Physics 3041 (Spring 2021) Solutions to Homework Set 9

1. Problem 10.2.8. (20 points)

(i) $x(0) = 1$, $\dot{x}(0) = 0$

$$(D^2 + 2D + 1)x(t) = 0 \Rightarrow \alpha^2 + 2\alpha + 1 = (\alpha + 1)^2 = 0,\ \alpha = -1 \text{ (repeated)}$$
$$x(t) = (A + Bt)e^{-t} \Rightarrow x(0) = A = 1$$
$$\dot{x} = Be^{-t} - (A + Bt)e^{-t} \Rightarrow \dot{x}(0) = B - A = 0,\ B = A = 1$$
$$x(t) = (1 + t)e^{-t}$$

(ii) $(D^4 + 1)x(t) = 0$

$$\alpha^4 + 1 = 0 \Rightarrow \alpha^4 = -1 = e^{i(2n+1)\pi},\ \alpha = e^{i(2n+1)\pi/4},\ n = 0,\ 1,\ 2,\ 3$$
$$\alpha_0 = e^{i\pi/4} = \frac{1+i}{\sqrt{2}},\ \alpha_1 = e^{i3\pi/4} = \frac{-1+i}{\sqrt{2}},\ \alpha_2 = e^{i5\pi/4} = -\frac{1+i}{\sqrt{2}},\ \alpha_3 = e^{i7\pi/4} = \frac{1-i}{\sqrt{2}}$$
$$x(t) = A_0 e^{\alpha_0 t} + A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + A_3 e^{\alpha_3 t} = A_0 e^{\frac{1+i}{\sqrt{2}}t} + A_1 e^{\frac{-1+i}{\sqrt{2}}t} + A_2 e^{-\frac{1+i}{\sqrt{2}}t} + A_3 e^{\frac{1-i}{\sqrt{2}}t}$$

(iii) $(D^3 - 3D^2 - 9D - 5)x(t) = 0$

$$\alpha^3 - 3\alpha^2 - 9\alpha - 5 = \alpha^3 + \alpha^2 - (4\alpha^2 + 9\alpha + 5) = \alpha^2(\alpha + 1) - (4\alpha + 5)(\alpha + 1) = 0$$
$$(\alpha + 1)(\alpha^2 - 4\alpha - 5) = (\alpha + 1)^2(\alpha - 5) = 0 \Rightarrow \alpha = -1 \text{ (repeated)},\ 5$$
$$x(t) = (A + Bt)e^{-t} + Ce^{5t}$$

(iv) $(D + 1)^2(D^4 - 256)x(t) = 0$

$$(\alpha + 1)^2(\alpha^4 - 256) = (\alpha + 1)^2(\alpha^2 - 16)(\alpha^2 + 16) = (\alpha + 1)^2(\alpha + 4)(\alpha - 4)(\alpha + 4i)(\alpha - 4i) = 0$$
$$\alpha = -1 \text{ (repeated)},\ -4,\ 4,\ -4i,\ 4i \Rightarrow x(t) = (A + Bt)e^{-t} + Ce^{-4t} + Fe^{4t} + Ge^{-i4t} + He^{i4t}$$

2. Problem 10.2.11. (40 points)

$y(0) = 1$, $\dot{y}(0) = 0$

(i) $\ddot{y} - \dot{y} - 2y = e^{2x}$

$$\alpha^2 - \alpha - 2 = (\alpha + 1)(\alpha - 2) = 0 \Rightarrow \alpha = -1,\ 2,\ y_c(x) = Ae^{-x} + Be^{2x}$$
$$y_p(x) = Cxe^{2x},\ (D^2 - D - 2)y_p(t) = C(D + 1)(D - 2)xe^{2x} = C(D + 1)e^{2x} = 3Ce^{2x} = e^{2x},\ C = \frac{1}{3}$$
$$y(x) = y_c(x) + y_p(x) = Ae^{-x} + Be^{2x} + \frac{xe^{2x}}{3} \Rightarrow y(0) = A + B = 1$$
$$y'(x) = -Ae^{-x} + 2Be^{2x} + \frac{(1 + 2x)e^{2x}}{3} \Rightarrow y'(0) = -A + 2B + \frac{1}{3} = 0$$
$$A = \frac{7}{9},\ B = \frac{2}{9},\ y(x) = \frac{7e^{-x} + 2e^{2x}}{9} + \frac{xe^{2x}}{3}$$

(ii) $(D^2-2D+1)y = 2\cos x = 2\text{Re}(e^{ix})$

$$\alpha^2-2\alpha+1=(\alpha-1)^2=0 \Rightarrow \alpha=1 \text{ (repeated)},\ y_c(x)=(A+Bx)e^x$$

$$y_p(x)=Ce^{ix},\ (D^2-2D+1)y_p(x)=C(-1-2i+1)e^{ix}=-2iCe^{ix}=2e^{ix}\Rightarrow C=i$$

$$y(x)=y_c(x)+\text{Re}[y_p(x)]=(A+Bx)e^x+\text{Re}(ie^{ix})=(A+Bx)e^x+\text{Re}(i\cos x-\sin x)$$

$$=(A+Bx)e^x-\sin x \Rightarrow y(0)=A=1$$

$$y'(x)=Be^x+(A+Bx)e^x-\cos x \Rightarrow y'(0)=B+A-1=0,\ B=0$$

$$y(x)=e^x-\sin x$$

(iii) $y''+16y=16\cos 4x=16\text{Re}(e^{i4x})$

$$\alpha^2+16=(\alpha+4i)(\alpha-4i)=0\Rightarrow \alpha=4i,\ -4i,\ y_c(x)=Ae^{i4x}+Be^{-i4x}$$

$$\begin{aligned}y_p(x)=Cxe^{i4x},\ (D^2+16)y_p(x)&=C(D+4i)(D-4i)xe^{i4x}\\&=C(D+4i)e^{i4x}=i8Ce^{i4x}=16e^{i4x},\ C=-2i\\ y(x)=y_c(x)+\text{Re}[y_p(x)]&=Ae^{i4x}+Be^{-i4x}+\text{Re}(-2ixe^{i4x})\\&=Ae^{i4x}+Be^{-i4x}+\text{Re}(-2ix\cos 4x+2x\sin 4x)\\&=Ae^{i4x}+Be^{-i4x}+2x\sin 4x\Rightarrow y(0)=A+B=1\end{aligned}$$

$$y'(x)=i4Ae^{i4x}-i4Be^{-i4x}+2\sin 4x+8x\cos 4x\Rightarrow y'(0)=i4A-i4B=0,\ A=B=\frac{1}{2}$$

$$y(x)=\frac{e^{i4x}+e^{-i4x}}{2}+2x\sin 4x=\cos 4x+2x\sin 4x$$

(iv) $y''-y=\cosh x=\frac{e^x+e^{-x}}{2}$

$$\alpha^2-1=(\alpha+1)(\alpha-1)\Rightarrow \alpha=-1,\ 1,\ y_c(x)=Ae^{-x}+Be^x$$

$$\begin{aligned}y_{p+}(x)=C_+xe^x,\ (D^2-1)y_{p+}(x)&=C_+(D+1)(D-1)xe^x\\&=C_+(D+1)e^x=2C_+e^x=\frac{e^x}{2},\ C_+=\frac{1}{4}\\ y_{p-}(x)=C_-xe^{-x},\ (D^2-1)y_{p-}(x)&=C_-(D-1)(D+1)xe^{-x}\\&=C_-(D-1)e^{-x}=-2C_-e^{-x}=\frac{e^{-x}}{2},\ C_-=-\frac{1}{4}\\ y(x)=y_c(x)+y_{p+}(x)+y_{p-}(x)&=Ae^{-x}+Be^x+\frac{x(e^x-e^{-x})}{4}\\&=Ae^{-x}+Be^x+\frac{x\sinh x}{2},\ y(0)=A+B=1\end{aligned}$$

$$y'(x)=-Ae^{-x}+Be^x+\frac{\sinh x+x\cosh x}{2},\ y'(0)=-A+B=0,\ A=B=\frac{1}{2}$$

$$y(x)=\frac{e^{-x}+e^x}{2}+\frac{x\sinh x}{2}=\cosh x+\frac{x\sinh x}{2}$$

3. Problem 10.3.5. (10 points)

$$x^2y'+2xy=\sinh x,\ y(1)=2$$

$$x^2y'+2xy=(x^2y)'=\sinh x\Rightarrow d(x^2y)=\sinh x\,dx,\ \int_{x=1}^{x=x}d(x^2y)=\int_1^x\sinh x\,dx$$

$$x^2y-(x^2y)_{x=1}=\cosh x-\cosh 1,\ x^2y-2=\cosh x-\cosh 1,\ y(x)=\frac{\cosh x-\cosh 1+2}{x^2}$$

4. Problem 10.3.8. (10 points)

$$(1+x^2)y' = 1+xy$$

$$y' - \frac{x}{1+x^2}y = \frac{1}{1+x^2},\ P(x) = -\int \frac{xdx}{1+x^2} = -\frac{1}{2}\int \frac{d(1+x^2)}{1+x^2} = -\frac{\ln(1+x^2)}{2} + C'$$

$$e^{-P(x)}\frac{d}{dx}[e^{P(x)}y(x)] = e^{\frac{1}{2}\ln(1+x^2)}\frac{d}{dx}[e^{-\frac{1}{2}\ln(1+x^2)}y(x)]$$

$$= \sqrt{1+x^2}\frac{d}{dx}\frac{y(x)}{\sqrt{1+x^2}} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\frac{y(x)}{\sqrt{1+x^2}} = \frac{1}{(1+x^2)^{3/2}},\ \frac{y(x)}{\sqrt{1+x^2}} = \int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{dx}{x^3(1+x^{-2})^{3/2}}$$

$$= -\frac{1}{2}\int \frac{d(1+x^{-2})}{(1+x^{-2})^{3/2}} = \frac{1}{\sqrt{1+x^{-2}}} + C$$

$$y(x) = \frac{\sqrt{1+x^2}}{\sqrt{1+x^{-2}}} + C\sqrt{1+x^2} = x + C\sqrt{1+x^2}$$

5. Problem 10.3.9. (20 points)

$$y' + p(x)y = q(x)y^m \Rightarrow y^{-m}y' + p(x)y^{1-m} = q(x),\ \frac{(y^{1-m})'}{1-m} + p(x)y^{1-m} = q(x)$$

$$v = y^{1-m} \Rightarrow v' + (1-m)p(x)v = (1-m)q(x)$$

(a) $y' + xy = xy^2$

$$\frac{y'}{y^2} + \frac{x}{y} = x,\ -\frac{d}{dx}\frac{1}{y} + \frac{x}{y} = x,\ v = \frac{1}{y} \Rightarrow v' - xv = -x$$

$$P(x) = -\int xdx = -\frac{x^2}{2} + C',\ e^{-P(x)}\frac{d}{dx}[e^{P(x)}v(x)] = e^{\frac{x^2}{2}}\frac{d}{dx}[e^{-\frac{x^2}{2}}v(x)] = -x$$

$$\frac{d}{dx}[e^{-\frac{x^2}{2}}v(x)] = -xe^{-\frac{x^2}{2}},\ e^{-\frac{x^2}{2}}v(x) = -\int xe^{-\frac{x^2}{2}}dx = \int e^{-\frac{x^2}{2}}d(-\frac{x^2}{2}) = e^{-\frac{x^2}{2}} + C$$

$$v(x) = 1 + Ce^{\frac{x^2}{2}} \Rightarrow y(x) = \frac{1}{v(x)} = \frac{1}{1+Ce^{\frac{x^2}{2}}}$$

(b) $3xy' + y + x^2y^4 = 0$

$$3x\frac{y'}{y^4} + \frac{1}{y^3} = -x^2,\ -x\frac{d}{dx}\frac{1}{y^3} + \frac{1}{y^3} = -x^2,\ v(x) = \frac{1}{y^3} \Rightarrow v' - \frac{v}{x} = x$$

$$P(x) = -\int \frac{dx}{x} = -\ln x + C',\ e^{-P(x)}\frac{d}{dx}[e^{P(x)}v(x)] = e^{\ln x}\frac{d}{dx}[e^{-\ln x}v(x)] = x\frac{d}{dx}\frac{v(x)}{x} = x$$

$$\frac{d}{dx}\frac{v(x)}{x} = 1,\ \frac{v(x)}{x} = \int dx = x + C,\ v(x) = x^2 + Cx \Rightarrow y(x) = \frac{1}{v(x)^{1/3}} = \frac{1}{(x^2+Cx)^{1/3}}$$