My study notes

PHYS 3041
Mathematical Methods for Physicists

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Contents

1 Using potential energy

There are two types of problems related to using potential energy. We can be given $V(x)$ but not at the equilibrium point, or given $V(x)$ at the equilibrium point. If $V(x)$ given is not at the equilibrium point, then we first need to find x_0 which is the equilibrium point. This is done by solving $V'(x) = 0$. Then expand $V(x)$ near x_0 using Taylor series and obtain new $V(x)$ which is now centered around x_0.

The other type of problem, is where we need to find $V(x)$ at equilibrium, from the physics of the problem. See MC2 as example. For the vertical pendulum problem $V(x) = \frac{1}{2}kx^2 - mgx$. This is the potential energy at equilibrium.

We need to convert the above to $V(y) = \frac{1}{2}ky^2 + V(0)$ and only now we can write

$$F = -V'(y) = -m\omega^2 y$$

From the above, ω can be found.

$$\begin{aligned} ky &= m\omega^2 y \\ \omega^2 &= \frac{k}{m} \end{aligned}$$

Remember, we can only use $F = -V'(y) = -m\omega^2 y$ when $V(y)$ has form $\frac{1}{2}ky^2 + V(0)$. Do not use $\frac{1}{2}kx^2 - mgx$. There should not be linear term in $V(x)$.

$V(y)$ should always be 0 at equilibrium. And $V\left(y\right) = \frac{1}{2}m\omega^2 y^2$ so $V'(y) = m\omega^2 y$

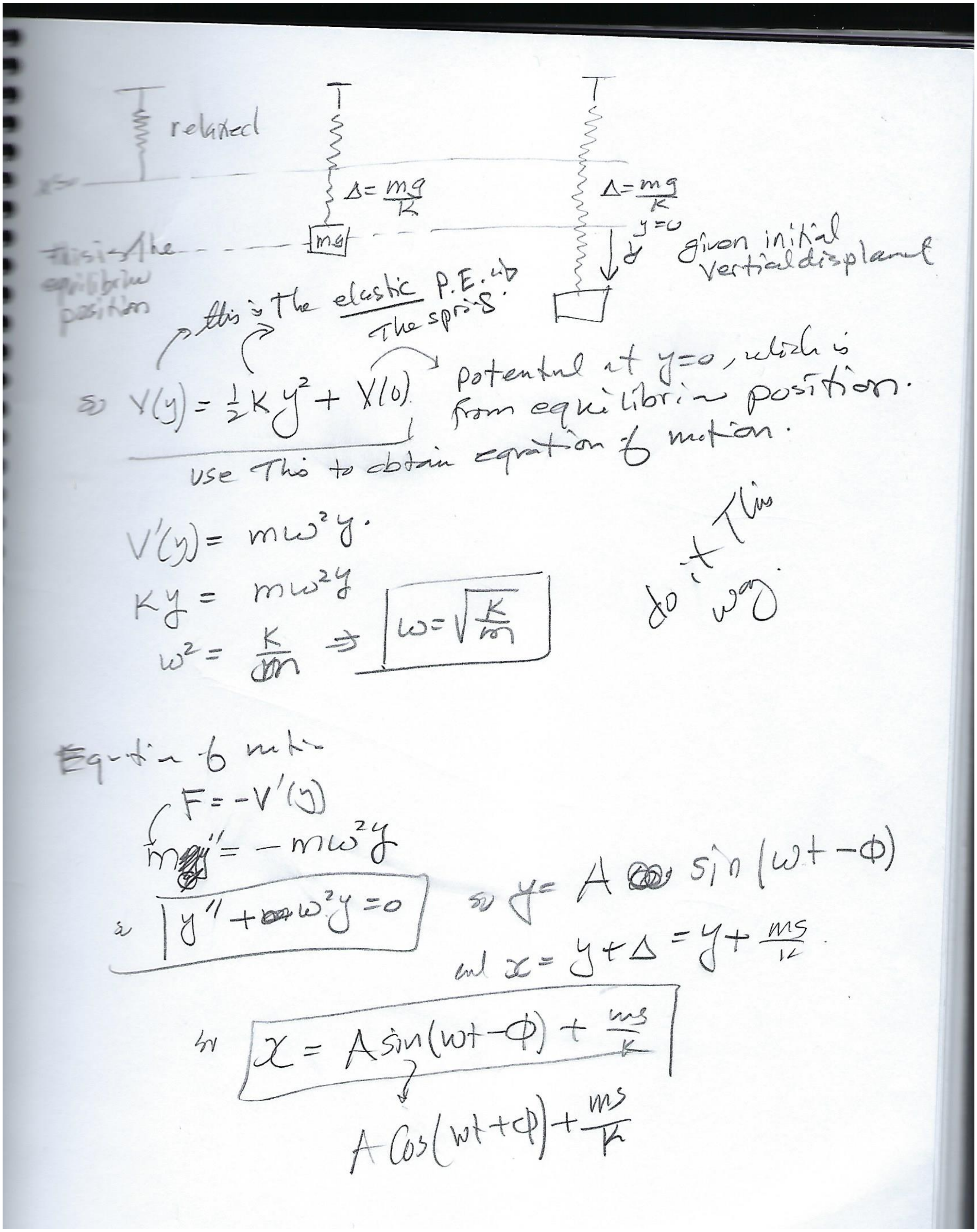


Figure 1: How to do the Vibration problems

2 Sterling approximation

$$
\begin{aligned}
\int_0^\infty t^n e^{-t} dt &= n! \\
\int_0^\infty t^n e^{-t} dt &= \int_0^\infty e^{n \ln t} e^{-t} dt \\
&= \int_0^\infty e^{(n \ln(t) - t)} dt \\
&= \int_0^\infty e^{f(t)} dt \qquad (1)
\end{aligned}
$$

Where $f(t) = n\ln(t) - t$. Contribution to integral comes mostly from where $f(t)$ is maximum.

$$
\begin{aligned}
f'(t) &= 0 \\
\frac{n}{t} - 1 &= 0 \\
t_{\max} &= n
\end{aligned}
$$

Approximating $f(t)$ around t_0

$$
f(t) = f(t_{\max}) + (t - t_{\max}) f'(t_{\max}) + \frac{1}{2}(t - t_{\max})^2 f''(t_{\max}) + \cdots
$$

But $f'(t_{\max}) = 0$ and $f''(t) = -\frac{n}{t^2}$. Hence the above becomes

$$
f(t) = f(t_{\max}) - \frac{1}{2}(t - t_{\max})^2 \frac{n}{t_{\max}^2} + \cdots
$$

Replacing $t_{\max} = n$ in the above gives

$$
\begin{aligned}
f(t) &= (n \ln(n) - n) - \frac{1}{2}(t - n)^2 \frac{n}{n^2} + \cdots \\
&= (n \ln(n) - n) - \frac{1}{2}(t - n)^2 \frac{1}{n} + \cdots \qquad (2)
\end{aligned}
$$

Substituting (2) into (1) gives

$$
\begin{aligned}
n! &\approx \int_0^\infty e^{(n\ln(n) - n) - \frac{1}{2}(t-n)^2 \frac{1}{n}} dt \\
&\approx e^{(n \ln(n) - n)} \int_0^\infty e^{-\frac{1}{2}(t-n)^2 \frac{1}{n}} dt \\
&\approx n^n e^{-n} \int_0^\infty e^{-\frac{1}{2}(t-n)^2 \frac{1}{n}} dt
\end{aligned}
$$

Let $u = \frac{t-n}{\sqrt{2n}}$. When $t = 0, u = -\frac{n}{\sqrt{2n}}$ and when $t = \infty, u = \infty$. And $du = \frac{1}{\sqrt{2n}} dt$. The above now becomes

$$
n! \approx n^n e^{-n} \int_{-\frac{n}{\sqrt{2n}}}^\infty e^{-u^2} \sqrt{2n}\, du
$$

When $n \gg 1$, the lower limit of the integral $\to -\infty$. Hence

$$\begin{aligned} n! &\approx n^n e^{-n} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2n}\, du \\ &\approx \sqrt{2n}\, n^n e^{-n} \sqrt{\pi} \\ &\approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \end{aligned}$$

3 Taylor series, convergence

Used to approximate function $f(x)$ at some x knowing its values and all its derivatives at some point x_0, called the expansion point.

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \cdots$$

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \end{aligned}$$

To find series for $\ln(1 + x)$, do this

$$\begin{aligned} \int \frac{1}{1+x} dx &= \ln(1+x) + C \\ \int (1 - x + x^2 - x^3 + \cdots) dx &= \ln(1+x) + C \\ x - \frac{x^2}{2} + \frac{x^3}{3!} - \cdots &= \ln(1+x) + C \qquad |x| < 1 \end{aligned}$$

To find C, let $x = 0$. Hence $0 = \ln(1) + C$. So $C = -\ln(1)$. Therefore

$$\ln(1+x) = \ln(1) + x - \frac{x^2}{2} + \frac{x^3}{3!} - \cdots \qquad |x| < 1$$

And

$$\begin{aligned} \int \frac{1}{1-x} dx &= -\ln(1-x) + C \\ -\int (1 + x + x^2 + x^3 + \cdots) dx &= \ln(1-x) + C \\ -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) &= \ln(1-x) + C \\ -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots &= \ln(1-x) + C \qquad |x| < 1 \end{aligned}$$

To find C, let $x = 0$. Hence $0 = \ln(1) + C$. So $C = -\ln(1)$. Therefore

$$\ln(1-x) = \ln(1) - x - \frac{x^2}{2} - \frac{x^3}{3!} + \cdots$$

And $\ln(1+2x)$ series is found as follows

$$\int \frac{1}{1+2x} dx = \frac{1}{2}\ln(1+2x) + C$$

$$\int (1 - 2x + (2x)^2 - (2x)^3 + \cdots) dx = \frac{1}{2}\ln(1+2x) + C$$

$$\left(x - \frac{2x^2}{2} + \frac{4x^3}{3} - \frac{8x^4}{4}\cdots\right) = \frac{1}{2}\ln(1+2x) + C \qquad |x| < 1$$

To find C, let $x = 0$. Hence $0 = \ln(1) + C$. So $C = -\ln(1)$. Therefore

$$\ln(1+2x) = 2\ln(1) + 2\left(x - \frac{2x^2}{2} + \frac{4x^3}{3} - \frac{8x^4}{4}\cdots\right)$$

And

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots$$

Some others

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad |x| < 1$$

$$(1+x)^a = \sum \binom{a}{n} x^n$$

Where $\binom{a}{n}$ is binomial coefficient $\binom{a}{n} = \frac{a!}{n!(a-n)!}$. General Binomial

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

This works for positive and negative n, rational or not. The sum converges only for $|x| < 1$. So, for $n = -1$ the above becomes

$$\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + \cdots$$

And

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

And

$$(1+x)^p = 1 + px + p(p-1)x^2 \cdots$$

For small x the above approximates to

$$(1+x)^p = 1 + px$$

3.1 Convergence

First test, check if $\lim_{n\to\infty} a_n$ goes to zero. If not, then no need to do anything. Series does not converge. Then use ratio test. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$

Then converges. if result is > 1 then diverges. If result is one, then more testing is needed. If converges, then radius of convergence R is

$$\begin{aligned}R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\ |x| &< R\end{aligned}$$

3.2 Closed sums

$$\begin{aligned}\sum_{n=1}^{N} n &= \frac{1}{2}N(N+1) \\ \sum_{n=1}^{N} a_n &= N\left(\frac{a_1 + a_N}{2}\right)\end{aligned}$$

i.e. the sum is N times the arithmetic mean.

Geometric series.

$$\begin{aligned}S &= a + ar + ar^2 + ar^3 + \cdots \\ &= \sum_{k=0}^{N} ar^k \\ &= a\left(\frac{1 - r^{N+1}}{1 - r}\right)\end{aligned}$$

For $|r| < 1$

$$S = \frac{a}{1-r}$$

4 Derivatives of inverse trig functions

To find $y = \arcsin(x)$, always write as $x = \sin(y)$. Then $\frac{dx}{dy} = \cos(y) = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$. Then $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, Hence

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1 - x^2}}$$

To find $\underline{y = \arccos(x)}$, write as $x = \cos\left(y\right)$. Then $\frac{dx}{dy} = -\sin(y) = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$. Then $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$, Hence

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$$

To find $\underline{y = \arctan(x)}$, write as $x = \tan\left(y\right)$. Then $\frac{dx}{dy} = \frac{1}{\cos^2 y}$, now need to use trick that $\cos^2 y + \sin^2 y = 1$ and divide both sides by $\cos^2 y$, hence $1 + \tan^2 y = \frac{1}{\cos^2 y}$. Then $\frac{dx}{dy} = 1 + \tan^2 y$. Hence $\frac{dy}{dx} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$. Therefore

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$$

5 Slit interference formulas

k is wave number.

$$k = \frac{2\pi}{\lambda}$$

6 Identities

6.0.1 trig and Hyper trig identities

$$\begin{aligned}\cos(i\theta) &= \cosh(\theta)\\ \sin(i\theta) &= i\sinh(\theta)\end{aligned}$$

$$\begin{aligned}\cos^2(\theta) + \sin^2(\theta) &= 1\\ \tan^2(\theta) &= \frac{1}{\cos^2(\theta)} - 1\\ &= \sec^2(\theta) - 1\\ \frac{\cos^2(\theta)}{\sin^2(\theta)} + 1 &= \frac{1}{\sin^2(\theta)}\\ \frac{1}{\tan^2(\theta)} &= \frac{1}{\sin^2(\theta)} - 1\\ \cot^2(\theta) &= \csc^2(\theta) - 1\\ \cosh^2(\theta) - \sinh^2(\theta) &= 1\end{aligned}$$

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ &= 2\cos^2(\theta) - 1\\ &= 1 - 2\sin^2(\theta)\\ \tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}\\ \sinh(2\theta) &= 2\sinh(\theta)\cosh(\theta)\\ \cosh(2\theta) &= 2\cosh^2(\theta) - 1\\ \tanh(2\theta) &= \frac{2\tanh(\theta)}{1+\tanh^2(\theta)}\end{aligned}$$

$$\begin{aligned}\sin(\theta) &= \cos\left(\frac{\pi}{2} - \theta\right)\\ \cos(\theta) &= \sin\left(\frac{\pi}{2} - \theta\right)\end{aligned}$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A + \tan B}{1+\tan A\tan B}$$

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$$
$$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$$
$$\tan^2(\theta) = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$
$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$\cos A\cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$
$$\sin A\cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$
$$\cos A\sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))$$

$$\begin{aligned} a\cos(\omega t) + b\sin(\omega t) &= A\sin\left(\omega t + \phi\right) \\ &= A\cos\left(\omega t - \phi\right) \\ A &= \sqrt{a^2+b^2} \\ \phi &= \arctan\left(\frac{B}{A}\right) \end{aligned}$$
$$\cos x + \sin x = \sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right)$$
$$\cos x + \sin x = \sqrt{2}\,\cos\left(x - \frac{\pi}{4}\right)$$

Laws of sines (a, b, c) are lengths of triangle sides and A, B, C are facing angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

laws of cosine

$$a^2 = b^2 + c^2 - 2bc\cos A$$

6.0.2 GAMMA function

$$\begin{aligned} \Gamma(n) &= (n-1)! \\ \Gamma(n+1) &= n(n-1)! \\ &= n\Gamma(n) \end{aligned}$$

6.0.3 Sterling

For $n \gg 1$

$$\Gamma(n+1) = n! = \sqrt{2\pi}\,n^{n+\frac{1}{2}}e^{-n}$$

7 Integrals

7.1 Integrals from 0 to infinity

$$\int_0^\infty x^n e^{-x} dx = n!$$
$$\int_0^\infty x^n e^{-ax} dx = n! \frac{1}{a^{n+1}} \qquad \text{use } y = ax$$
$$\int_0^\infty x^3 e^{-x} dx = 3!$$
$$\int_0^\infty \frac{x^3}{e^x - 1} dx = 3! \xi(4)$$

Start by multiplying numerator and denominator by e^{-x} using $\frac{1}{1-y} = 1+y+y^2+\cdots$ which becomes $\int_0^\infty x^3 \sum_{n=1}^\infty e^{-nx} dx$ or $\sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx} dx$, then use $z = nx$, this gives $\sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty z^3 e^{-z} dx$ or $(3!) \sum_{n=1}^\infty \frac{1}{n^4}$ or $3!\xi(4)$

$$\int_0^\infty e^{-x^4} dx = \frac{1}{4} \Gamma\left(\frac{1}{4}\right)$$

Start by using $x^4 = y$ or $x = y^{\frac{1}{4}}$. then $\frac{dy}{dx} = \frac{1}{4} y^{\left(\frac{1}{4}-1\right)}$, now the integral becomes $\frac{1}{4} \int_0^\infty y^{\left(\frac{1}{4}-1\right)} e^{-y} dy$ and compare this to $\int_0^\infty y^{(s-1)} e^{-x} dx = \Gamma(s)$

$$\int_0^\infty e^{-\sqrt{x}} dx = \int_0^\infty e^{-x^{\frac{1}{2}}} dx$$

Use same method as above. Will get $2\Gamma(2) = 2$

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{x^{(s-1)}}{e^x - 1} dx \qquad s > 1$$
$$\zeta(n+1)(n!) = \int_0^\infty \frac{x^n}{e^x - 1} dx \qquad n > 0$$
$$\zeta(2) = \frac{\pi^2}{6}$$
$$\zeta(4) = \frac{\pi^4}{90}$$
$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} \qquad s > 1$$
$$\zeta(4) = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

So given $\int_0^\infty \frac{x^3}{e^x-1}dx$, write as $\int_0^\infty \frac{x^{(4-1)}}{e^x-1}dx = \zeta(4)\Gamma(4)$ or $(3!)\zeta(4)$

$$\int_0^\infty x^n e^{-x} dx = n!$$
$$\int_0^\infty x^{1-n} e^{-x} dx = \Gamma(n) = (n-1)!$$

$$I = \int \frac{dx}{\sqrt{a^2 - x^2}} \qquad \text{use } x = a \sin\theta$$
$$I = \int \frac{dx}{x^2 + a^2} \qquad \text{use } x = a \tan\theta$$
$$I = \int_0^\infty x e^{-ax^2} dx \qquad \text{use } u = x^2$$
$$I = \int_0^\infty e^{-ax^2} dx \qquad \text{use } I = \frac{1}{2}\int_{-\infty}^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

For $I = \int_0^\infty x^n e^{-ax^2} dx$ or $I = \int_{-\infty}^\infty x^n e^{-ax^2} dx$. If n is even, use the trick of $I(a) = \int_{-\infty}^\infty e^{-ax^2} dx$ and repeated $I'(a)$. if n is odd, use $I(a) = \int_{-\infty}^\infty x e^{-ax^2} dx = \frac{1}{2a}$ (integration by parts) and then repeated $I'(a)$.

GAMMA:

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$
$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{\frac{-1}{2}} e^{-x} dx$$

use $u = x^{\frac{1}{2}}$, then $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ and the integral becomes $\int_0^\infty x^{\frac{-1}{2}} e^{-x} dx = \int_0^\infty \frac{1}{u} e^{-u^2}(2u du) = 2\int_0^\infty e^{-u^2} du = \sqrt{\pi}$

$$I = \int_0^\infty x e^{-ax} \sin kx \, dx$$
$$I = \int_0^\infty x e^{-ax} \cos kx \, dx$$

For these, we will be given $I = \int_0^\infty e^{-ax} \sin kx \, dx$ and then use $I(a) = \int_0^\infty e^{-ax} \sin kx \, dx$ and then do the $I'(a)$ method.

7.2 Integrals from -infinity to infinity

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$
$$\int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}} \qquad a > 0$$
$$\int_{-\infty}^{\infty} x^n e^{-ax^2} dx = I \qquad \text{for } n \text{ even, use the } I'(a) \text{ method}$$

8 Lorentz transformation

Lorentz transformation is given by

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

Where θ is called the rapidity. Also

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$
$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

And

$$v = c \tanh\theta$$

9 Rotation matrices and coordinates transformations

Rotation matrix 2D

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation matrix 3D

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$
$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$
$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is how to find the above. First row, is the projection of x', y', z' on x. Second row is projection of x', y', z' on y and so on.

Spherical coordinates

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

10 Matrices and linear algebra

Commutator is defined as
$$[M, N] = MN - NM$$
Where N, M are matrices.

Anti-commutator is when
$$[M, N]_+ = MN + NM$$
Two matrices commute means $MN - NM = 0$. Matrices that commute share an eigenbasis.

Properties of commutators

$$\begin{aligned} [A+B, C] &= [A, C] + [B, C] \\ [A, B+C] &= [A, B] + [B, C] \\ [A, A] &= 0 \\ \left[A^2, B\right] &= A[A, B] + [A, B]A \\ [AB, C] &= A[B, C] + [A, C]B \\ [A, BC] &= [A, B]C + B[A, C] \end{aligned}$$

Matrices are generally noncommutative. i.e.

$$MN \neq NM$$

Matrix Inverse

$$A^{-1} = \frac{1}{|A|} A_c^T$$

Where A_c is the cofactor matrix.

Matrix inverse satisfies

$$A^{-1}A = I = AA^{-1}$$

Matrix adjoint is same as Transpose for real matrix. If Matrix is complex, then Matrix adjoint does conjugate in addition to transposing. This is also called dagger.

$$A_{ij}^{\dagger} = A_{ji}^{*}$$

So dagger is just transpose but for complex, we also do conjugate after transposing. That is all.

If $A_{ij} = A_{ji}$ then matrix is symmetric. If $A_{ij} = -A_{ji}$ then antisymmetric.

Hermitian matrix is one which $A^{\dagger} = A$. If $A^{\dagger} = -A$ then it is antiHermitian.

Any real symmetric matrix is always Hermitian. But for complex matrix, non-symmetric can still be Hermitian. An example is $\begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$.

Unitary matrix Is one whose dagger is same as its inverse. i.e.

$$A^{\dagger} = A^{-1}$$
$$A^{\dagger}A = I$$

Remember, dagger is just transpose followed by conjugate if complex. Example of unitary matrix is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. Determinant of a unitrary matrix must be complex number whose magnitude is 1.

Also $|Av| = |v|$ if A is unitary. This means A maps vector of some norm, to vector which must have same length as the original vector.

A unitary operator looks the same in any basis.

Orthogonal matrix One which satisfies

$$AA^T = I$$
$$A^TA = I$$
$$A^{-1} = A^T$$

<u>commute</u> means $[MN] = MN - NM$. Also $[MN]_+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Another property is that $\det(\alpha_i) = -1$. Since they are Hermitian and unitary, then $\alpha_i^{-1} = \alpha_i$.

<u>If H is Hermitian, then $U = e^{iH}$ is unitary.</u>

When moving a number out of a BRA, make sure to complex conjugate it. For example $\langle 3v_1|v_2\rangle = 3^*\langle v_1|v_2\rangle$. But for the ket, no need to. For example $\langle v_1|3v_2\rangle = 3\langle v_1|v_2\rangle$

<u>item</u> $\langle f|\Omega|g\rangle^* = \langle \left(\Omega|g\right)^*|f\rangle = \langle g|\Omega^\dagger|f\rangle$

<u>item</u> when moving operator from ket to bra, remember to dagger it. $\langle u|Tv\rangle = \langle T^\dagger u|v\rangle$

<u>item</u> if given set of vectors and asked to show L.I., then set up $Ax = 0$ system, and check $|A|$. If determinant is zero, then there exist non-trivial solution, which means Linearly dependent. Otherwise, L.I.

<u>item</u> if given A, then to represent it in say basis e_i, we say $A_{ki}^{(e)} = \langle e_k, Ae_i\rangle = \langle e_k|A|e_i\rangle$. i.e $A_{1,1} = \langle e_1, Ae_1\rangle$ and $A_{1,2} = \langle e_1, Ae_2\rangle$ and so on.

11 Gram-Schmidt

Let the input $V_1, V_2, \cdots, V_n$ be a set of n linearly independent vectors. We want to use Grame-Schmidt to obtain set of n orthonormal vectors, called $v_1, v_2, \cdots, v_n$. The notation $\langle V_1, V_2\rangle$ is used to mean the inner product between any two vectors. The first vector v_1 is easy to find777

$$v_1 = \frac{V_1}{\sqrt{\langle V_1, V_1\rangle}} \tag{1}$$

The second

$$v_2' = V_2 - v_1\langle v_1, V_2\rangle$$

Where v_2' means v_2 but not yet normalized. Before we normalize v_2', we need to show that $\langle v_1, v_2'\rangle = 0$. But

$$\langle v_1, v_2'\rangle = \langle v_1, (V_2 - v_1\langle v_1, V_2\rangle)\rangle$$

Expanding the above gives

$$\langle v_1, v_2'\rangle = \langle v_1, V_2\rangle - \langle v_1, v_1\langle v_1, V_2\rangle\rangle$$

But $\langle v_1, V_2\rangle$ above is just a number. We can take it out of the second inner product term above. The above becomes

$$\langle v_1, v_2'\rangle = \langle v_1, V_2\rangle - \langle v_1, V_2\rangle\langle v_1, v_1\rangle$$

But $\langle v_1, v_1\rangle = 1$, since v_1 is normalized vector. The above becomes

$$\begin{aligned}\langle v_1, v_2'\rangle &= \langle v_1, V_2\rangle - \langle v_1, V_2\rangle \\ &= 0\end{aligned}$$

Now we normalized v_2'

$$v_2 = \frac{v_2'}{\sqrt{\langle v_2', v_2'\rangle}}$$

Now we find v_3

$$\begin{aligned}v_3' &= V_3 - (v_1\langle v_1, V_3\rangle + v_2\langle v_2, V_3\rangle) \\ v_3 &= \frac{v_3'}{\sqrt{\langle v_3', v_3'\rangle}}\end{aligned}$$

And so on.

12 Modal analysis

given $|\ddot{x}(t)\rangle + M|x(t)\rangle = 0$, find the eigenvectors and eigenvalues of M. Then $\Phi = [V_2, V_2]$ is 2×2 matrix, transformation matrix. where each column is the eigenvector of M. Then $|X(t)\rangle = \Phi^T|x(t)\rangle$ and $|x(t)\rangle = \Phi\,|X(t)\rangle$. The new system becomes $|\ddot{X}(t)\rangle + \Omega|X(t)\rangle = 0$ where Ω is now diagonal matrix with eigenvalues of M on the diagonal. Solve using this. First transform initial conditions to $X(t)$. Then trandform solution back to $|x(t)\rangle$ using $|x(t)\rangle = \Phi$ $|X(t)\rangle$.

13 Complex Fourier series and Fourier transform

Given $f(x)$ which is periodic on $0 < x < L$, so period is L, then Fourier series is

$$f(x) \sim \frac{1}{\sqrt{L}}\sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{L}x}$$

Where

$$\begin{aligned}c_n &= \langle n|f\rangle \\ &= \frac{1}{\sqrt{L}}\int_0^L f(x)e^{-in\frac{2\pi}{L}x}dx\end{aligned}$$

The basis are $|n\rangle = \frac{1}{\sqrt{L}}e^{-in\frac{2\pi}{L}x}$ and L is the period.

Fourier transform for non periodic $f(x)$ is (sum above becomes integral)

$$\begin{aligned}f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} c_k e^{ikx}dk \\ c_k &= \int_{-\infty}^{\infty} f(x)e^{-ikx}dx\end{aligned}$$

This gives rise to

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}dk$$

14 RLC circuit

$$\begin{aligned} V(s) &= I(s)\left(R + Ls + \frac{1}{Cs}\right) \\ I(s) &= \frac{1}{R + Ls + \frac{1}{Cs}}V(s) \end{aligned}$$

As differential equation for current

$$I''(t) + 2\frac{R}{2L}I'(t) + \frac{1}{LC}I(t) = 0$$

15 Time evaluation of spin state

$$\begin{aligned} H &= -\mu \cdot B \\ &= \frac{eB}{m_e}S_z \\ &= \frac{eB\hbar}{2m_e}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

The eigenvalues are $E_+ = \frac{eB\hbar}{2m_e}m, E_- = -\frac{eB\hbar}{2m_e}m$

$$\begin{aligned} i\hbar\frac{d}{dt}|X\rangle &= H|X\rangle \\ &= \frac{eB\hbar}{2m_e}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}|X\rangle \end{aligned}$$

Hence

$$\begin{aligned} i\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \frac{eB}{2m_e}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ \hbar\dot{x}_1(t) &= \frac{eB}{2m_e}x_1(t) \\ \hbar\dot{x}_2(t) &= -\frac{eB}{2m_e}x_2(t) \end{aligned}$$

The solution is

$$x_1(t) = \frac{1}{\sqrt{2}} e^{-i\gamma t}$$
$$x_2(t) = \frac{1}{\sqrt{2}} e^{i\gamma t}$$

Or

$$|X\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\gamma t} \\ e^{i\gamma t} \end{bmatrix}$$

Where $\gamma = \frac{eB}{2m_e}$

$$\begin{aligned}
|X\rangle &= c_+ |S_x = \frac{\hbar}{2}\rangle + c_- |S_x = -\frac{\hbar}{2}\rangle \\
c_+ &= \langle S_x = \frac{\hbar}{2} |X\rangle \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\gamma t} \\ e^{i\gamma t} \end{bmatrix} \\
&= \frac{1}{2}\left(e^{i\gamma t} + e^{-i\gamma t}\right) \\
&= \cos \gamma t
\end{aligned}$$

Probability to measure $S_x = \frac{\hbar}{2}$ at $t > 0$ is $P(t) = |c_+|^2 = \cos^2 \gamma t$. And

$$\begin{aligned}
|X\rangle &= c_+ |S_x = \frac{\hbar}{2}\rangle + c_- |S_x = -\frac{\hbar}{2}\rangle \\
c_- &= \langle S_x = -\frac{\hbar}{2} |X\rangle \\
&= \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\gamma t} \\ e^{i\gamma t} \end{bmatrix} \\
&= \frac{1}{2}\left(e^{i\gamma t} - e^{i\gamma t}\right) \\
&= i \sin \gamma t
\end{aligned}$$

Probability to measure $S_x = -\frac{\hbar}{2}$ at $t > 0$ is $P(t) = |c_-|^2 = \sin^2 \gamma t$

16 Pauli matrices, Spin matrices

Pauli matrices There are 3 of these. They are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

There are also sometimes called $\alpha_x, \alpha_y, \alpha_z$. Not to be confused by component x, y, z of an ordinary vector. Important property is that $\sigma_i^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. Also they are all Hermitians (i.e. $A^\dagger = A$). This is obvious for the first and last matrix, since there are symmetric and real (we know if a matrix is real and also symmetric, it is also Hermitian.). Another important property is that they are unitary. i.e. $A^\dagger = A^{-1}$. Also any two anticommute. This means $[M, N]_+ = MN + NM$.

$$\left[\sigma_x, \sigma_y\right] = 2i\sigma_z$$

For Pauli matrices, $\left[\sigma_i, \sigma_j\right] = 2i \sum \epsilon_{ijk}\sigma_k$. Hence

$$\begin{aligned}
[\sigma_1, \sigma_2] &= 2i\sigma_3 \\
[\sigma_2, \sigma_1] &= -2i\sigma_3 \\
[\sigma_1, \sigma_3] &= -2i\sigma_2 \\
[\sigma_3, \sigma_1] &= 2i\sigma_2 \\
[\sigma_2, \sigma_3] &= 2i\sigma_1 \\
[\sigma_3, \sigma_2] &= -2i\sigma_1
\end{aligned}$$

Eigenvalues of Pauli matrices can be only $1, -1$.

$$\mathrm{Tr}(\sigma_i) = 0$$

And Pauli matrices do not commute. This means $\sigma_x\sigma_y \neq \sigma_y\sigma_x$.

Electron $\frac{1}{2}$ spin matrices

Spin matrix	Eigenvalues	Eigenvectors
$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\frac{\hbar}{2}, -\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$	$\frac{\hbar}{2}, -\frac{\hbar}{2}$	$\frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$ $\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$
$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$\frac{\hbar}{2}, -\frac{\hbar}{2}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

And using $\left[S_i, S_j\right] = i\hbar \sum_k \epsilon_{ijk} S_k$. Hence $[S_1, S_2] = i\hbar S_3$ and $[S_1, S_3] = -i\hbar S_2$ and $[S_2, S_1] =$

$-i\hbar S_3$ and $[S_2, S_3] = i\hbar S_1$ and $[S_3, S_1] = -i\hbar S_2$ and $[S_3, S_2] = -i\hbar S_1$. Hence

$$\left[S_x, S_y\right] = i\hbar S_z$$
$$\left[S_y, S_x\right] = -i\hbar S_z$$
$$[S_x, S_z] = -i\hbar S_y$$
$$[S_z, S_x] = i\hbar S_y$$
$$\left[S_y, S_z\right] = i\hbar S_x$$
$$\left[S_z, S_y\right] = -i\hbar S_x$$

And

$$S_i = \frac{\hbar}{2}\sigma_i$$

And

$$\sigma_i^2 = I$$

And

$$\begin{aligned} S_+^\dagger S_+ &= S^2 - S_z^2 - \hbar S_z \\ &= \hbar^2 \\ S_-^\dagger S_- &= S^2 - S_z^2 + \hbar S_z \\ &= \hbar^2 \end{aligned}$$

Where $S^2 = \frac{3}{4}\hbar^2 I$.

<u>Electron 1 spin matrices</u>

Spin matrix	Eigenvalues	Eigenvectors
$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$	$1, 0, -1$	$\begin{bmatrix}\frac{1}{2}\\\frac{1}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$ $\begin{bmatrix}-\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$ $\begin{bmatrix}\frac{1}{2}\\\frac{-1}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$
$S_y = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\i & 0 & -i\\0 & i & 0\end{bmatrix}$	$1, 0, -1$	$\begin{bmatrix}-\frac{1}{2}\\-\frac{i}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$ $\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix}$ $\begin{bmatrix}-\frac{1}{2}\\\frac{i}{\sqrt{2}}\\\frac{1}{2}\end{bmatrix}$
$S_z = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & -1\end{bmatrix}$	$\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\end{bmatrix}$

And

$$\begin{aligned} S_+^\dagger S_+ &= S^2 - S_z^2 - \hbar S_z \\ &= \hbar^2 \\ S_-^\dagger S_- &= S^2 - S_z^2 + \hbar S_z \\ &= \hbar^2 \end{aligned}$$

Where $S^2 = 2\hbar^2 I = \hbar^2\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$.

If we are given state vector V and asked to find expectation value when measuring along x axis, then do $\langle V|S_x|V\rangle$

17 Quantum mechanics cheat sheet

17.1 Hermitian operator in function spaces

If Ω is Hermitian operator, then it satisfies

$$\begin{aligned} \langle u|\Omega|v\rangle^* &= \langle v|\Omega|u\rangle \\ \left(\int u^*(x)\Omega[v(x)]dx\right)^* &= \int v^*(x)\Omega[u(x)]dx \\ \int u(x)\Omega[v^*(x)]dx &= \int v^*(x)\Omega[u(x)]dx \end{aligned}$$

For this, the boundary terms must vanish. For example, for the operator $\Omega = -i\frac{d}{dx}$

17.2 Dirac delta relation to integral

$$\delta(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx}dx$$

17.3 Normalization condition

$$\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)dt = 1$$

17.4 Expectation (or average value)

If a system is in state of Ψ , then we apply operator $\hat{A}$, then the average value of the observable quantity is the expectation integral

$$\begin{aligned} \left\langle \hat{A} \right\rangle &= \langle\psi|\hat{A}|\psi\rangle \\ &= \frac{\int_{-\infty}^{\infty} \Psi^*\hat{A}\Psi dx}{\int_{-\infty}^{\infty} \Psi\Psi dx} \end{aligned}$$

Note that $\int_{-\infty}^{\infty} \Psi(x)\Psi(x)dx = 1$ if the state wave function is already normalized.

Given an operator $\hat{X}$, acting on $\Psi(x,t)$ then

$$\hat{X}\Psi(x,t) = x\Psi(x,t)$$

The expectation of measuring x is (assuming everything is normalized)

$$\begin{aligned} \left\langle \hat{X} \right\rangle &= \int_{-\infty}^{\infty} \Psi^*(x,t)\hat{X}\Psi(x,t)dx \\ &= \int_{-\infty}^{\infty} \Psi^*(x,t)x\Psi(x,t)dx \\ &= \langle x\rangle \end{aligned}$$

Given system is in state $\psi(x)$. What is the expectation value for x measurement. Is this same as writing $\langle X\rangle$. Yes. it is

$$\langle\psi|x|\psi\rangle$$

17.5 Probability

The probability that position x of particle is between x and $x + dx$ is $|\Psi(x,t)|^2 dx$. Hence $|\Psi(x,t)|^2$ is the probability density.

Note that

$$\langle\Psi|\Psi\rangle = \int_{-\infty}^{\infty} |\Psi(x)|^2 dx$$
$$\langle\Psi_1|\Psi_2\rangle = \int_{-\infty}^{\infty} \Psi_1^*(x)\Psi_2(x)dx$$

Given $|\Psi\rangle = a|\Psi_1\rangle + b|\Psi_2\rangle$ then the probabilities to measure a or b are

$$P(a) = \frac{|a|^2}{|a|^2 + |b|^2}$$
$$P(b) = \frac{|b|^2}{|a|^2 + |b|^2}$$

17.6 Position operator $\hat{x}$

eigenvalue/eigenfunction	$\hat{x}|x\rangle = x|x\rangle$ Where x is eigenvalue and $|x\rangle$ is position vector.
orthonormal eigenbasis	$\{|x\rangle\} \rightarrow \begin{cases} \langle x|x'\rangle = \delta(x-x') \\ \int_{-\infty}^{\infty} |x\rangle\langle x|\, dx = 1 \end{cases}$ for $-\infty < x < \infty$
Vector form to function form	$\langle x|\psi\rangle \equiv \psi(x)$ probability at position x
Expansion of state vector $|\psi\rangle$	$|\psi\rangle = \int_{-\infty}^{\infty} |x'\rangle\langle x'|\psi\rangle dx' = \int_{-\infty}^{\infty} |x'\rangle\psi(x')dx'$
Eigenfunctions in deep well	Not defined for position operator
Operator matrix elements	$\langle x|\hat{x}|x'\rangle = x'\delta(x-x')$ Operator is diagonal matrix.

17.7 Momentum operator $\hat{p}$

eigenvalue/eigenfunction	$\hat{p}\vert\phi_p\rangle = p\vert\phi_p\rangle$ Where p is eigenvalue and $\vert\phi_p\rangle$ is momentum eigenstate
orthonormal eigenbasis	$\{\vert\phi_p\rangle\} \to \begin{cases} \langle\phi_p\vert\phi_{p'}\rangle = \delta\left(p - p'\right) \\ \int_{-\infty}^{\infty} \vert\phi_p\rangle\langle\phi_p\vert\, dp = 1 \end{cases}$ for $-\infty < p < \infty$
Vector form to function form	$\langle x\vert\phi_p\rangle \equiv \phi_p(x)$
Expansion of state vector $\vert\psi\rangle$	$\vert\psi\rangle = \int_{-\infty}^{\infty} \vert\phi_p\rangle\langle\phi_p\vert\psi\rangle dp$
General Eigenfunction	$\langle x\vert\phi_p\rangle \equiv \phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$
Operator matrix elements	$\langle x\vert\hat{p}\vert x'\rangle = -i\hbar\delta(x - x')\frac{d}{dx'}$ Operator is not diagonal matrix.

17.8 Hamilitonian operator $\hat{H}$

$$\hat{H} = \hat{T} + \hat{V}$$

Where $\hat{T}$ is K.E. operator and $\hat{V}$ is P.E. operator. Recall that $p = mv$ and $T = \frac{1}{2}mv^2$. Hence $\hat{T} = \frac{\hat{p}^2}{2m}$.

eigenvalue/eigenfunction	$\hat{H}\vert\psi_{E_n}\rangle = E_n\vert\psi_E\rangle$ Where E_n is eigenvalue (energy level)
Orthonormal basis of operator	$\{\vert\psi_{E_n}\rangle\} \to \begin{cases} \langle\psi_{E_n}(x)\vert\psi_{E_m}(x)\rangle = \delta(E_n - E_m) \\ \int_{-\infty}^{\infty} \vert\psi_{E_n}\rangle\langle\psi_{E_n}\vert\, dE = 1 \end{cases}$ for $n = 1, 2, \cdots$ (check)
Vector form to function form	$\langle x\vert\psi_{E_n}\rangle \equiv \psi_{E_n}(x)$
Expansion of state vector $\vert\psi\rangle$	$\vert\psi_E\rangle = \sum_n \vert\psi_{E_n}\rangle\langle\psi_{E_n}\vert\psi\rangle$
Eigenfunctions for deep well problem	$\langle x\vert\psi_E\rangle = \psi(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$, $E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$
Operator matrix elements	$\langle x\vert\hat{H}\vert x'\rangle = \frac{1}{2}mv^2 + V(x) = \delta(x - x')\left(\frac{\hat{p}^2}{2m} + \hat{V}(x')\right) = \delta(x - x')\left(\frac{-\hbar^2}{2m}\frac{d^2}{dx'^2} + \hat{V}(x')\right)$

The ODE for deep well is derived as follows.

$$\hat{H}\psi = E_n\psi$$
$$\left(\hat{T} + \hat{V}\right)\psi = E_n\psi$$

But $\hat{V} = 0$ inside and $\hat{T} = \frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2}$. Hence the above becomes

$$\begin{aligned}\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) &= E\psi(x)\\ \frac{d^2}{dx^2}\psi(x) + \frac{2mE}{\hbar^2}\psi(x) &= 0\\ \frac{d^2}{dx^2}\psi(x) + k^2\psi(x) &= 0\end{aligned}$$

Where $k = \sqrt{\frac{2mE}{\hbar^2}}$. The eigenvalues are k_n from solving for boundary conditions at $x = L$. Now solve as standard second order ODE, with BC $\psi(0) = 0, \psi(L) = 0$. The solution becomes

$$\psi(x) = \psi(x) = \begin{cases}\sqrt{\frac{2}{L}}\sin(k_n x) & 0 < x < L\\ 0 & \text{otherwise}\end{cases}$$

Where eigenvalues are $k_n = \frac{n\pi}{L}, n = 1, 2, 3, \cdots$

18 Questions and answers

18.1 Question 1

Problem says that the system is in some general state $\psi(x)$ and asks what is the probability distribution to measure momentum p ?

solution

The probability is $\left|\langle\phi_p|\psi\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current state*.

$$\begin{aligned}\langle\phi_p|\psi\rangle &= \int_{-\infty}^{\infty}\langle\phi_p|x\rangle\langle x|\psi\rangle dx\\ &= \int_{-\infty}^{\infty}\langle x|\phi_p\rangle^*\langle x|\psi\rangle dx\\ &= \int_{-\infty}^{\infty}\phi_p^*(x)\psi(x) dx\end{aligned}$$

Now, for the deep well problem for $0 < x < L$, we should know that $\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{ipx}{\hbar}}$ and $\psi(x)$ will be given. For example $\psi_E(x) = \begin{cases}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} & 0 < x < L\\ 0 & \text{otherwise}\end{cases}$. Hence

$$\langle\phi_p|\psi\rangle = \int_0^L \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{-ipx}{\hbar}}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} dx$$

Now evaluate this integral and at the end take the square of the modulus. This will give the probability distribution to measure p. The above was problem 4, in HW7.

18.2 Question 2

Problem says that the system is in some general state $\psi(x)$ and asks what is the probability distribution to measure position x ?

solution

The probability is $\left|\langle x|\psi\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current state*.

$$\begin{aligned}
\langle x|\psi\rangle &= \int_{-\infty}^{\infty} \langle x|x'\rangle\langle x'|\psi\rangle dx' \\
&= \int_{-\infty}^{\infty} \delta(x-x')\psi(x')\rangle dx \\
&= \psi(x)
\end{aligned}$$

Hence $prob(x) = \left|\langle x|\psi\rangle\right|^2 = |\psi(x)|^2$

18.3 Question 3

Problem says that the system is in some general state $\psi_E(x)$ and asks what is the probability distribution to measure position x ?

solution

The probability is $\left|\langle x|\psi\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current or given eigenstate*.

$$\begin{aligned}
\langle x|\psi\rangle &= \int_{0}^{L} \langle x|x'\rangle\langle x'|\psi\rangle dx' \\
&= \int_{0}^{L} \delta(x-x')\psi(x') dx' \\
&= \psi(x)
\end{aligned}$$

Hence the probability is $\left|\psi(x)\right|^2$. Now, for the deep well problem for $0 < x < L$, we know that $\psi_{E_n}(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} & 0 < x < L \\ 0 & \text{otherwise} \end{cases}$ then

$$\begin{aligned} \left|\psi_{E_n}(x)\right|^2 &= \left|\sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}\right|^2 \\ &= \frac{2}{L} \sin^2\left(\frac{n\pi x}{L}\right) \end{aligned}$$

Is this correct? Checked, yes correct.

18.4 Question 4

Problem gives that the system is in some general state $\phi_p(x)$ (i.e. momentum eigenstate, not energy eigenstate as above, due to having done momentum measurement done before) and then problem asks what is the probability distribution to measure position x ?

solution

The probability is $\left|\langle x|\phi_p\rangle\right|^2$. What goes in the bra is the eigenstate *being measured*. What goes in the ket is the *current eigenstate*.

$$\begin{aligned} \langle x|\phi_p\rangle &= \int_0^L \langle x|x'\rangle\langle x'|\phi_p\rangle dx' \\ &= \int_0^L \delta(x - x')\phi_p(x')dx' \\ &= \phi_p(x) \end{aligned}$$

Hence the probability is $\left|\phi_p(x)\right|^2$. we know that $\phi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$ then

$$\begin{aligned} \left|\phi_p(x)\right|^2 &= \left|\frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}\right|^2 \\ &= \frac{1}{2\pi\hbar} \end{aligned}$$

Which is constant. So if we measure momentum first, then ask for probability of measuring position x next, it will be the above. Same probability to measure any position? Is this correct? yes.

18.5 Question 5

Problem gives that the system is in some general state $\phi_p(x)$ and asks what is the probability to measure momentum p'?

The probability of measuring momentum p' given that system is already in state $|\psi_p\rangle \equiv |\phi_p\rangle$ is $\left|\langle\phi_{p'}|\phi_p\rangle\right|^2$ where

$$\begin{aligned}
\langle\phi_{p'}|\phi_p\rangle &= \int_{-\infty}^{\infty} \langle\phi_{p'}|x\rangle\langle x|\phi_p\rangle dx \\
&= \int_{-\infty}^{\infty} \langle x|\phi_{p'}\rangle^* \langle x|\phi_p\rangle dx \\
&= \int_{-\infty}^{\infty} \phi_{p'}^*(x)\phi_p(x) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{-ip'x}{\hbar}\right) \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right) dx \\
&= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \exp\left(\frac{i\left(p-p'\right)x}{\hbar}\right) dx
\end{aligned}$$

but $\delta(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ipx} dx$, therefore $\delta\left(p-p'\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(p-p'\right)x} dx$.

Let $u = \frac{x}{\hbar}$, then $du = \frac{1}{\hbar}dx$. The integral becomes

$$\begin{aligned}
\langle\phi_{p'}|\phi_p\rangle &= \frac{\hbar}{2\pi\hbar} \int_{-\infty}^{\infty} e^{i\left(p-p'\right)u} du \\
&= \frac{1}{2\pi}\left(2\pi\delta\left(p-p'\right)\right) \\
&= \delta\left(p-p'\right)
\end{aligned}$$

19 Position, velocity and acc in different coordinates system

In polar, just remember these

$$\begin{aligned}
\vec{r} &= \rho\hat{e}_\rho \\
d\vec{r} &= \hat{e}_\rho d\rho + \hat{e}_\phi \rho d\phi \\
\vec{v} &= \frac{d\vec{r}}{dt} \\
&= \hat{e}_\rho \frac{d\rho}{dt} + \hat{e}_\phi \rho \frac{d\phi}{dt} \\
\frac{d}{dt}\hat{e}_\rho &= \dot{\phi}\hat{e}_\phi \\
\frac{d}{dt}\hat{e}_\phi &= -\dot{\phi}\hat{e}_\rho
\end{aligned}$$

Given $\vec{r} = \rho\hat{e}_\rho$, then

$$\begin{aligned}\vec{v} &= \dot{\rho}\hat{e}_\rho + \rho\frac{d}{dt}\hat{e}_\rho \\ &= \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi\end{aligned}$$

And similarly for $\vec{a}$.

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{e}_\rho + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{e}_\phi$$

This is much better than the alternatives.

In <u>Cylindrical</u>

$$\begin{aligned}d\hat{e}_\rho &= \hat{e}_\phi d\phi \\ d\hat{e}_\phi &= -\hat{e}_\rho d\phi \\ d\hat{e}_z &= 0\end{aligned}$$

<u>dr is different coordinates</u>

Cartesian

$$dr = \hat{e}_x dx + \hat{e}_y dy + \hat{e}_z dz$$

Cylindrical

$$dr = \hat{e}_\rho d\rho + \hat{e}_\phi \rho d\phi + \hat{e}_z dz$$

Spherical

$$dr = \hat{e}_r dr + \hat{e}_\theta r d\theta + \hat{e}_\phi r\sin\theta d\phi$$

<u>v is different coordinates</u>

Use these for finding Lagrangian.

In Cartesian

$$\vec{v} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y + \dot{z}\hat{e}_z$$

Polar

$$\vec{v} = \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi$$

Spherical

$$\vec{v} = \dot{\rho}\hat{e}_\rho + \rho\dot{\theta}\hat{e}_\theta + \rho\sin\theta\dot{\phi}\hat{e}_\phi$$

$$\nabla V\left(\rho,\theta,\phi\right) = \hat{e}_\rho V_r + \hat{e}_\theta\frac{1}{\rho}V_\theta + \hat{e}_\phi\frac{1}{\rho\sin\theta}V_\phi$$

20 Gradient, Curl, divergence, Gauss flux law, Stokes

The gradient ∇ is vector operator. In Cartesian

$$\nabla = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

In Cylindrical

$$\nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\phi \rho \frac{\partial}{\partial \phi} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \rho \frac{\partial f}{\partial \phi} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

In spherical

$$\nabla = \hat{e}_\rho \frac{\partial}{\partial \rho} + \hat{e}_\theta \frac{1}{\rho} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{\rho \sin\theta} \frac{\partial}{\partial \phi}$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial \rho} \\ \frac{1}{\rho} \frac{\partial f}{\partial \theta} \\ \frac{1}{\rho \sin\theta} \frac{\partial f}{\partial \phi} \end{pmatrix}$$

For conservative force

$$F = -\nabla V$$

Notice that $-\int \bar{F} \cdot d\bar{r} = \int \nabla V \cdot d\bar{r} = \int_{from}^{to} dV = V(to) - V\big(from\big)$ also $\oint \bar{F} \cdot d\bar{r} = 0$ for conservative force.

The <u>curl</u> in Cartesian

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

In Cylinderical

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & F_\phi & F_z \end{vmatrix}$$

In Spherical

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{e}_\rho & \hat{e}_\phi & \hat{e}_\theta \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho \sin\theta}\frac{\partial}{\partial \phi} & \frac{1}{\rho}\frac{\partial}{\partial \theta} \\ F_\rho & F_\phi & F_\theta \end{vmatrix}$$

<u>Divergence</u> This is scalar. see cha7b.pdf

$$\nabla \cdot \bar{F}$$

<u>Gauss law</u>

From Wiki

It states that the flux of the electric field out of an arbitrary closed surface is proportional to the electric charge enclosed by the surface.

Gauss's law can be used in its differential form, which states that the divergence of the electric field is proportional to the local density of charge.

$$\overbrace{\int\int \bar{F}\cdot d\bar{s}}^{\text{surface integral}} = \int_V \left(\nabla \cdot \bar{F}\right) dV$$

<u>Stoke's theorem</u>

$$\overbrace{\oint \bar{F}\cdot d\bar{r}}^{\text{line integral}} = \int_S \left(\nabla \times \bar{F}\right)\cdot d\bar{s}$$

Also divergence of the curl is zero.

$$\nabla \cdot \left(\nabla \times \bar{F}\right) = 0$$

From the net

The characteristic of a conservative field is that the line integral around every simple closed contour is zero. Since the curl is defined as a particular closed contour line integral, it follows that $curl(gradF)$ equals zero.

And curl of a gradient is the zero vector.

$$\nabla \times \left(\nabla \bar{F}\right) = \bar{0}$$

21 Gas pressure

average speed of gas particles is v_{rms} or take avergae of the squares of each particle velocity and then take the square root at end. Or

$$\bar{v} = \sqrt{\frac{3RT}{m}}$$

Where R is the gas constant, T is gas absolute temperature and m is molar mass of each gas particle in kg/mol.

dn

$$dn = f(v) dv_x dv_y dv_z$$

Where dn is the number denity of gas particles (how many particles per unit volume with velocity between v and $v + dv$)

<u>Average speed of particles</u>

$$\begin{aligned}
\bar{v} &= \frac{\int v dn}{\int dn} \\
&= \frac{\int\int\int v f(v) dv_x dv_y dv_z}{n} \\
&= \frac{1}{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v f(v) dv_x dv_y dv_z \\
&= \frac{1}{n}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{v=0}^{\infty} v f(v)\left(v^2\sin\theta\right) dv d\theta d\phi \\
&= \frac{1}{n}\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi}\sin\theta d\theta \int_{v=0}^{\infty} f(v) v^3 dv \\
&= \frac{1}{n}(2\pi)(-\cos\theta)_0^{\pi}\int_{v=0}^{\infty} f(v) v^3 dv \\
&= -\frac{1}{n}(2\pi)(-1-1)\int_{v=0}^{\infty} f(v) v^3 dv \\
&= \frac{4\pi}{n}\int_{v=0}^{\infty} f(v) v^3 dv
\end{aligned}$$

<u>Pressure</u>

$$\begin{aligned}
dF &= F_1 dN \\
&= \left(\frac{2mv_z}{\Delta t}\right) dn \Delta A v_z \Delta t \\
&= 2mv_z^2 dn \Delta A
\end{aligned}$$

Hence

$$\begin{aligned}
P &= \frac{\int dF}{\Delta A} \\
&= 2m\int v_z^2 dn \\
&= 2m\int dv_x \int dv_y \int f(v) v_z^2 dv_z
\end{aligned}$$

This integral can be evaluated in spherical coordinates.

net energy density of gas

$$
\begin{aligned}
E &= \int \frac{1}{2} m v^2 dn \\
&= \frac{1}{2} m \int \int \int v^2 dn \\
&= \frac{1}{2} m \int \int \int \left(v_x^2 + v_y^2 + v_z^2\right) dn \\
&= \frac{3}{2} m \int \int \int v_z^2 dn \\
&= \frac{3}{2} m \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} v_z^2 f(v) dv_z \\
&= 3m \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{0}^{\infty} v_z^2 f(v) dv_z
\end{aligned}
$$

Hence

$$
P = \frac{2}{3} E
$$

And $E = \frac{3}{2} nKT \rightarrow P = nKT$ for ideal gas.

22 Table of study guide

chapter	topics
ch7c.pdf	PDE's, seperation of variables, Lagrangian method
ch7b.pdf	Position, velocity and acc in different coordinates. Gradient, Curl and Div.
ch7a.pdf	Multivariable calculus. Jacobian. Gravitional field for shell, Pressure and energy of gas
ch6b.pdf	First order ODE's. Second order Constant coefficients. under,over and critical damping
ch6a.pdf	Second order ODE's. Variable coefficient. Power series methods. Hermite ODE.
ch5c.pdf	Function spaces. Hermitian operators. Complex Fourier series. Fourier transform. Deep well probem

ch5b.pdf	Linear vector spaces and QM. Probability when making measurements. Commutation. Schrodinger equation. Spin operators S_x, S_y, S_z. Pauli matrices. Time evolution of spin state. Solving mass/spring problem using normal modes.
ch5a.pdf	Linear vector spaces. Linear independence. Gram-Schmidt. Linear operators. Finding eigenvalues and eigenvectors for matrices. Coordinates transformation between orthonormal basis.
ch4.pdf	Matrices and Determinants. 2D rotation matrix. Lorentz transformation. Pauli matrices. Levi-civita. Properties of determinants. Solution to linear equations. Cramer rule. Dimensional analysis.
ch3.pdf	Complex numbers. Taylor series expansion. Solving $x^n = 1$. Integrals. Completing the squares for $\int_{-\infty}^{\infty} e^{(x+ia)^2}\, dx$. Gaussian integral, N slit intererence. Single slit diffraction.
ch2.pdf	Gaussian and exponential integrals. Evaluating Gaussian integral. Evaluating $\int_0^{\infty} x^n e^{-x}\, dx = n!$. Zeta function. Gamma function. Sterling formula.
ch1.pdf	Taylor series. Convergence test. Taylor series of common functions. Using Taylor series to find equilibrium point for small oscillations. Pendulum.

23 Questions

1. Do all spin matrices always have same eigenvalues? this is the case for S_x, S_y, S_z for electron. NO. depends spin number.

2. How do we get the probability of measuring $S_y = -\frac{h}{2}$ or $S_y = \frac{h}{2}$ to be $\frac{1}{2}$? is it because there are two eigenvalues, and it is 50% each? see class notes lecture 5b. page 9. Answer: Current state vector is $|S_x = \frac{h}{2}\rangle$.

3. Does the order matter? In page 5, lecture 5B, could we do $C_+ = \langle S_x = \frac{h}{2} | S_z = \frac{h}{2}\rangle$ or $C_+ = \langle S_z = \frac{h}{2} | S_x = \frac{h}{2}\rangle$? Resolved.

4. Why is $\langle V|S_x|V\rangle$ gives the The statistical average of measuring S_x given current state vector is $|V\rangle$? Resolved.

5. Can we just move the H operator to RHS, as in $x'' + Mx = 0$ instead of $x'' = -Mx$. This way no need to work with negative eigenvalues? Yes.

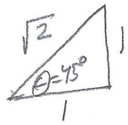
6. HW 5, last problem, I do not see how M, N share all the 3 eigenvectors. I get only one common eigenvector. I also do not understand the comment in my solution to refer to set of vectors as basis? What does this mean? Also, we know M, N commute, and so they share a common basis, but the question is asking which ones they share? Resolved.

7. For Pauli matrices, $\left[\sigma_i, \sigma_j\right] = 2i \sum \epsilon_{ijk}\sigma_k$. and for spin $\frac{1}{2}$ it is $\left[S_i, S_j\right] = i\hbar \sum_k \epsilon_{ijk} S_k$. So what is it for spin 1? is it still $\left[S_i, S_j\right] = i\hbar \sum_k \epsilon_{ijk} S_k$? Yes.

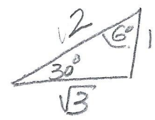
8. I think $\Psi(x,t)$ is just the eigenfunction corresponding to the eigenvalue just measured. So if the operator used is the position operator X, then it is called $\Psi(x)$. If the operator used is momentum operator P, we call it $\phi_p(x)$, but should it be really be $\Psi_p(x)$? If the operator is Hamiltonian $\hat{H}$, then the eigenvalue is the energy level E and the Ψ is called $\Psi_E(x)$. Any of these are also called the wave function $\Psi(x)$. Is this correct? I think so.

24 Appendix

1



$\cos\theta = \frac{1}{\sqrt{2}}$
$\sin\theta = \frac{1}{\sqrt{2}}$
$\tan\theta = 1$



$\sin 30^\circ = \frac{1}{2}$ $\qquad \sin 60^\circ = \frac{\sqrt{3}}{2}$
$\cos 30^\circ = \frac{\sqrt{3}}{2}$ $\qquad \cos 60^\circ = \frac{1}{2}$
$\tan 30^\circ = \frac{1}{\sqrt{3}}$ $\qquad \tan 60^\circ = \frac{1}{\sqrt{3}}$

Taylor Series $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ $\quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$|x| < 1$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots$$

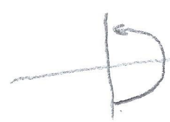
to remember: Think of exp(x) but with sign flip and no extra !.

$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots$$

to check convergence use ratio test $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \Rightarrow$ yes.

to find R: $\quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \Rightarrow |x| < R$ is radius of convergence.

For inverse trig



use this range of angle $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. so inverse is unique.

To Find derivative of inverse trigs let $y = \arcsin(x)$ Then

$$x = \sin(y) \Rightarrow \frac{dx}{dy} = \cos(y) = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

hence $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$ hence $\boxed{\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}}$

$y = \arccos(x)$, $x = \cos(y) \Rightarrow \frac{dx}{dy} = -\sin(y) = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2}$

so $\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \Rightarrow \boxed{\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}}$

$y = \arctan(x)$ $\quad x = \tan(y) \Rightarrow \frac{dx}{dy} = \frac{1}{\cos^2 y}$. but $\cos^2 y + \sin^2 y = 1 \Rightarrow 1 + \tan^2 y = \frac{1}{\cos^2 y}$.

but $\tan^2 y = x^2$. hence $\frac{dx}{dy} = 1 + x^2 \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$.

exponential and interest rate

interest

① $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$

$$\boxed{\lim_{N\to\infty}\left(1+\frac{r}{N}\right)^N = e^r;\ \lim_{N\to\infty}\left(1+\frac{r}{N}\right)^{Nt} = e^{rt}}$$

$$\int_1^y \frac{dt}{t} = x \Rightarrow y = e^x$$

compound interest formula $\boxed{A = P\left(1+\frac{r}{n}\right)^{nt}}$ ②

where P is principle, r is interest rate, n is number of times interest is applied per year, t is number of years.

here is the connection to exp as $n\to\infty$ Then ② becomes $\boxed{A = Pe^{rt}}$ $\lim_{n\to\infty}$

$$e^x = 1+x+\frac{x^2}{2}+\frac{x^3}{3!}\cdots$$

now we can use exp. to find A. for example.

$$A = P\left(1+(rt)+\frac{(rt)^2}{2!}+\cdots\right)$$

t is # of years and r is interest rate as in 0.01.

---

e^{-x} when multiplied by any x^n and as $x\to\infty$ will bring result to zero so e^{-x} will subdue any powers of x. i.e $\lim_{x\to\infty} x^n e^{-x} \to 0$ — $\lim_{x\to\infty}\frac{x^n}{e^{+x}}$. apply L'hopitals n-times n. to proof.

$\ln(x)$ is weaker than positive powers of x

i.e as $n\to\infty$ $x^n \gg \ln(x)$

To proof the above use $\lim_{x\to\infty}\frac{\ln(x)}{x^n}$ and use L'hopital.

Implicit differentiation:

2

Given $3x^2+4y^2=R^2$, problem ask to find **slope** at (x_0, y_0). Then do

$$\frac{d}{dx}\left(3x^2+4y^2\right)=0 \Rightarrow 6x+8y\frac{dy}{dx}=0$$

so $\boxed{\frac{dy}{dx} = \frac{-6x}{8y}}$ now plug in x_0, y_0 in RHS to find slope.

Differential

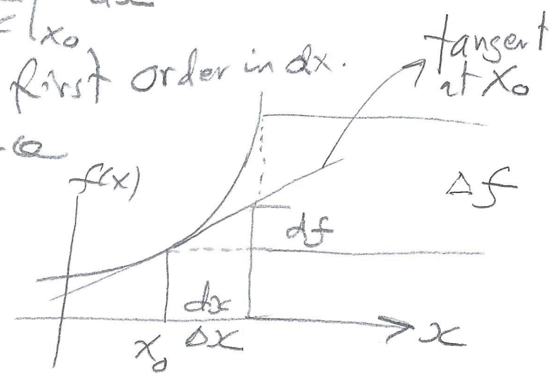
$$\Delta f = \left.\frac{df}{dx}\right|_{x_0}\Delta x + \left.\frac{d^2f}{dx^2}\right|_{x_0}\Delta x^2 + \cdots$$

but df called differential is

$$df = \left.\frac{df}{dx}\right|_{x_0} dx$$

so differential is always to first order in dx.

Picture to see difference



so df is change in $f(x)$ when moving along tangent distance dx

as $dx \to 0$, $df \to \Delta f$

Properties of complex Variables

$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{zw} = \bar{z}\,\bar{w}$

$\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$|\bar{z}| = |z|$

inner product

$\langle z_1, z_2\rangle = z_1\bar{z}_2 = r_1 r_2 e^{i(\theta_1-\theta_2)}$

$z\bar{z} = |z|^2$

$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

$\overline{(z^n)} = (\bar{z})^n$

$\exp(\bar{z}) = \overline{\exp(z)}$

$\ln(\bar{z}) = \overline{\ln(z)}$

$|z| = \sqrt{x^2+y^2}$ for $z = x+iy$.

$|\mathrm{Re}\, z| \le |z|$

$|\mathrm{I}z| \le |z|$

$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$

$|z_1+z_2| \le |z_1| + |z_2|$

$|z_1 z_2| = |z_1|\,|z_2|$

more complex

$e^{i\theta} = \cos\theta + i\sin\theta$

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

roots of unity

③

$z^N = 1$

$z = 1^{\frac{1}{N}} = \left(e^{i(2\pi)}\right)^{\frac{1}{N}}$

$= \left(\cos(2\pi + n2\pi) + i\sin\ldots\right)^{\frac{1}{N}}$

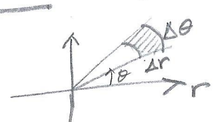
$= \cos\left(\frac{2\pi}{N} + \frac{n}{N}2\pi\right) + i\ldots$

$n = 0, 1, \ldots, N-1$

$$\boxed{(1-z^N) = (1-z)(1+z+z^2+\cdots+z^{N-1})}$$

Integrals

Gaussian $\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ $a>0$

proof:  area is $\boxed{(r\,d\theta)(dr)}$ so $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int e^{-y^2}dy\right)}$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dx \overset{\text{Polar}}{\Longrightarrow} \boxed{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta} = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r\,dr$

$= 2\pi\int_0^{\infty} e^{-r^2} r\,dr$. let $r^2 = u$. $\frac{du}{dr} = 2r$. $\Rightarrow 2\pi\int_0^{\infty} e^{-u}\, r\frac{du}{2r} = \pi\int_0^{\infty} e^{-u}du$

$= \pi\left[-e^{-u}\right]_0^{\infty} = \pi\left[0+1\right] = \pi$. $\Rightarrow \boxed{\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}}$

$\int_{-\infty}^{\infty} x^n e^{-ax^2}dx$. For odd $n \Rightarrow 0$. so we only have to do even n. for $n=2$ $I = \int x^2 e^{-ax^2}dx \Rightarrow I_1 = \int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$.

Then $I_1'(\alpha) = \int_{-\infty}^{\infty} x^2 e^{-ax^2}dx = \frac{1}{2}\frac{\sqrt{\pi}}{\alpha^{3/2}}$ QED.

for $n=4$ $I_1''(\alpha)$ etc...

$\int_{-\infty}^{\infty} e^{-ax^2}dx$. start by using $\boxed{y = \sqrt{a}\,x} \Rightarrow \frac{dy}{dx} = \sqrt{a}$. The integral becomes

$\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} e^{-y^2}dy = \boxed{\frac{1}{\sqrt{a}}\sqrt{\pi}}$

4

$$\int_0^\infty x^n e^{-x}\,dx = n! \qquad n \geq 0$$

watch out. starts at 0.

$$\int_0^\infty x^n e^{-ax}\,dx = n!\,\frac{1}{a^{n+1}}$$ (use $y = ax$ substitution).

$\boxed{\text{Zeta}}$

$$\int_0^\infty \frac{x^n}{e^x - 1}\,dx = n!\,\zeta(n+1). \qquad \zeta(2) = \frac{\pi^2}{6} \qquad \zeta(4) = \frac{\pi^4}{90}.$$

works only for odd n. since $\zeta(n)$ is known only for even.

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = 3!\,\zeta(4). \qquad \zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\text{so } \zeta(4) = \frac{1}{1} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \boxed{\frac{\pi^4}{90}}$$

Gamma $\boxed{\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\,dt}$

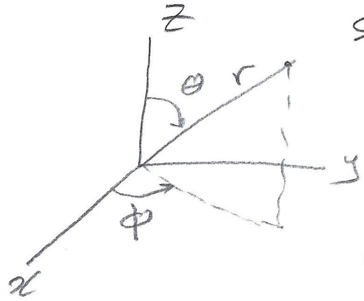
$\Gamma(\tfrac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-t}\,dt$. let $u = \sqrt{t}$. $\frac{du}{dt} = \frac{1}{2}\frac{1}{\sqrt{t}}$. the integral becomes

$$= 2\int_0^\infty e^{-u^2}\,du = 2\left(\frac{1}{2}\int_{-\infty}^{\infty} e^{-u^2}\,du\right) = \sqrt{\pi}.$$

for n integers $\Gamma(n) = (n-1)!$ and $\Gamma(n+1) = n!$

so $\boxed{\Gamma(n+1) = n(n-1)! = n\Gamma(n)}$

sterling for $n \gg 1$, $\boxed{\Gamma(n+1) = n! \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n}}$



spherical. 35

Volume $\int_0^\infty r\,dr \int_0^{2\pi} d\phi \int_0^{\pi} r\sin\theta\, d\theta\, f(r,\theta,\phi)$.

$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$

so Jacobian $= h_r h_\theta h_\phi = r^2 \sin\theta$

$$V = \int_0^\infty \int_0^{2\pi} \int_0^{\pi} f(\;)\, \underbrace{r^2 \sin\theta}_{\text{Jacobian}}\, d\theta\, d\phi\, dr$$

Volume of sphere $= \frac{4}{3}\pi r^3$, $f = 1$

area of sphere $= 4\pi r^2$

Taylor

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y +$$
$$\frac{1}{2}\left[f_{xx}\Delta x^2 + f_{yy}\Delta y^2 + f_{xy}\Delta x \Delta y + f_{yx}\Delta y \Delta x \right] + \cdots$$

$$f(x_0+\Delta x) - f(x_0) = \frac{\partial f}{\partial x} + \frac{1}{2} f_{xx}\Delta^2 x + \frac{1}{3!} f_{xxx}\Delta^3 x + \cdots$$

or $f(x) = f(x_0) + x f'(x_0) + \frac{x^2}{2} f''(x_0) + \frac{x^3}{3!} f'''(x_0) + \cdots$

geometric series each term is some positive ratio r of last term

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ so $\boxed{r = \frac{1}{2}}$

so $S_N = 1 + r + r^2 + r^3 + \cdots + r^N$

$r S_N = r + r^2 + \cdots + r^{N+1}$

$S_N - r S_N = 1 - r^{N+1} \Rightarrow \boxed{S_N = \frac{1 - r^{N+1}}{1 - r}}$

if $r < 1$ then $\boxed{\lim_{N\to\infty} = \frac{1}{1-r}}$ so $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ $|r| < 1$ geometric

if series converges absolutely, then it converges.

so absolute convergence is stronger.

Power series $\boxed{S = \sum_{n=0}^{\infty} a_n x^n}$

6

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \quad |x| < 1.$$

$$\exp(x) = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad |x| < 1$$

take deriv.

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots} \quad |x| < 1.$$

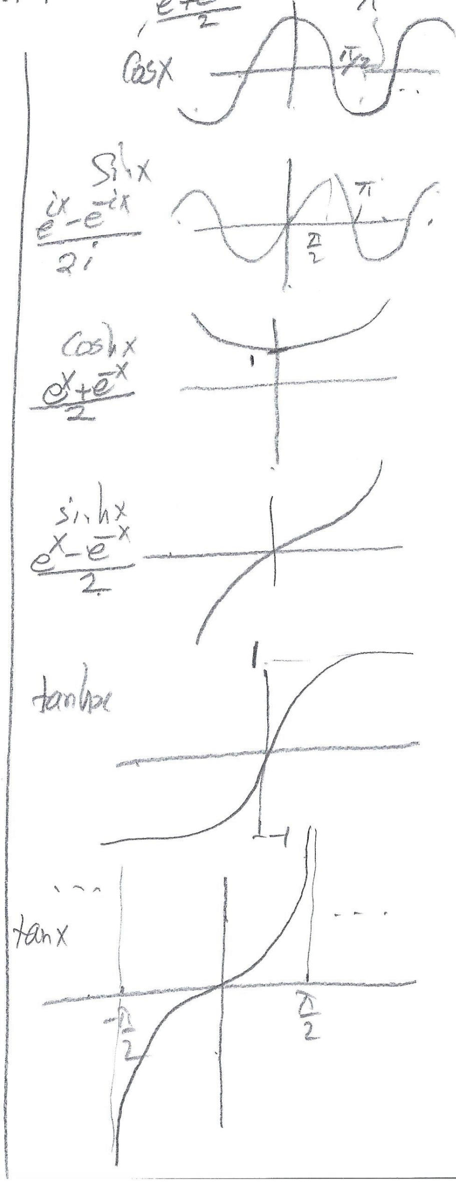
Trig

sin(x) domain 0..2π, Range −1..+1

Cos(x) " , Range −1..+1

tan 0..2π , −∞ ... +∞

arcsin	domain −1..1	Range −90°..+90°
arcos	domain −1..1	0..180°
arctan	−∞..+∞	−90°..+90°



7

integrals

Gaussian $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$

$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx = (n-1)!$

$\int_0^{\infty} x^n e^{-x} dx = n! \quad n>0$

$\int_0^{\infty} \frac{x^n}{e^x - 1} dx = n!\, \zeta(n+1)$

Sterling

$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx = n! = \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \quad n \gg 1.$

tricks $\int \frac{dx}{\sqrt{a^2 - x^2}}$ use $x = a\sin\theta \cdot \frac{dx}{d\theta} = a\cos\theta$

$\int \frac{a\cos\theta\, d\theta}{a\sqrt{1-\sin^2\theta}} = \int d\theta$

$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{d}{da}\left(\sqrt{\frac{\pi}{a}}\right), \int_{-\infty}^{\infty} x^4 e^{-ax^2} dx = \frac{d^2}{da}\sqrt{\frac{\pi}{a}}$

etc. $\boxed{\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = \frac{d^n}{da^n}\sqrt{\frac{\pi}{a}}}$ this only works for even 2n

odd $\int_{-\infty}^{\infty} x^3 e^{-x^2} dx = 0$

<u>integrals</u> $\boxed{\int_0^\infty x^3 e^{-x} dx} = (3!)$ ← ①

$\boxed{\int_0^\infty \frac{x^3}{e^x - 1}} dx$.. start by converting to $\int_0^\infty \frac{x^3 e^{-x}}{1-e^{-x}} dx \Rightarrow \int_0^\infty x^3 \sum_{n=1}^\infty e^{-nx} dx$

$\Rightarrow \sum_{n=1}^\infty \int_0^\infty x^3 e^{-nx} dx$. then use $z = nx \Rightarrow \sum_{n=1}^\infty \frac{1}{n^4} \int_0^\infty z^3 e^{-z} dz$

$\Rightarrow (3!) \sum_{n=1}^\infty \frac{1}{n^4} = \underline{(3!)\,\zeta(4)}$ keep like this.

$\boxed{\int_0^\infty e^{-x^4} dx}$ use $x^4 = y$ or $x = y^{\frac{1}{4}} \Rightarrow \frac{dx}{dy} = \frac{1}{4} y^{(\frac{1}{4}-1)}$

convert $\Rightarrow \frac{1}{4} \int_0^\infty e^{-y} y^{(\frac{1}{4}-1)} dy$ or $\frac{1}{4} \int_0^\infty y^{(\frac{1}{4}-1)} e^{-y} dy$. now compare to

$\boxed{\int_0^\infty y^{(s-1)} e^{-y} dy} = \Gamma(s)$. so $\boxed{\int_0^\infty y^{(\frac{1}{4}-1)} e^{-y} dy = \Gamma(\tfrac{1}{4})}$

Gamma

$\int_0^\infty e^{-\sqrt{x}} dx$. Similar to above. let $y = x^{\frac{1}{2}} \Rightarrow x = y^2 \Rightarrow \frac{dx}{dy} = 2y^{(2-1)}$

so $I = 2 \int_0^\infty y^{(2-1)} e^{-y} dy = 2\,\Gamma(2)$. but $\Gamma(n) = (n-1)! \Rightarrow \boxed{I = 2}$

$\zeta(s)\,\Gamma(s) = \int_0^\infty \frac{x^{(s-1)}}{e^x - 1} dx$. note $\underline{s > 1}$

Zeta $\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}$ $\underline{s > 1}$

So given $\int_0^\infty \frac{x^3}{e^x - 1} dx$, write as $\int_0^\infty \frac{x^{(4-1)}}{e^x - 1} dx = \Gamma(4)\,\zeta(4)$

$= \boxed{(3!)\,\zeta(4)}$

$\int_0^\infty x^n e^{-x} dx = n!$ | $\int_0^\infty x^{1-n} e^{-x} dx = \Gamma(n) = (n-1)!$

2

$\int \frac{dx}{\sqrt{a^2-x^2}} dx$. use $x = a\sin\theta$.

$\int \frac{dx}{x^2+a^2}$. use $x = a\tan\theta$.

$\int_0^\infty x e^{-ax^2} dx$. use $u = x^2$.

Gaussian

$\int_0^\infty e^{-ax^2} dx$. use $I = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ax^2} dx = \boxed{\frac{1}{2}\sqrt{\frac{\pi}{a}}}$

$\boxed{\int_0^\infty x^n e^{-ax^2} dx \text{ or } \int_{-\infty}^{\infty} x^n e^{-ax^2} dx}$ These are done by knowing that $I(a) = \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$. and then doing repeated $I'(a)$. on both sides. This works only for $\boxed{\text{even } n}$.

Example $\int_0^\infty x^2 e^{-ax^2} dx$. let $I(a) = \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

Then $I'(a) = -\int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{2}\frac{d}{da}\left(\sqrt{\pi}\, a^{-\frac{1}{2}}\right) = -\frac{1}{4}\sqrt{\pi}\, a^{-\frac{3}{2}}$.

so $\boxed{\int_0^\infty x^2 e^{-ax^2} dx = \frac{1}{4}\sqrt{\pi}\, a^{-\frac{3}{2}}}$

for odd n $\int_0^\infty x^3 e^{-ax^2} dx$. we know that $I(a) = \int_0^\infty x e^{-ax^2} dx = \frac{1}{2a}$.

Then $I'(a) = \int_0^\infty -x^3 e^{-ax^2} dx = \frac{d}{da}\frac{1}{2a}$. ✓

so $\int_0^\infty x^n e^{-ax^2} dx$. can be done for both odd and even n integrals of form $\left[\int_0^\infty x e^{-ax} \sin kx\,dx,\ \int_0^\infty x e^{-ax} \cos kx\,dx\right]$ ③

we start with $I(a) = \int_0^\infty e^{-ax} \sin kx\,dx = \frac{k}{a^2+k^2}$. Can be done using integration by parts.

Then evaluate $\underline{I'(a)}$.

Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.