Physics 3041 (Spring 2021) Solutions to Homework Set 8

1. Problem 10.4.3. (20 points)

$$H_n(y) = \sum_{m=0}^{n} a_m y^m,\ a_{n-1} = 0,\ \epsilon = n + \frac{1}{2},\ a_{m+2} = \frac{1+2m-2\epsilon}{(m+2)(m+1)} a_m = \frac{2(m-n)}{(m+2)(m+1)} a_m$$

$$n = 0 \Rightarrow H_0 = a_0 \rightarrow H_0 = 1,\ n = 1 \Rightarrow H_1 = a_1 y \rightarrow H_1 = 2y$$

$$n = 2 \Rightarrow a_2 = \frac{2(-2)}{2} a_0 = -2a_0,\ H_2 = (1-2y^2)a_0 \rightarrow H_2 = -2(1-2y^2)$$

$$n = 3 \Rightarrow a_3 = \frac{2(1-3)}{3\cdot 2} a_1 = -\frac{2}{3} a_1,\ H_3 = (y - \frac{2}{3}y^3)a_1 \rightarrow H_3 = -12(y - \frac{2}{3}y^3)$$

$$\int_{-\infty}^{\infty} [H_0(y)]^2 e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} [H_1(y)]^2 e^{-\alpha y^2} dy = 4\int_{-\infty}^{\infty} y^2 e^{-\alpha y^2} dy = -4\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty} e^{-\alpha y^2} dy = -4\frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty} e^{-z^2} d\frac{z}{\sqrt{\alpha}}$$

$$= -4\frac{\partial}{\partial\alpha}\frac{\sqrt{\pi}}{\sqrt{\alpha}} = \frac{4\sqrt{\pi}}{2\alpha^{3/2}} = \frac{2\sqrt{\pi}}{\alpha^{3/2}} \Rightarrow \int_{-\infty}^{\infty} [H_1(y)]^2 e^{-y^2} dy = 2\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} [H_2(y)]^2 e^{-y^2} dy = 4\int_{-\infty}^{\infty} (1-2y^2)^2 e^{-y^2} dy = 4\int_{-\infty}^{\infty} (1-4y^2+4y^4) e^{-y^2} dy$$

$$= 4[\sqrt{\pi} - 2\sqrt{\pi} + 4(\frac{\partial^2}{\partial\alpha^2}\frac{\sqrt{\pi}}{\sqrt{\alpha}})_{\alpha=1}] = 4\sqrt{\pi}(1 - 2 + \frac{4\cdot 3}{2\cdot 2}) = 8\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} H_0(y)H_2(y)e^{-y^2} dy = -2\int_{-\infty}^{\infty} (1-2y^2)e^{-y^2} dy = -2(\sqrt{\pi} - \frac{2\sqrt{\pi}}{2}) = 0$$

$$\int_{-\infty}^{\infty} H_0(y)H_1(y)e^{-y^2} dy = 0,\ \int_{-\infty}^{\infty} H_1(y)H_2(y)e^{-y^2} dy = 0$$

The last two results are straightforward because the integrands are odd functions of $y$. The general result is $\int_{-\infty}^{\infty} H_m(y)H_n(y)e^{-y^2} dy = \delta_{mn}(2^n n!\sqrt{\pi}\,)$.

2. Problem 10.4.4. (30 points)

$$y = \sum_{m=0}^{n} c_m x^m,\ \ y' = \sum_{m=0}^{n} m c_m x^{m-1},\ \ y'' = \sum_{m=0}^{n} m(m-1) c_m x^{m-2}$$

$$\begin{aligned}
(1-x^2)y'' - 2xy' + l(l+1)y &= \sum_{m=0}^{n} m(m-1)c_m x^{m-2} + \sum_{m=0}^{n}[-m(m-1) - 2m + l(l+1)]c_m x^m \\
&= \sum_{m=2}^{n} m(m-1)c_m x^{m-2} + \sum_{m=0}^{n}[-m(m-1) - 2m + l(l+1)]c_m x^m \\
&= \sum_{m'=0}^{n-2} (m'+2)(m'+1)c_{m'+2} x^{m'} + \sum_{m=0}^{n}[-m(m-1) - 2m + l(l+1)]c_m x^m \\
&= \sum_{m=0}^{n-2}\{(m+2)(m+1)c_{m+2} + [l(l+1) - m(m+1)]c_m\}x^m \\
&+ \sum_{m=n-1}^{n} [l(l+1) - m(m+1)]c_m x^m = 0
\end{aligned}$$

For $m = n$, $[l(l+1) - n(n+1)]c_n = 0 \Rightarrow n = l$ because $c_n \neq 0$ by the definition of $x^n$ as the term of the highest power. The other possibility of $n = -l-1$ is discarded because we are looking for solutions of $n \geq 0$.

For $m = n-1$, $[l(l+1) - n(n-1)]c_{n-1} = [l(l+1) - l(l-1)]c_{n-1} = 2lc_{n-1} = 0 \Rightarrow c_{n-1} = 0$.

For $0 \leq m \leq n-2$,

$$(m+2)(m+1)c_{m+2} + [l(l+1) - m(m+1)]c_m = 0 \Rightarrow c_{m+2} = \frac{m(m+1) - l(l+1)}{(m+2)(m+1)} c_m.$$

Therefore, for an even (odd) $l$, all the odd (even) terms vanish because $c_{l-1} = 0$ and the solution is a polynomial of the $l$th order with even (odd) terms only, defined as $P_l$.

$$\begin{aligned}
&l = 0 \Rightarrow P_0 = c_0 \to P_0 = 1 \\
&l = 1 \Rightarrow P_1 = c_1 x \to P_1 = x \\
&l = 2 \Rightarrow c_2 = \frac{-2\cdot 3}{2}c_0 = -3c_0,\ P_2 = c_0(1-3x^2) \to P_2 = \frac{1}{2}(3x^2-1) \\
&l = 3 \Rightarrow c_3 = \frac{2 - 3\cdot 4}{3\cdot 2}c_1 = -\frac{5}{3}c_1,\ P_3 = c_1(x - \frac{5}{3}x^3) \to P_3 = \frac{1}{2}(5x^3 - 3x)
\end{aligned}$$

Clearly, $\int_{-1}^{1} P_l(x)P_{l'}(x)dx = 0$ for even-odd or odd-even pairs of $l$ and $l'$.

$$\int_{-1}^{1} P_0(x)P_2(x)dx = \frac{1}{2}\int_{-1}^{1}(3x^2-1)dx = \int_{0}^{1}(3x^2-1)dx = x^3|_0^1 - x|_0^1 = 0$$
$$\int_{-1}^{1} P_1(x)P_3(x)dx = \frac{1}{2}\int_{-1}^{1}x(5x^3-3x)dx = \int_{0}^{1}x(5x^3-3x)dx = x^5|_0^1 - x^3|_0^1 = 0$$

3. Problem 10.4.5. (20 points)

$$
\begin{aligned}
|I\rangle &= 1 \Rightarrow \langle I|I\rangle = \int_{-1}^{1} dx = 2,\ |1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}} = \frac{1}{\sqrt{2}} \to P_0 = 1 \\
|II\rangle &= x \Rightarrow |II'\rangle = |II\rangle - |1\rangle\langle 1|II\rangle = x - \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{x}{\sqrt{2}} dx = x \\
\langle II'|II'\rangle &= \int_{-1}^{1} x^2 dx = \frac{2}{3},\ |2\rangle = \frac{|II'\rangle}{\sqrt{\langle II'|II'\rangle}} = x\sqrt{\frac{3}{2}} \to P_1 = x \\
|III\rangle &= x^2 \Rightarrow |III'\rangle = |III\rangle - |1\rangle\langle 1|III\rangle - |2\rangle\langle 2|III\rangle \\
&= x^2 - \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{x^2}{\sqrt{2}} dx - x\sqrt{\frac{3}{2}} \int_{-1}^{1} x^3 \sqrt{\frac{3}{2}}\, dx = x^2 - \frac{1}{3} \\
\langle III'|III'\rangle &= \int_{-1}^{1} (x^2 - \frac{1}{3})^2 dx = \int_{-1}^{1} (x^4 - \frac{2}{3}x^2 + \frac{1}{9}) dx = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45} \\
|3\rangle &= \frac{|III'\rangle}{\sqrt{\langle III'|III'\rangle}} = (x^2 - \frac{1}{3})\sqrt{\frac{45}{8}} = (3x^2 - 1)\sqrt{\frac{5}{8}} \to P_2 = \frac{1}{2}(3x^2 - 1) \\
|IV\rangle &= x^3 \Rightarrow |IV'\rangle = |IV\rangle - |1\rangle\langle 1|IV\rangle - |2\rangle\langle 2|IV\rangle - |3\rangle\langle 3|IV\rangle \\
&= x^3 - \frac{1}{\sqrt{2}} \int_{-1}^{1} \frac{x^3}{\sqrt{2}} dx - x\sqrt{\frac{3}{2}} \int_{-1}^{1} x^4 \sqrt{\frac{3}{2}}\, dx - (3x^2 - 1)\sqrt{\frac{5}{8}} \int_{-1}^{1} x^3 (3x^2 - 1)\sqrt{\frac{5}{8}}\, dx \\
&= x^3 - x\frac{3}{2}\frac{2}{5} = x^3 - \frac{3}{5}x \\
\langle IV'|IV'\rangle &= \int_{-1}^{1} (x^3 - \frac{3}{5}x)^2 dx = \int_{-1}^{1} (x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2) dx = \frac{2}{7} - \frac{12}{25} + \frac{6}{25} = \frac{8}{175} \\
|4\rangle &= \frac{|IV'\rangle}{\sqrt{\langle IV'|IV'\rangle}} = (x^3 - \frac{3}{5}x)\sqrt{\frac{175}{8}} = (5x^3 - 3x)\sqrt{\frac{7}{8}} \to P_3 = \frac{1}{2}(5x^3 - 3x)
\end{aligned}
$$

Following the above procedure, the general result is $|i = l + 1\rangle = P_l \sqrt{\frac{2l+1}{2}}$.

4. Problem 10.4.10. (30 points)

$$y = x^s \sum_{n=0}^{\infty} c_n x^n,\ y' = \sum_{n=0}^{\infty} (n+s) c_n x^{n+s-1},\ y'' = \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n x^{n+s-2}$$

$$\begin{aligned} xy'' + (1-x)y' + my &= \sum_{n=0}^{\infty} [(n+s)(n+s-1) + (n+s)] c_n x^{n+s-1} + \sum_{n=0}^{\infty} [-(n+s) + m] c_n x^{n+s} \\ &= \sum_{n=0}^{\infty} (n+s)^2 c_n x^{n+s-1} + \sum_{n=0}^{\infty} [m - (n+s)] c_n x^{n+s} = 0 \end{aligned}$$

For $n = 0$, $c_n = c_0 \neq 0$ for the lowest term $x^{s-1}$, so $(n+s)^2 = s^2 = 0 \Rightarrow s = 0$ (repeated). The regular solution is

$$\begin{aligned} y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow \sum_{n=0}^{\infty} n^2 c_n x^{n-1} + \sum_{n=0}^{\infty} (m-n) c_n x^n &= \sum_{n=1}^{\infty} n^2 c_n x^{n-1} + \sum_{n=0}^{\infty} (m-n) c_n x^n \\ = \sum_{n'=0}^{\infty} (n'+1)^2 c_{n'+1} x^{n'} + \sum_{n=0}^{\infty} (m-n) c_n x^n &= \sum_{n=0}^{\infty} [(n+1)^2 c_{n+1} + (m-n) c_n] x^n = 0 \end{aligned}$$

So we have

$$c_{n+1} = \frac{n-m}{(n+1)^2} c_n,$$

which indicates that the series stops at $n = m$ with the highest term $x^m$ for integer $m$.

$$m = 0 \Rightarrow y = c_0,\ \int_0^\infty y^2 e^{-x} dx = c_0^2 = 1,\ c_0 = 1 \to L_0 = 1$$

$$m = 1 \Rightarrow c_1 = -c_0,\ y = c_0(1-x)$$

$$\int_0^\infty y^2 e^{-x} dx = c_0^2 \int_0^\infty (1 - 2x + x^2) e^{-x} dx = c_0^2 (1 - 2 + 2) = c_0^2 = 1,\ c_0 = 1 \to L_1 = 1 - x$$

$$m = 2 \Rightarrow c_1 = -2c_0,\ c_2 = -\frac{1}{4} c_1 = \frac{1}{2} c_0,\ y = c_0(1 - 2x + \frac{1}{2}x^2)$$

$$\begin{aligned} \int_0^\infty y^2 e^{-x} dx &= c_0^2 \int_0^\infty (1 - 2x + \frac{1}{2}x^2)^2 e^{-x} dx = c_0^2 \int_0^\infty (1 - 4x + 5x^2 - 2x^3 + \frac{1}{4}x^4) e^{-x} dx \\ &= c_0^2 (1 - 4 + 5 \cdot 2 - 2 \cdot 6 + \frac{24}{4}) = c_0^2 = 1,\ c_0 = 1 \to L_2 = 1 - 2x + \frac{1}{2}x^2 \end{aligned}$$

$$m = 3 \Rightarrow c_1 = -3c_0,\ c_2 = -\frac{1}{2} c_1 = \frac{3}{2} c_0,\ c_3 = -\frac{1}{9} c_2 = -\frac{1}{6} c_0,\ y = c_0(1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3)$$

$$\begin{aligned} \int_0^\infty y^2 e^{-x} dx &= c_0^2 \int_0^\infty (1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3)^2 e^{-x} dx \\ &= c_0^2 \int_0^\infty (1 - 6x + 12x^2 - \frac{28}{3}x^3 + \frac{13}{4}x^4 - \frac{1}{2}x^5 + \frac{1}{36}x^6) e^{-x} dx \\ &= c_0^2 (1 - 6 + 12 \cdot 2 - \frac{28}{3} \cdot 6 + \frac{13}{4} \cdot 24 - \frac{1}{2} \cdot 120 + \frac{1}{36} \cdot 720) = c_0^2 = 1,\ c_0 = 1 \\ &\to L_3 = 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3 \end{aligned}$$

$$\begin{aligned} \int_0^\infty L_1 L_2 e^{-x} dx &= \int_0^\infty (1-x)(1 - 2x + \frac{1}{2}x^2) e^{-x} dx = \int_0^\infty (1 - 3x + \frac{5}{2}x^2 - \frac{1}{2}x^3) e^{-x} dx \\ &= 1 - 3 + \frac{5}{2} \cdot 2 - \frac{1}{2} \cdot 6 = 0 \end{aligned}$$