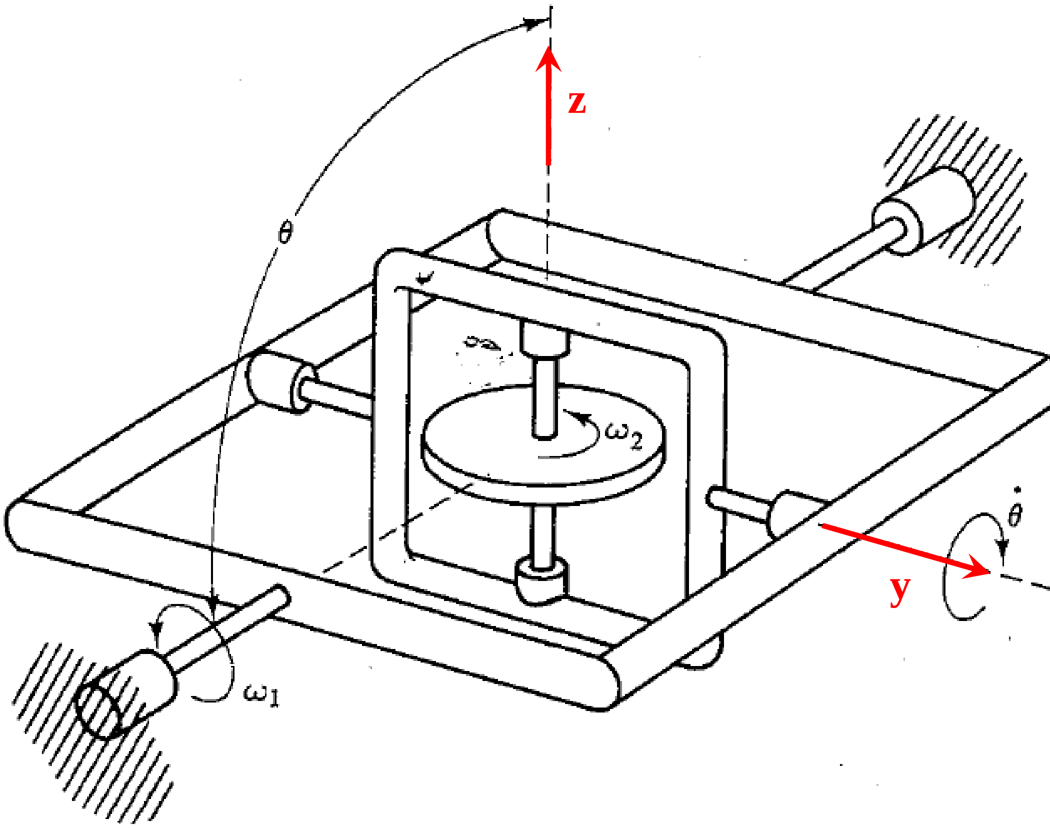
(c) $K = 0.683$ m^{-1}

(d) $\dot{\theta} = 2.077$ r/s

(e) BOTH $\vec{V}_P$ AND $\vec{a}_P$; THE VELOCITY AND ACCELERATION OF P, RESPECTIVELY, ALWAYS LIE IN OSCULATING PLANE. THEIR CROSS PRODUCT IS THEREFORE NORMAL TO THE OSCULATING PLANE AND PARALLEL TO $\vec{e}_b$

Homework #1
EMA 542, Fall 2007

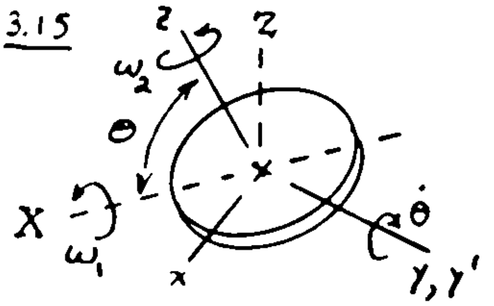
Problem #1



3.15 The flywheel of the gyroscope rotates about its own axis at $\omega_2 = 6{,}000$ rev/min. At the instant when $\theta = 120°$, the inner gimbal support is rotating relative to the outer gimbal at $\dot{\theta} = 6$ rad/s and $\ddot{\theta} = -90$ rad/s^2. The corresponding rotation of the outer gimbal about the horizontal axis is $\omega_1 = 10$ rad/s, $\dot{\omega}_1 = 100$ rad/s^2. Determine the angular velocity and angular acceleration of the flywheel at this instant.

Please give your solution in terms of components in the reference frame illustrated above that is attached to the inner gimbal. The y axis is oriented along the axis of the pin joint connecting the inner gimbal to the outer gimbal and the z axis is aligned with the axis of rotation of the flywheel.

3.15



Given: $\omega_2 = 6000\left(\frac{2\pi}{60}\right)$ rad/s, $\dot\omega_2 = 0$, when $\theta = 120°$: $\dot\theta = 6$ rad/s, $\ddot\theta = -90$ rad/s^2, $\omega_1 = 10$ rad/s, $\dot\omega_1 = 100$ rad/s^2. Find $\bar\omega$ & $\bar\alpha$ at this instant.

Solution: Fix xyz to the flywheel, and fix $x'y'z'$ to the outer gimbal. Select instantaneous orientations such that $\bar j' = \bar j$ & $\bar I \cdot \bar j = 0$. Then

$\bar\omega = \omega_1 \bar I - \dot\theta \bar j' + \omega_2 \bar k$, $\bar\omega' = \omega_1 \bar I$, $\bar\alpha = \dot\omega_1 \bar I - \ddot\theta \bar j' - \dot\theta(\bar\omega' \times \bar j') + \omega_2(\bar\omega \times \bar k)$

Set $\theta = 120° \Rightarrow \bar I = \cos 30° \bar\imath - \sin 30° \bar k$, $\bar j' = \bar j$

$\bar\omega = \omega_1(0.8660\,\bar\imath - 0.50\,\bar k) - \dot\theta \bar j + \omega_2 \bar k = 8.66\bar\imath - 6\bar j + 623.3\bar k$ rad/s ◁

$\bar\omega' = \omega_1(0.8660\,\bar\imath - 0.50\,\bar k) = 8.66\,\bar\imath - 5.0\bar k$

$\bar\alpha = \dot\omega_1(0.8660\bar\imath - 0.50\bar k) - \ddot\theta \bar j - \dot\theta(8.66\bar\imath - 5.0\bar k) \times \bar j$

$\quad + (8.66\bar\imath - 6\bar j + 623.3\bar k) \times \bar k = -3713\bar\imath - 5351\bar j - 102\bar k$ rad/s^2 ◁