HW1, ECE 3341 Stochastic processes, Northeastern Univ. Boston

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Contents

1 problem 1

to find $f_Y(y)$ given $f_X(x)$ and $Y = g(X)$, divide the $g(X)$ region into 3 parts:

part 1: $1.5 \leq x \leq 2$

part 2: $.5 \leq x < 1.5$

part 3: $0 \leq x < .5$

and then use the fundemental theorm of probabilities (page 93 of notes), which says:

$$f_Y(y) = \frac{f_X(x_1)}{|g'(X_1)|} + \frac{f_X(x_2)}{|g'(X_2)|} + \cdots + \frac{f_X(x_n)}{|g'(X_n)|}$$

where n is the number of parts over which region $g(X)$ was divided, here $n = 3$ and $f_X(x) = \frac{3x}{2} - \frac{3x^2}{4}$ over $0 \leq x \leq 2$ and 0 everywhere else.

part1:

$x_1 \equiv 1.5 \leq x \leq 2 \implies -1 \leq y \leq 0$

$$g(X_1) = 2x - 4 = y$$

so

$$x = \frac{y+4}{2} \tag{1}$$

now

$$g'(X_1) = 2$$

so

$$f_Y(y) = \frac{f(x_1)}{|g(X_1)|} = \frac{\frac{3x}{2} - \frac{3x^2}{4}}{2} = 3x - \frac{3x^2}{2} \tag{2}$$

from 0.1 and 0.2 we get

$$\begin{aligned} f_Y(y) &= 3x - \tfrac{3x^2}{2} \\ &= 3\left(\tfrac{y+4}{2}\right) - \tfrac{3}{2}\left(\tfrac{y+4}{2}\right)^2 \\ &= \tfrac{3}{2}(y+4) - \tfrac{3}{8}(y+4)^2 \\ &= \tfrac{3}{2}y + 6 - \tfrac{3}{8}\left(y^2 + 16 + 8y\right) \\ &= \tfrac{3}{2}y + 6 - \tfrac{3}{8}y^2 - 6 - 3y \\ &= -\tfrac{3}{8}y^2 - \tfrac{3}{2}y \end{aligned}$$

so over $-1 \leq y \leq 0$

$$f_Y(y) = -\frac{3}{8}y^2 - \frac{3}{2}y$$

part2:

$x_2 \equiv 0.5 \leq x < 1.5 \implies y = 0$

over this part, since $g(X_2) = 0$ then $f_Y(y)$ is an impulse

$$f_Y(y) = \frac{f(x_2)}{|g'(X_2)|} = P(.5 \le x \le 1.5)\,\delta(y)$$

but

$$P(.5 \le x \le 1.5) = F_X(1.5) - F_X(.5)$$

$$= \int_{-\infty}^{1.5} f_X(x)\ dx\ - \int_{-\infty}^{.5} f_X(x)\ dx$$

$$= \int_0^{1.5} \tfrac{3x}{2} - \tfrac{3x^2}{4}\ dx - \int_0^{.5} \tfrac{3x}{2} - \tfrac{3x^2}{4} dx$$

$$= 0.84375 - 0.15625$$

$$= 0.6875$$

so at $y = 0$

$$f_Y(y) = 0.6875\ \delta(y)$$

part3:

$x_3 \equiv 0 \le x < 0.5 \Longrightarrow 0 \le y \le 1$

$$g(X_3) = -2x + 1 = y$$

so

$$x = \frac{1 - y}{2} \tag{3}$$

now

$$g'(X_3) = -2$$

so

$$f_Y(y) = \frac{f(x_3)}{|g(X_3)|} = \frac{\frac{3x}{2} - \frac{3x^2}{4}}{2} = 3x - \frac{3x^2}{2} \tag{4}$$

from 0.3 and 0.4 we get

$$f_Y(y) = 3x - \tfrac{3x^2}{2}$$

$$= 3\left(\tfrac{1-y}{2}\right) - \tfrac{3}{2}\left(\tfrac{1-y}{2}\right)^2$$

$$= \tfrac{3}{2}(1 - y) - \tfrac{3}{8}(1 - y)^2$$

$$= \tfrac{3}{2} - \tfrac{3}{2}y - \tfrac{3}{8}\left(y^2 + 1 - 2y\right)$$

$$= \tfrac{3}{2} - \tfrac{3}{2}y - \tfrac{3}{8}y^2 - \tfrac{3}{8} + \tfrac{3}{4}y$$

$$= -\tfrac{3}{8}y^2 - \tfrac{3}{4}y + \tfrac{9}{8}$$

so over $0 \le y \le 1$

$$f_Y(y) = -\frac{3}{8}y^2 - \frac{3}{4}y + \frac{9}{8}$$