HW 9

Physics 3041
Mathematical Methods for Physicists

Spring 2021
University of Minnesota, Twin Cities

Nasser M. Abbasi

May 13, 2021 Compiled on May 13, 2021 at 10:35am

Contents

1 Problem 1 (10.2.8)

Problem 10.2.8. *Find the solutions to*

$$\begin{array}{lll} \text{(i)} & (D^2+2D+1)x(t)=0 & \text{with } x(0)=1,\ \dot{x}(0)=0 \\ \text{(ii)} & (D^4+1)x(t)=0 & \\ \text{(iii)} & (D^3-3D^2-9D-5)x(t)=0 & (5 \text{ is a root}) \\ \text{(iv)} & (D+1)^2(D^4-256)x(t)=0 & \end{array}$$

Figure 1: Problem statement

Solution

1.1 Part 1

The ode to solve is

$$\begin{aligned} x''(t)+2x'(t)+x(t)&=0 \\ x(0)&=1 \\ x'(0)&=0 \end{aligned} \tag{1}$$

This is a constant coefficient ODE. Assuming the solution has the form $x = Ae^{\lambda t}$ and substituting this back in (1) gives the characteristic equation (the constant A drops out)

$$\begin{aligned} \lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} &= 0 \\ \left(\lambda^2 + 42\lambda + 1\right)e^{\lambda t} &= 0 \end{aligned}$$

Since $e^{\lambda t} \neq 0$, the above gives

$$\begin{aligned} \lambda^2 + 2\lambda + 1 &= 0 \\ (\lambda+1)^2 &= 0 \end{aligned}$$

Therefore $\lambda = -1$. (double root). Since the root is double, then the basis solutions are $x_1(t) = e^{\lambda t}, x_2(t) = te^{\lambda t}$ and the general solution is a linear combination of these basis solutions. Therefore the general solution is

$$x(t) = Ae^{-t} + Bte^{-t} \tag{2}$$

The constants A, B are found from initial conditions. At $t = 0$ and using $x(0) = 1$ gives

$$1 = A \tag{3}$$

Solution (2) becomes

$$x(t) = e^{-t} + Bte^{-t} \tag{4}$$

Taking derivative of (4) gives

$$x'(t) = -e^{-t} + Be^{t} - Bte^{-t}$$

Using $x'(0) = 0$ on the above gives

$$\begin{aligned} 0 &= -1 + B \\ B &= 1 \end{aligned} \tag{5}$$

Substituting (3,5) in (4) gives the final solution

$$\begin{aligned} x(t) &= e^{-t} + te^{-t} \\ &= (1+t)e^{-t} \end{aligned}$$

1.2 Part 2

The ode to solve is

$$x''''(t) + x(t) = 0$$

As was done in the above part, substituting $x = Ae^{\lambda t}$ in the above and simplifying gives the characteristic equation

$$\lambda^4 + 1 = 0$$

Hence the roots are $\lambda^4 = -1$ or $\lambda^4 = e^{-i\frac{\pi}{2}}$. There are 4 roots that divide the unit circle equally, each is 90 degrees phase shifted (anti clockwise)from the other, starting from first root at phase $-\frac{\pi}{2} = -45$ degrees. Hence the roots are

$$\begin{aligned}
\lambda_1 &= \cos(-45) + i\sin(-45)\\
\lambda_2 &= \cos(45) + i\sin(45)\\
\lambda_3 &= \cos(135) + i\sin(135)\\
\lambda_4 &= \cos(225) + i\sin(225)
\end{aligned}$$

or

$$\begin{aligned}
\lambda_1 &= \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\\
\lambda_2 &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\\
\lambda_3 &= -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\\
\lambda_4 &= -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}
\end{aligned}$$

Therefore the basis solutions are

$$\begin{aligned}
x_1(t) &= e^{\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t}\\
x_2(t) &= e^{\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t}\\
x_3(t) &= e^{\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t}\\
x_4(t) &= e^{\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t}
\end{aligned}$$

The general solution is linear combination of the above basis solutions, which becomes

$$\begin{aligned}
x(t) &= c_1e^{\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t} + c_2e^{\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t} + c_3e^{\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)t} + c_4e^{\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)t}\\
&= c_1e^{\frac{\sqrt{2}}{2}t}e^{-i\frac{\sqrt{2}}{2}t} + c_2e^{\frac{\sqrt{2}}{2}t}e^{i\frac{\sqrt{2}}{2}t} + c_3e^{-\frac{\sqrt{2}}{2}t}e^{i\frac{\sqrt{2}}{2}t} + c_4e^{-\frac{\sqrt{2}}{2}t}e^{-i\frac{\sqrt{2}}{2}t}\\
&= e^{\frac{\sqrt{2}}{2}t}\left(c_1e^{-i\frac{\sqrt{2}}{2}t} + c_2e^{i\frac{\sqrt{2}}{2}t}\right) + e^{-\frac{\sqrt{2}}{2}t}\left(c_3e^{i\frac{\sqrt{2}}{2}t} + c_4e^{-i\frac{\sqrt{2}}{2}t}\right)
\end{aligned}$$

Using Euler relation, the above can be rewritten as

$$x(t) = e^{\frac{\sqrt{2}}{2}t}\left(c_1\sin\left(\frac{\sqrt{2}}{2}t\right) + c_2\cos\left(\frac{\sqrt{2}}{2}t\right)\right) + e^{-\frac{\sqrt{2}}{2}t}\left(c_3\sin\left(\frac{\sqrt{2}}{2}t\right) + c_4\cos\left(\frac{\sqrt{2}}{2}t\right)\right)$$

1.3 Part 3

The ode to solve is

$$x'''(t) - 3x''(t) - 9x'(t) - 5x(t) = 0$$

As was done in the above part, substituting $x = Ae^{\lambda t}$ in the above and simplifying gives the characteristic equation

$$\lambda^3 - 3\lambda^2 - 9\lambda - 5 = 0$$

Since one root is 5, then the above can be written as

$$(\lambda - 5)(\Delta) = 0$$

Where

$$\Delta = \frac{\lambda^3 - 3\lambda^2 - 9\lambda - 5}{\lambda - 5}$$

Using long division gives

$$\Delta = (\lambda + 1)^2$$

Therefore the roots of the characteristic equation are

$$\begin{aligned}\lambda_1 &= 5\\ \lambda_2 &= -1\\ \lambda_3 &= -1\end{aligned}$$

roots λ_2, λ_3 are the same. $\lambda = -1$ is a double root. Therefore the basis solutions are

$$\begin{aligned}x_1(t) &= e^{5t}\\ x_2(t) &= e^{t}\\ x_3(t) &= te^{t}\end{aligned}$$

Where t multiplies the last basis $x_3(t)$ due to the double root. The general solution is linear combination of the above basis solutions, which gives

$$\begin{aligned}x(t) &= c_1x_1(t) + c_2x_2(t) + c_3x_3(t)\\ &= c_1e^{5t} + c_2e^{t} + c_3te^{t}\end{aligned}$$

1.4 Part 4

The ode to solve is

$$(D+1)^2\left(D^4 - 256\right)x(t) = 0$$

This has the characteristic equation equation $(\lambda + 1)^2\left(\lambda^4 - 256\right) = 0$. The roots of $\left(\lambda^4 - 256\right)$ are given by $\lambda^4 = 256$. Let $\lambda^2 = \omega$. Therefore $\omega^2 = 256$ which gives $\omega = \pm 16$.

When $\omega = 16$, then $\lambda^2 = 16$ which gives $\lambda = \pm 4$ and when $\omega = -16$, then $\lambda^2 = -16$ which gives $\lambda = \pm 4i$.

The other part $(\lambda + 1)^2 = 0$ gives $\lambda = -1$, double root. Therefore the roots of the characteristic equation are

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= -4\\ \lambda_3 &= 4i\\ \lambda_4 &= -4i\\ \lambda_5 &= -1\\ \lambda_6 &= -1\end{aligned}$$

Root $\lambda = -1$ is a double root. Therefore the basis solutions as

$$\begin{aligned}x_1(t) &= e^{4t}\\ x_2(t) &= e^{-4t}\\ x_3(t) &= e^{4it}\\ x_4(t) &= e^{-4it}\\ x_5(t) &= e^{-t}\\ x_6(t) &= te^{-t}\end{aligned}$$

Where t was multiplied by e^{-t} in $x_6(t)$ since the root is double. The solution is linear combination of the above basis solutions, which gives

$$\begin{aligned}x(t) &= c_1x_1(t) + c_2x_2(t) + c_3x_3(t) + c_4x_4(t) + c_5x_5(t) + c_6x_6(t)\\ &= c_1e^{4t} + c_2e^{-4t} + c_3e^{4it} + c_4e^{-4it} + c_5e^{-t} + c_6te^{-t}\\ &= e^{-t}(c_5 + tc_6) + c_1e^{4t} + c_2e^{-4t} + c_3\sin(4t) + c_4\cos(4t)\end{aligned}$$

Where Euler relation was used in the last step above to rewrite $c_3e^{4it} + c_4e^{-4it}$.

2 Problem 2 (10.2.11)

Problem 10.2.11. *Solve the following subject to* $y(0) = 1,\ \dot{y}(0) = 0$

$$
\begin{array}{ll}
\textit{(i)} & \ddot{y} - \dot{y} - 2y = e^{2x} \\
\textit{(ii)} & (D^2 - 2D + 1)y = 2\cos x \\
\textit{(iii)} & y'' + 16y = 16\cos 4x \\
\textit{(iv)} & y'' - y = \cosh x
\end{array}
$$

Figure 2: Problem statement

<u>Solution</u>

2.1 Part 1

The ode to solve is

$$y'' - y' - 2y = e^{2x} \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. The general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' - y' - 2y = 0$ and $y_p(x)$ is any particular solution to $y'' - y' - 2y = e^{2x}$. The homogenous solution is found using the characteristic polynomial method as was done in the above problems. Substituting $y = Ae^{\lambda x}$ in $y'' - y' - 2y = 0$ and simplifying gives

$$
\begin{aligned}
\lambda^2 - \lambda - 2 &= 0 \\
(\lambda + 1)(\lambda - 2) &= 0
\end{aligned}
$$

The roots are $\lambda_1 = -1, \lambda_2 = 2$. Therefore the basis solutions are

$$
\begin{aligned}
y_1(x) &= e^{-x} \\
y_1(x) &= e^{2x}
\end{aligned} \tag{3}
$$

Hence $y_h(x)$ is linear combination of the above, which gives

$$y_h(x) = c_1 e^{-x} + c_2 e^{2x}$$

The particular solution is now found. Assuming $y_p = Ae^{2x}$. But e^{2x} is a basis solution of the homogeneous ode. Therefore y_p is multiplied by x giving

$$y_p = Axe^{2x}$$

Substituting this back in (1) and solving for A gives

$$
\begin{aligned}
y_p' &= Ae^{2x} + 2Axe^{2x} \\
y_p'' &= 2Ae^{2x} + 2Ae^{2x} + 4Axe^{2x} \\
&= 4Ae^{2x} + 4Axe^{2x}
\end{aligned}
$$

Eq (1) becomes

$$
\begin{aligned}
\left(4Ae^{2x} + 4Axe^{2x}\right) - \left(Ae^{2x} + 2Axe^{2x}\right) - 2\left(Axe^{2x}\right) &= e^{2x} \\
4Ae^{2x} + 4Axe^{2x} - Ae^{2x} - 2Axe^{2x} - 2Axe^{2x} &= e^{2x} \\
4A + 4Ax - A - 2Ax - 2Ax &= 1 \\
3A &= 1 \\
A &= \frac{1}{3}
\end{aligned}
$$

Hence the particular solution is

$$y_p(x) = \frac{1}{3}xe^{2x}$$

Therefore from (2) the general solution is

$$y(x) = c_1e^{-x} + c_2e^{2x} + \frac{1}{3}xe^{2x} \tag{4}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (4) becomes

$$1 = c_1 + c_2 \tag{5}$$

Taking derivative of (4) gives

$$y'(x) = -c_1e^{-x} + 2c_2e^{2x} + \frac{1}{3}e^{2x} + \frac{2}{3}xe^{2x}$$

At $x = 0$ the above gives

$$0 = -c_1 + 2c_2 + \frac{1}{3} \tag{6}$$

Eq (5,6) are now solved for c_1, c_2. From (5)

$$c_1 = 1 - c_2$$

Substituting this back in (6) gives

$$\begin{aligned} 0 &= -(1 - c_2) + 2c_2 + \frac{1}{3} \\ c_2 &= \frac{2}{9} \end{aligned}$$

Therefore $c_1 = 1 - \frac{2}{9} = \frac{7}{9}$. The final solution (4) becomes

$$y(x) = \frac{7}{9}e^{-x} + \frac{2}{9}e^{2x} + \frac{1}{3}xe^{2x}$$

2.2 Part 2

The ode to solve is

$$y'' - 2y' + y = 2\cos x \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. Hence the general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' - 2y' + y = 0$ and $y_p(x)$ is any particular solution to $y'' - 2y' + y = 2\cos x$. The homogenous is found using the characteristic polynomial method. Substituting $y = Ae^{\lambda x}$ in $y'' - 2y' + y = 0$ and simplifying gives

$$\begin{aligned} \lambda^2 - 2\lambda + 1 &= 0 \\ (\lambda - 1)(\lambda - 1) &= 0 \end{aligned}$$

roots are $\lambda_1 = 1, \lambda_2 = 1$. (double root). The basis solutions are therefore

$$\begin{aligned} y_1(x) &= e^x \\ y_1(x) &= xe^x \end{aligned} \tag{3}$$

$y_h(x)$ is linear combination of the the above which gives

$$y_h(x) = c_1e^x + c_2xe^x$$

The particular solution is now found. Assuming $y_p = A\cos x$. Taking all derivatives of this solution gives the set $\{\cos x, \sin x\}$. Therefore

$$y_p = A\cos x + B\sin x$$

Substituting this back in (1) to solve for A, B gives

$$\begin{aligned} y_p' &= -A\sin x + B\cos x \\ y_p'' &= -A\cos x - B\sin x \end{aligned}$$

Hence (1) becomes

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2\cos x \\ (-A\cos x - B\sin x) - 2(-A\sin x + B\cos x) + (A\cos x + B\sin x) &= 2\cos x \\ -A\cos x - B\sin x + 2A\sin x - 2B\cos x + A\cos x + B\sin x &= 2\cos x \\ \cos x(-A - 2B + A) + \sin x(-B + 2A + B) &= 2\cos x \\ -2B\cos x + 2A\sin x &= 2\cos x \end{aligned}$$

Hence $A = 0$ and $B = -1$. Therefore the particular solution is

$$y_p(x) = -\sin x$$

Eq (2) becomes

$$y(x) = c_1 e^x + c_2 x e^x - \sin x \tag{4}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (4) becomes

$$1 = c_1 \tag{5}$$

The solution (4) becomes

$$y(x) = e^x + c_2 x e^x - \sin x \tag{6}$$

Taking derivative of (6) gives

$$y'(x) = e^x + c_2 e^x + c_2 x e^x - \cos x$$

At $x = 0$ the above gives

$$0 = 1 + c_2 - 1 \tag{6}$$

Therefore $c_2 = 0$ and now Eq (6) gives the final solution as

$$y(x) = e^x - \sin x$$

2.3 Part 3

The ode to solve is

$$y'' + 16y = 16\cos 4x \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. Hence the general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' + 16y = 0$ and $y_p(x)$ is any particular solution to $y'' + 16y = 16\cos 4x$. The homogenous is found using the characteristic polynomial method. Substituting $y = Ae^{\lambda x}$ in $y'' + 16y = 0$ and simplifying gives

$$\begin{aligned} \lambda^2 + 16 &= 0 \\ \lambda &= \pm 4i \end{aligned}$$

The roots are $\lambda_1 = 4i, \lambda_2 = -4i$. The basis solutions are therefore

$$\begin{aligned} y_1(x) &= e^{i4x} \\ y_2(x) &= e^{-i4x} \end{aligned} \tag{3}$$

Therefore $y_h(x)$ is linear combination of the the above.

$$y_h(x) = c_1 e^{i4x} + c_2 e^{-i4x}$$

Which can be written, using Euler formula as

$$y_h(x) = c_1 \cos 4x + c_2 \sin 4x$$

The particular solution is now found. Assuming $y_p = A\cos 4x$. Taking all derivatives of this, the basis for y_p becomes $\{\cos 4x, \sin 4x\}$. But $\cos 4x$ is a basis of y_h. Therefore this set is multiplied by x. The whole set is multiplied by x and not just $\cos 4x$ because the set was generated by taking derivative of $\cos 4x$.

The basis set for y_p now becomes $\{x\cos 4x, x\sin 4x\}$. Hence y_p is linear combination of these basis, giving trial y_p as

$$y_p = Ax\cos 4x + Bx\sin 4x \tag{4}$$

Therefore

$$\begin{aligned} y_p' &= (A\cos 4x - 4Ax\sin 4x) + (B\sin 4x + 4Bx\cos 4x) \\ y_p'' &= (-4A\sin 4x - 4A\sin 4x - 16Ax\cos 4x) + (4B\cos 4x + 4B\cos 4x - 16Bx\sin 4x) \\ &= -8A\sin 4x - 16Ax\cos 4x + 8B\cos 4x - 16Bx\sin 4x \end{aligned}$$

Substituting the above back in (1) gives

$$\begin{aligned} (-8A\sin 4x - 16Ax\cos 4x + 8B\cos 4x - 16Bx\sin 4x) + 16(Ax\cos 4x + Bx\sin 4x) &= 16\cos 4x \\ \sin 4x(-8A - 16Bx + 16Bx) + \cos 4x(-16Ax + 8B + 16Ax) &= 16\cos 4x \end{aligned}$$

Hence

$$\begin{aligned} -16Ax + 8B + 16Ax &= 16 \\ -8A - 16Bx + 16Bx &= 0 \end{aligned}$$

Or

$$\begin{aligned} 8B &= 16 \\ -8A &= 0 \end{aligned}$$

First equation gives $B = 2$. Second equation gives $A = 0$. Therefore the particular solution (4) becomes

$$y_p = 2x\sin 4x$$

From (2), the general solution becomes

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1\cos 4x + c_2\sin 4x + 2x\sin 4x \end{aligned} \tag{5}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (5) becomes

$$1 = c_1$$

Hence the solution (5) becomes

$$y(x) = \cos 4x + c_2\sin 4x + 2x\sin 4x \tag{6}$$

Taking derivative of the above

$$y'(x) = -4\sin 4x + 4c_2\cos 4x + 2\sin 4x + 8x\cos 4x$$

At $t = 0$ the above gives

$$0 = 4c_2$$

Hence $c_2 = 0$ and the final solution (6) becomes

$$y(x) = \cos 4x + 2x\sin 4x$$

2.4 Part 4

The ode to solve is

$$y'' - y = \cosh x \tag{1}$$

This is second order constant coefficients inhomogeneous ODE. Hence the general solution is

$$y(x) = y_h(x) + y_p(x) \tag{2}$$

Where $y_h(x)$ is the solution to $y'' + y = 0$ and $y_p(x)$ is any particular solution to $y'' + y = \cosh x$. The homogenous is found using the characteristic polynomial method. Substituting $y = Ae^{\lambda x}$ in $y'' + y = 0$ and simplifying gives

$$\begin{aligned}\lambda^2 - 1 &= 0\\ \lambda &= \pm 1\end{aligned}$$

roots are $\lambda_1 = 1, \lambda_2 = -1$. The basis solutions are therefore

$$\begin{aligned}y_1(x) &= e^x\\ y_2(x) &= e^{-x}\end{aligned} \tag{3}$$

Therefore $y_h(x)$ is linear combination of the the above.

$$y_h(x) = c_1 e^x + c_2 e^{-x}$$

Which can be written, using Euler formula as

$$y_h(x) = c_1 \cosh x + c_2 \sinh x$$

The particular solution is now found. Assuming $y_p = A\cosh x$. Taking all derivatives of this, the basis for y_p becomes $\{\cosh x, \sinh x\}$. But $\cosh x$ is basis of y_h. Therefore this set is multiplied by x. The whole set is multiplied by x and not just $\cosh x$ because the set was generated by taking derivative of $\cosh x$.

The basis set for y_p becomes $\{x\cosh x, x\sinh x\}$. Hence y_p is linear combination of these basis, giving trial y_p as

$$y_p = Ax\cosh x + Bx\sinh x \tag{4}$$

Therefore

$$\begin{aligned}y_p' &= A\cosh x + Ax\sinh x + B\sinh x + Bx\cosh x\\ y_p'' &= A\sinh x + A\sinh x + Ax\cosh x + B\cosh x + B\cosh x + Bx\sinh x\\ &= 2A\sinh x + Ax\cosh x + 2B\cosh x + Bx\sinh x\end{aligned}$$

Substituting the above back in (1) gives

$$\begin{aligned}(2A\sinh x + Ax\cosh x + 2B\cosh x + Bx\sinh x) - (Ax\cosh x + Bx\sinh x) &= \cosh x\\ \sinh x(2A + Bx - Bx) + \cosh x(Ax + 2B - Ax) &= \cosh x\end{aligned}$$

Hence

$$\begin{aligned}2B &= 1\\ 2A &= 0\end{aligned}$$

Therefore $B = \frac{1}{2}, A = 0$ and (4) becomes

$$y_p = \frac{1}{2}x\sinh x$$

From (2), the general solution becomes

$$\begin{aligned}y(x) &= y_h(x) + y_p(x)\\ &= c_1\cosh x + c_2\sinh x + \frac{1}{2}x\sinh x\end{aligned} \tag{5}$$

c_1, c_2 are now found from initial conditions. At $x = 0$, (5) becomes

$$1 = c_1$$

Hence the solution (5) becomes

$$y(x) = \cosh x + c_2 \sinh x + \frac{1}{2} x \sinh x \tag{6}$$

Taking derivative of the above

$$y'(x) = \sinh x + c_2 \cosh x + \frac{1}{2} \sinh x + \frac{1}{2} x \cosh x$$

At $t = 0$ the above gives

$$0 = c_2 \cosh x$$

Hence $c_2 = 0$ and the final solution (6) becomes

$$y(x) = \cosh x + \frac{1}{2} x \sinh x$$

3 Problem 3 (10.3.5)

Solve $x^2y' + 2xy = \sinh x$ with $y(1) = 2$

Solution

Dividing by $x \neq 0$

$$y' + 2\frac{y}{x} = \frac{\sinh x}{x^2}$$

The integrating factor is $I = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$. Multiplying both sides by this integration factor makes the left side a complete differential

$$\begin{aligned}\frac{d}{dx}\left(yx^2\right) &= x^2\frac{\sinh x}{x^2}\\ \frac{d}{dx}\left(yx^2\right) &= \sinh x\end{aligned}$$

Integrating gives

$$\begin{aligned}yx^2 &= \int \sinh x dx + C\\ yx^2 &= \cosh x + C\\ y &= \frac{\cosh x}{x^2} + \frac{C}{x^2}\end{aligned} \tag{1}$$

At $x = 1$ the above becomes

$$\begin{aligned}2 &= \cosh 1 + C\\ C &= 2 - \cosh 1\end{aligned}$$

Hence the solution (1) becomes

$$\begin{aligned}y(x) &= \frac{\cosh x}{x^2} + \frac{2 - \cosh 1}{x^2}\\ &= \frac{1}{x^2}(\cosh x + 2 - \cosh 1)\end{aligned}$$

Where $x \neq 0$

4 Problem 4 (10.3.8)

Solve

$$\left(1+x^2\right)y' = 1+xy$$

Solution

$$\begin{aligned} y' &= \frac{1+xy}{1+x^2} \\ &= \frac{1}{1+x^2} + \frac{xy}{1+x^2} \end{aligned}$$

Therefore

$$y' - y\frac{x}{1+x^2} = \frac{1}{1+x^2} \tag{1}$$

This is linear in y first order ODE. It has the form $y' + p(x)y = q(x)$. The integration factor is

$$\begin{aligned} I &= e^{\int p(x)dx} \\ &= e^{-\int \frac{x}{1+x^2}dx} \end{aligned}$$

But $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln\left(1+x^2\right)$. Therefore

$$\begin{aligned} I &= e^{-\frac{1}{2}\ln\left(1+x^2\right)} \\ &= e^{\ln\left(1+x^2\right)^{-\frac{1}{2}}} \\ &= \left(1+x^2\right)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{1+x^2}} \end{aligned}$$

Multiplying both sides of (1) by this integrating factor makes the left side a complete differential

$$\begin{aligned} \frac{d}{dx}\left(y\frac{1}{\sqrt{1+x^2}}\right) &= \frac{1}{\sqrt{1+x^2}}\frac{1}{1+x^2} \\ \frac{d}{dx}\left(y\frac{1}{\sqrt{1+x^2}}\right) &= \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}} \\ &= \left(1+x^2\right)^{-\frac{3}{2}} \end{aligned}$$

Integrating gives

$$y\frac{1}{\sqrt{1+x^2}} = \int \left(1+x^2\right)^{-\frac{3}{2}}dx + C \tag{2}$$

To integrate $\int \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}} dx$, let $x = \tan u$, then $dx = \left(1 + \tan^2 u\right) du$. Hence

$$
\begin{aligned}
\int \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}} dx &= \int \frac{1}{\left(1+\tan^2 u\right)^{\frac{3}{2}}} \left(1 + \tan^2 u\right) du \\
&= \int \frac{1}{\left(1+\tan^2 u\right)^{\frac{1}{2}}} du \\
&= \int \frac{1}{\left(1+\frac{\sin^2 u}{\cos^2 u}\right)^{\frac{1}{2}}} du \\
&= \int \frac{\cos u}{\left(\cos^2 u + \sin^2 u\right)^{\frac{1}{2}}} du \\
&= \int \cos u \, du \\
&= \sin u
\end{aligned}
$$

But $\sin u = \frac{\frac{\sin u}{\cos u}}{\sqrt{1+\frac{\sin^2 u}{\cos^2 u}}} = \frac{\tan u}{\sqrt{1+\tan^2 u}} = \frac{x}{\sqrt{1+x^2}}$. Hence

$$
\int \frac{1}{\left(1+x^2\right)^{\frac{3}{2}}} dx = \frac{x}{\sqrt{1+x^2}}
$$

Therefore the final solution (2) becomes

$$
\begin{aligned}
y \frac{1}{\sqrt{1+x^2}} &= \frac{x}{\sqrt{1+x^2}} + C \\
y &= x + C\sqrt{1+x^2} \qquad (3)
\end{aligned}
$$

5 Problem 5 (10.3.9)

Solve (a) $y' + xy = xy^2$ (b) $3xy' + y + x^2y^4 = 0$

Solution

5.1 Part a

The ode has the form

$$y' + p(x)y = q(x)y^m$$

Where $p(x) = x, q(x) = x$ and $m = 2$. Therefore this is Bernoulli ODE. The first step is to divide throughout by $y^m = y^2$ which gives

$$\frac{y'}{y^2} + p(x)y^{-1} = q(x) \tag{1}$$

Setting

$$v(x) = y^{-1} \tag{2}$$

Taking derivatives of the above w.r.t. x gives

$$v'(x) = \frac{-1}{y^2}y'(x) \tag{3}$$

Substituting (2,3) into (1) gives

$$-v'(x) + p(x)v(x) = q(x)$$

But here $p(x) = x$ and $q(x) = x$. The above becomes

$$\begin{aligned} -v'(x) + xv(x) &= x \\ v'(x) - xv(x) &= -x \end{aligned}$$

This is linear ODE in $v(x)$. The integrating factor is $e^{\int -xdx} = e^{-\frac{x^2}{2}}$. Multiplying both sides of the above by this integrating factor makes the left side a complete differential

$$\frac{d}{dx}\left(ve^{-\frac{x^2}{2}}\right) = -xe^{-\frac{x^2}{2}}$$

Integrating gives

$$ve^{-\frac{x^2}{2}} = -\int xe^{-\frac{x^2}{2}}dx + C \tag{4}$$

To integrate $\int xe^{-\frac{x^2}{2}}dx$, let $u = x^2$. Then $du = 2xdx$. Substituting gives

$$\begin{aligned} \int xe^{-\frac{x^2}{2}}dx &= \int xe^{-\frac{u}{2}}\frac{du}{2x} \\ &= \frac{1}{2}\int e^{-\frac{u}{2}}du \\ &= \frac{1}{2}\frac{e^{-\frac{u}{2}}}{-\frac{1}{2}} \\ &= -e^{-\frac{u}{2}} \end{aligned}$$

But $u = x^2$. Therefore

$$\int xe^{-\frac{x^2}{2}}dx = -e^{-\frac{x^2}{2}}$$

Substituting the above in (4) gives

$$
\begin{aligned}
ve^{-\frac{x^2}{2}} &= e^{-\frac{x^2}{2}} + C \\
v &= 1 + e^{\frac{x^2}{2}} C
\end{aligned}
$$

But $v = y^{-1}$, therefore

$$
\begin{aligned}
y^{-1} &= 1 + e^{\frac{x^2}{2}} C \\
y(x) &= \frac{1}{1 + e^{\frac{x^2}{2}} C}
\end{aligned}
$$

Where C is constant of integration.

5.2 Part b

The ode is

$$
3xy' + y + x^2y^4 = 0
$$

Dividing by $3x$ for $x \neq 0$ gives

$$
\begin{aligned}
y' + \frac{y}{3x} + \frac{x}{3}y^4 &= 0 \\
y' + \frac{1}{3x}y &= -\frac{x}{3}y^4
\end{aligned}
$$

Now this ODE has the Bernoulli form,

$$
y' + p(x)y = q(x)y^m
$$

Where $p(x) = \frac{1}{3x}, q(x) = -\frac{x}{3}$ and $m = 4$. Therefore this is Bernoulli ODE. The first step is to divide throughout by $y^m = y^4$ which gives

$$
\frac{y'}{y^4} + p(x)y^{-3} = q(x) \tag{1}
$$

Setting

$$
v(x) = y^{-3} \tag{2}
$$

Taking derivatives of the above w.r.t. x gives

$$
v'(x) = \frac{-3}{y^4}y'(x) \tag{3}
$$

Substituting (2,3) into (1) gives

$$
-\frac{1}{3}v'(x) + p(x)v(x) = q(x)
$$

But here $p(x) = \frac{1}{3x}, q(x) = -\frac{x}{3}$. The above becomes

$$
\begin{aligned}
-\frac{1}{3}v'(x) + \frac{1}{3x}v(x) &= -\frac{x}{3} \\
v'(x) - \frac{1}{x}v(x) &= x
\end{aligned}
$$

This is linear in $v(x)$. The integrating factor is $e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$. Multiplying both sides of the above by this integrating factor make the left side a complete differential

$$
\frac{d}{dx}\left(v\frac{1}{x}\right) = 1
$$

Integrating gives

$$\begin{aligned} v\frac{1}{x} &= x + C \\ v &= x^2 + xC \end{aligned} \tag{4}$$

But $v(x) = y^{-3}$. Therefore the above becomes

$$\begin{aligned} y^{-3} &= x^2 + xC \\ y^3(x) &= \frac{1}{x^2 + xC} \end{aligned}$$

Or

$$y(x) = \left(x^2 + xC\right)^{-\frac{1}{3}}$$