HW 8

Physics 3041
Mathematical Methods for Physicists

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Contents

1 Problem 1 (10.4.3)

Problem 10.4.3. *Show that the first four Hermite polynomials are*

$$H_0 = 1 \tag{10.4.35}$$
$$H_1 = 2y \tag{10.4.36}$$
$$H_2 = -2(1-2y^2) \tag{10.4.37}$$
$$H_3 = -12(y - \frac{2}{3}y^3) \tag{10.4.38}$$

where the overall normalization (choice of a_0 or a_1) is as per some convention we need not get into. To compare your answers to the above, choose the starting coefficients to agree with the above. Show that

$$\int_{-\infty}^{\infty} e^{-y^2} H_n(y) H_m(y) dy = \delta_{nm}(\sqrt{\pi}2^n n!) \tag{10.4.39}$$

for the cases $m, n \leq 2$. Notice that the Hermite polynomials are not themselves orthogonal or even normalizable, we need the weight function e^{-y^2} *in the integration measure. We understand this is follows: the exponential factor converts u's to ψ's, which are the eigenfunctions of a hermitian operator (hermitian with respect to normalizable function that vanished at infinity) and hence orthogonal for different eigenvalues.*

Figure 1: Problem statement

Solution

1.1 Part 1

Starting with ode (10.4.12) which is

$$\psi''(y) - y^2\psi(y) = -2\epsilon\psi(y) \tag{10.4.12}$$

Where $\epsilon = \frac{E}{\hbar\omega}$ the energy of the particle. Let the solution be

$$\begin{aligned}\psi(y) &= u(y)e^{\frac{-y^2}{2}} \\ &= e^{\frac{-y^2}{2}} \sum_{m=0}^{n} a_m y^m \end{aligned} \tag{1}$$

Where

$$u(y) = \sum_{m=0}^{n} a_m y^m \tag{1A}$$

Eq. (1) can be written as

$$\psi(y) = \begin{cases} e^{\frac{-y^2}{2}} \sum_{m=0}^{n} a_m y^m & \lim_{y\to 0} \\ a_m y^m e^{\frac{-y^2}{2}} & \lim_{y\to\infty} \end{cases}$$

Substituting (1) in 10.4.12 gives

$$\begin{aligned} \frac{d^2}{dy^2}\left(ue^{\frac{-y^2}{2}}\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}} \\ \frac{d}{dy}\left(u'e^{\frac{-y^2}{2}} - uye^{\frac{-y^2}{2}}\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}} \\ \left(u''e^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - u\left(e^{\frac{-y^2}{2}} - y^2e^{\frac{-y^2}{2}}\right)\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}} \\ \left(u''e^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - u'ye^{\frac{-y^2}{2}} - ue^{\frac{-y^2}{2}} + y^2ue^{\frac{-y^2}{2}}\right) - y^2ue^{\frac{-y^2}{2}} &= -2\epsilon ue^{\frac{-y^2}{2}} \end{aligned}$$

Dividing by $e^{\frac{-y^2}{2}} \neq 0$ gives

$$\begin{aligned} u'' - u'y - u'y - u + y^2u - y^2u &= -2\epsilon u \\ u'' - 2u'y - u &= -2\epsilon u \end{aligned}$$

Which becomes the Hermite ODE as given in 10.4.24

$$u''(y) - 2yu'(y) + (2\varepsilon - 1)u(y) = 0 \tag{10.4.24}$$

From (1A)

$$\begin{aligned} u' &= \sum_{m=0}^{n} ma_m y^{m-1} \\ u'' &= \sum_{m=0}^{n} m(m-1)a_m y^{m-2} \end{aligned}$$

Substituting the above in (10.4.24) gives

$$\begin{aligned} \sum_{m=0}^{n} m(m-1)a_m y^{m-2} - 2y\sum_{m=0}^{n} ma_m y^{m-1} + (2\varepsilon - 1)\sum_{m=0}^{n} a_m y^m &= 0 \\ \sum_{m=0}^{n} m(m-1)a_m y^{m-2} - \sum_{m=0}^{n} 2ma_m y^m + \sum_{m=0}^{n} (2\varepsilon - 1)a_m y^m &= 0 \\ \sum_{m=0}^{n} m(m-1)a_m y^{m-2} + \sum_{m=0}^{n} (2\varepsilon - 1 - 2m)a_m y^m &= 0 \end{aligned}$$

The first sum can start from $m = 2$ without affecting the sum, hence the above becomes

$$\sum_{m=2}^{n} m(m-1)a_m y^{m-2} + \sum_{m=0}^{n}(2\varepsilon - 1 - 2m)a_m y^m = 0$$

Let $m' = m - 2$ in the first sum, it becomes

$$\sum_{m'=0}^{n-2}(m'+2)(m'+1)a_{m'+2}y^{m'} + \sum_{m=0}^{n}(2\varepsilon - 1 - 2m)a_m y^m = 0$$

Changing the index in the first sum from m' back to m gives

$$\sum_{m=0}^{n-2}(m+2)(m+1)a_{m+2}y^{m} + \sum_{m=0}^{n}(2\varepsilon - 1 - 2m)a_m y^m = 0$$

Combining terms gives

$$\sum_{m=0}^{n-2}((m+2)(m+1)a_{m+2} + (2\varepsilon - 1 - 2m)a_m)y^m + \sum_{m=n-1}^{n}(2\varepsilon - 1 - 2m)a_m y^m = 0 \tag{1B}$$

Considering the second term above for now.

$$\sum_{m=n-1}^{n}(2\varepsilon - 1 - 2m)a_m y^m = 0$$
$$(2\varepsilon - 1 - 2m)a_m = 0 \qquad m = n, m = n-1$$

Looking at case <u>$m = n$</u>

$$(2\varepsilon - 1 - 2n)a_n = 0$$

but $a_n \neq 0$ since that is the highest order of the power series. If $a_n = 0$ then the dominant term of the power series is lost. This means $(2\varepsilon - 1 - 2n) = 0$ or

$$\varepsilon = n + \frac{1}{2} \tag{10.4.34}$$

Looking at case <u>$m = n - 1$</u>

$$(2\varepsilon - 1 - 2(n-1))a_{n-1} = 0$$
$$(2\varepsilon - 1 - 2n + 2)a_{n-1} = 0$$
$$(2\varepsilon + 1 - 2n)a_{n-1} = 0$$

But $\varepsilon = n + \frac{1}{2}$, hence the above becomes

$$\left(2\left(n + \frac{1}{2}\right) + 1 - 2n\right)a_{n-1} = 0$$
$$(2n + 1 + 1 - 2n)a_{n-1} = 0$$
$$2a_{n-1} = 0$$

This means

$$a_{n-1} = 0 \tag{2}$$

Now looking at case $\underline{m \leq n-2}$ from Eq. (1C) above

$$\begin{aligned}
\sum_{m=0}^{n-2}((m+2)(m+1)a_{m+2} + (2\varepsilon - 1 - 2m)a_m)y^m &= 0 \\
(m+2)(m+1)a_{m+2} + (2\varepsilon - 1 - 2m)a_m &= 0 \\
a_{m+2} &= \frac{-(2\varepsilon - 1 - 2m)}{(m+2)(m+1)}a_m
\end{aligned}$$

But $\varepsilon = n + \frac{1}{2}$, therefore the above becomes

$$\begin{aligned}
a_{m+2} &= \frac{-\left(2\left(n+\frac{1}{2}\right) - 1 - 2m\right)}{(m+2)(m+1)}a_m \\
&= \frac{-(2n-2m)}{(m+2)(m+1)}a_m \\
&= -\frac{2(n-m)}{(m+2)(m+1)}a_m
\end{aligned} \tag{3}$$

If n is even then $n-1$ is odd. Then $a_{n-1} = 0$ from (2). But due to the recursive formula (3), this implies $a_1 = a_3 = a_5 \cdots = 0$. Which means all odd terms in the solution polynomial vanish. And if n is odd, then $n-1$ is even. Therefore $a_{n-1} = 0$, But due to the recursive formula (3), this implies $a_0 = a_2 = a_4 \cdots = 0$. Which means all even terms in the solution polynomial vanish.

Now Eq. (3) is the recursive relation used to determine all coefficients a_i. For $m = 0$, (3) gives

$$a_2 = -na_0 \tag{4}$$

For $m = 1$, (3) gives

$$a_3 = \frac{-2(n-1)}{3!}a_1 \tag{5}$$

For $m = 2$, (3) gives

$$\begin{aligned}
a_4 &= \frac{-2(n-2)}{(4)(3)}a_2 \\
&= \frac{-2^2(n-2)}{4!}a_2 \\
&= \frac{2^2(n-2)n}{4!}a_0
\end{aligned} \tag{6}$$

For $m = 3$, (3) gives

$$
\begin{aligned}
a_5 &= \frac{-2(n-3)}{(3+2)(3+1)} a_3 \\
&= \frac{-2(n-3)}{(5)(4)} \frac{-2^2(n-1)}{3!} a_1 \\
&= \frac{2^3(n-3)(n-1)}{5!} a_1
\end{aligned} \tag{7}
$$

For $m = 4$, (3) gives

$$
\begin{aligned}
a_6 &= \frac{-2(n-4)}{(4+2)(4+1)} a_4 \\
&= \frac{-2(n-4)}{(6)(5)} \frac{2^2(n-2)n}{4!} a_0 \\
&= \frac{-2^3(n-4)(n-2)n}{6!} a_0
\end{aligned} \tag{8}
$$

And so on. Therefore the solution to the Hermite ODE (2) is

$$
\begin{aligned}
u &= \sum_{m=0}^{n} a_m y^m \\
&= a_0 + a_1 y + a_2 y^2 + a_3 y^3 + a_4 y^4 + a_5 y^5 + a_6 y^6 + \cdots \\
&= a_0 + a_1 y - n a_0 y^2 - \frac{2(n-1)}{3!} a_1 y^3 + \frac{2^2(n-2)n}{4!} a_0 y^4 + \frac{2^3(n-3)(n-1)}{5!} a_1 y^5 - \frac{2^3(n-4)(n-2)n}{6!} a_0 y^6 + \cdots
\end{aligned} \tag{9}
$$

Which can be written as

$$
\begin{aligned}
u(y) = a_0 &\left(1 - n y^2 + \frac{2^2(n-2)n}{4!} y^4 - \frac{2^3(n-4)(n-2)n}{6!} y^6 + \cdots\right) \\
&+ a_1 \left(y - \frac{2(n-1)}{3!} y^3 + \frac{2^3(n-3)(n-1)}{5!} a_1 y^5 + \cdots\right)
\end{aligned}
$$

Or

$$
u(y) = a_0 u_0 + a_1 u_1
$$

Where u_1, u_2 are two linearly independent solutions for the second order Hermite ODE where

$$
\begin{aligned}
u_0 &= 1 - n y^2 + \frac{2^2(n-2)n}{4!} y^4 - \frac{2^3(n-4)(n-2)n}{6!} y^6 + \cdots \\
u_1 &= y - \frac{2^2(n-1)}{3!} y^3 + \frac{2^3(n-3)(n-1)}{5!} a_1 y^5 + \cdots
\end{aligned}
$$

For even n the solution $u_0(y)$ will eventually terminates, and for odd n the solution $u_1(y)$ eventually terminates. The even Hermite polynomials $H_0, H_2, H_4, \cdots$ are found from $u_0(y)$

for $n=0,2,4,\cdots$ and the odd Hermite polynomials $H_1,H_3,H_5,\cdots$ are found from $u_1(y)$ for $n=1,3,5,\cdots$. The Hermite polynomials need to also be normalize at the end. The even Hermite polynomials are the following

For $n=0$

$$\begin{aligned}u_0(y) &= a_0\left(1-ny^2-\frac{2^2(n-2)n}{4!}y^4-\frac{2^3(n-4)(n-2)n}{6!}y^6+\cdots\right)_{n=0}\\ &= a_0\end{aligned}$$

Therefore

$$H_0(y)=a_0$$

To find a_0, the normalization $\int_{-\infty}^{\infty}e^{-y^2}H_{n'}(y)H_n(y)dy=2^n n!\sqrt{\pi}\,\delta_{n,n'}$ is used, where $H_0(y)=a_0$ in this case. This gives

$$\begin{aligned}\int_{-\infty}^{\infty}e^{-y^2}H_0(y)H_0(y)dy &= \sqrt{\pi}\\ \int_{-\infty}^{\infty}e^{-y^2}a_0^2dy &= \sqrt{\pi}\\ a_0^2\int_{-\infty}^{\infty}e^{-y^2}dy &= \sqrt{\pi}\\ a_0^2\sqrt{\pi} &= \sqrt{\pi}\\ a_0 &= 1\end{aligned}$$

Hence $a_0=1$ and

$$H_0(y)=1$$

For $n=2$

$$\begin{aligned}u_0(y) &= a_0\left(1-ny^2+\frac{2^2(n-2)n}{4!}y^4-\frac{2^3(n-4)(n-2)n}{6!}y^6+\cdots\right)_{n=2}\\ &= a_0\left(1-2y^2\right)\end{aligned}$$

Therefore

$$H_2(y)=a_0\left(1-2y^2\right)$$

To find a_0, There is an easier way to normalize $H_n(x)$ than using the normalization integral equation as was done above. This method will be used for the rest of the problem as it is simpler. It works as follows. $H_n(y)=\left(1-2y^2\right)$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^2. Hence $n=2$. Therefore the coefficient is $2^2=4$. But the coefficient is -2. Therefore the whole expression is multiplied by -2. This means $a_0=-2$. Hence

$$H_2(y)=-2\left(1-2y^2\right)$$

For $H_4(y)$ (This is not required to find, but found for verification)

For $n = 4$

$$
\begin{aligned}
u_0(y) &= a_0\left(1 - ny^2 + \frac{2^2(n-2)n}{4!}y^4 - \frac{2^3(n-4)(n-2)n}{6!}y^6 + \cdots\right)_{n=4} \\
&= a_0\left(1 - 4y^2 + \frac{2^2(4-2)4}{4!}y^4\right) \\
&= a_0\left(1 - 4y^2 + \frac{4}{3}y^4\right)
\end{aligned}
$$

Therefore

$$
H_4(y) = a_0\left(1 - 4y^2 + \frac{4}{3}y^4\right)
$$

$H_4(y) = a_0\left(1 - 4y^2 + \frac{4}{3}y^4\right)$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^4. Hence $n = 4$. Therefore the coefficient is $2^4 = 16$. But the coefficient is $\frac{4}{3}$. Therefore the whole expression is multiplied by 12. This means $a_0 = 12$. Hence

$$
H_4(y) = 12\left(1 - 4y^2 + \frac{4}{3}y^4\right)
$$

Now the odd Hermite polynomials are found. These are found from $u_1(y)$.

For $n = 1$

$$
\begin{aligned}
u_1(y) &= a_1\left(y - \frac{2(n-1)}{3!}y^3 + \frac{2^3(n-3)(n-1)}{5!}a_1y^5 + \cdots\right)_{n=1} \\
&= a_1 y
\end{aligned}
$$

Hence

$$
H_1(y) = a_1 y
$$

$H_1(y) = a_1 y$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^1. Hence $n = 1$. Therefore the coefficient is $2^1 = 2$. But the coefficient is 1. Therefore the whole expression is multiplied by 2. This means $a_1 = 2$. Hence

$$
H_1(y) = 2y
$$

For $n = 3$

$$
\begin{aligned}
u_2(y) &= a_1\left(y - \frac{2(n-1)}{3!}y^3 + \frac{2^3(n-3)(n-1)}{5!}a_1y^5 + \cdots\right)_{n=3} \\
&= a_1\left(y - \frac{2(3-1)}{3!}y^3\right)
\end{aligned}
$$

Hence

$$H_3(y) = a_1\left(y - \frac{2}{3}y^3\right)$$

$H_3(y) = a_1\left(y - \frac{2}{3}y^3\right)$ is normalized as follows. The coefficient in front of the largest power in y^n is forced to be 2^n. In the above, the largest power is y^3. Hence $n = 3$. Therefore the coefficient is $2^3 = 8$. But the coefficient is $-\frac{2}{3}$. Therefore the whole expression is multiplied by −12. This means $a_1 = -12$. Hence

$$H_3(y) = -12\left(y - \frac{2}{3}y^3\right)$$

The following gives the final results

$$\begin{aligned}
H_0(y) &= 1\\
H_1(y) &= 2y\\
H_2(y) &= -2\left(1 - 2y^2\right)\\
H_3(y) &= -12\left(y - \frac{2}{3}y^3\right)\\
H_4(y) &= 12\left(1 - 4y^2 + \frac{4}{3}y^4\right)
\end{aligned}$$

1.2 Part 2

This part verifies the results obtained in part 1 above for $m, n \leq 2$ using

$$\int_{-\infty}^{\infty} e^{-y^2} H_n(y) H_m(y) dy = 2^n n! \sqrt{\pi}\, \delta_{n,m} \tag{1}$$

<u>For $n = 0, m = 0$</u>

Eq (1) becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-y^2} H_0(y) H_0(y) dy &= \sqrt{\pi}\\
\int_{-\infty}^{\infty} e^{-y^2} dy &= \sqrt{\pi}
\end{aligned}$$

But $\int_{-\infty}^{\infty} e^{-y^2} dy$ is the Gaussian integral which is $\sqrt{\pi}$. Hence

$$\sqrt{\pi} = \sqrt{\pi}$$

Verified.

<u>For $n = 0, m = 1$</u>

Eq (1) becomes

$$\int_{-\infty}^{\infty} e^{-y^2} H_0(y) H_1(y) dy = 0$$
$$\int_{-\infty}^{\infty} e^{-y^2} \left(2y\right) dy = 0$$
$$2 \int_{-\infty}^{\infty} y e^{-y^2} dy = 0$$

But y is odd, and e^{-y^2} is even. Hence the LHS is integral over odd function. Hence it must be zero. Therefore

$$0 = 0$$

Verified.

For $n = 0, m = 2$

Eq (1) becomes

$$\int_{-\infty}^{\infty} e^{-y^2} H_0(y) H_2(y) dy = 0$$
$$\int_{-\infty}^{\infty} e^{-y^2} \left(-2\left(1 - 2y^2\right)\right) dy = 0$$
$$\int_{-\infty}^{\infty} e^{-y^2} \left(-2 + 4y^2\right) dy = 0$$
$$-2 \int_{-\infty}^{\infty} e^{-y^2} dy + 4 \int_{-\infty}^{\infty} y^2 e^{-y^2} dy = 0$$

But $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$ and $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$, therefore the above becomes

$$-2\sqrt{\pi} + 4\left(\frac{\sqrt{\pi}}{2}\right) = 0$$
$$-2\sqrt{\pi} + 2\sqrt{\pi} = 0$$
$$0 = 0$$

Verified.

For $n = 1, m = 1$

Eq (1) becomes

$$\int_{-\infty}^{\infty} e^{-y^2} H_1(y) H_1(y) dy = 2\sqrt{\pi}$$
$$\int_{-\infty}^{\infty} e^{-y^2} \left(2y\right)\left(2y\right) dy = 2\sqrt{\pi}$$
$$4 \int_{-\infty}^{\infty} y^2 e^{-y^2} dy = 2\sqrt{\pi}$$

But $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$. The above becomes

$$\begin{aligned} 4\frac{\sqrt{\pi}}{2} &= 2\sqrt{\pi} \\ 2\sqrt{\pi} &= 2\sqrt{\pi} \end{aligned}$$

Verified.

<u>For $n = 1, m = 2$</u>

Eq (1) becomes

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-y^2} H_1(y) H_2(y) dy &= 0 \\ \int_{-\infty}^{\infty} e^{-y^2} (2y)\left(-2\left(1 - 2y^2\right)\right) dy &= 0 \\ \int_{-\infty}^{\infty} e^{-y^2} \left(8y^3 - 4y\right) dy &= 0 \\ 8\int_{-\infty}^{\infty} y^3 e^{-y^2} dy - 4\int_{-\infty}^{\infty} y e^{-y^2} dy &= 0 \end{aligned}$$

Both integrals in the LHS are zero, since both are odd functions. Therefore

$$0 = 0$$

Verified.

<u>For $n = 2, m = 2$</u>

Eq (1) becomes

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-y^2} H_2(y) H_2(y) dy &= (4)2!\sqrt{\pi} \\ \int_{-\infty}^{\infty} e^{-y^2} \left(\left(-2\left(1 - 2y^2\right)\right)\right)\left(-2\left(1 - 2y^2\right)\right) dy &= 8\sqrt{\pi} \\ \int_{-\infty}^{\infty} \left(16y^4 - 16y^2 + 4\right) e^{-y^2} dy &= 8\sqrt{\pi} \\ 16\int_{-\infty}^{\infty} y^4 e^{-y^2} dy - 16\int_{-\infty}^{\infty} y^2 e^{-y^2} dy + 4\int_{-\infty}^{\infty} e^{-y^2} dy &= 8\sqrt{\pi} \end{aligned}$$

But $\int_{-\infty}^{\infty} y^4 e^{-y^2} dy = \frac{3}{4}\sqrt{\pi}$ and $\int_{-\infty}^{\infty} y^2 e^{-y^2} dy = \frac{1}{2}\sqrt{\pi}$ and $\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}$. The above becomes

$$\begin{aligned} 16\left(\frac{3}{4}\sqrt{\pi}\right) - 16\left(\frac{1}{2}\sqrt{\pi}\right) + 4\sqrt{\pi} &= 8\sqrt{\pi} \\ 12\sqrt{\pi} - 8\sqrt{\pi} + 4\sqrt{\pi} &= 8\sqrt{\pi} \\ 8\sqrt{\pi} &= 8\sqrt{\pi} \end{aligned}$$

Verified. This completes the solution.

2 Problem 3 (10.4.4)

Problem 10.4.4. *Consider the* Legendre Equation

$$(1-x^2)y'' - 2xy + l(l+1)y = 0 \tag{10.4.40}$$

Argue that the power series method will lead to a two term recursion relation and find the latter. Show that if l is an even (odd) integer, the even(odd) series will reduce to polynomials, called P_l, the Legendre polynomials *of order l. Show that*

$$\begin{aligned} P_0 &= 1 & (10.4.41)\\ P_1 &= x & (10.4.42)\\ P_2 &= \frac{1}{2}(3x^2-1) & (10.4.43)\\ P_3 &= \frac{1}{2}(5x^3-3x) & (10.4.44) \end{aligned}$$

(The overall scale of these functions is not defined by the equation, but by convention as above.) Pick any two of the above and show that they are orthogonal over the interval $-1 \le x \le 1$.

Figure 2: Problem statement

Solution

2.1 Part 1

The Legendre ODE is given by 10.4.40 as (L is used instead of l as it is more clear because l looks like 1, depending on font used.)

$$\left(1-x^2\right)y'' - 2xy + L(L+1)y = 0 \tag{10.4.40}$$

Let the solution be

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \end{aligned}$$

And

$$\begin{aligned} y'' &= \sum_{n=1}^{\infty} n(n-1)a_n x^{n-2} \\ &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \end{aligned}$$

Substituting the above results back in (10.4.40) gives

$$\begin{aligned} \left(1-x^2\right)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2x\sum_{n=1}^{\infty} na_n x^{n-1} + L(L+1)\sum_{n=0}^{\infty} a_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} L(L+1)a_n x^n &= 0 \\ \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n - \sum_{n=1}^{\infty} 2na_n x^n + \sum_{n=0}^{\infty} L(L+1)a_n x^n &= 0 \qquad (1) \end{aligned}$$

For $n = 0$ only the above gives

$$\begin{aligned} (n+2)(n+1)a_{n+2}x^n + L(L+1)a_n x^n &= 0 \\ 2a_2 + L(L+1)a_0 &= 0 \\ a_2 &= -\frac{L(L+1)}{2}a_0 \end{aligned}$$

For $n = 1$ only Eq (1) gives

$$\begin{aligned} (n+2)(n+1)a_{n+2}x^n - 2na_n x^n + L(L+1)a_n x^n &= 0 \\ (3)(2)a_3 - 2a_1 + L(L+1)a_1 &= 0 \\ a_3 &= \frac{2a_1 - L(L+1)a_1}{6} \\ &= \frac{2 - L(L+1)}{6}a_1 \end{aligned}$$

And for $n \geq 2$, Eq(1) gives the recusive relation

$$\begin{aligned} ((n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + L(L+1)a_n)x^n &= 0 \\ (n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + L(L+1)a_n &= 0 \\ (n+2)(n+1)a_{n+2} &= (n(n-1) + 2n - L(L+1))a_n \end{aligned}$$

Hence the <u>two term recursive</u> is

$$a_{n+2} = \frac{n(n-1) + 2n - L(L+1)}{(n+2)(n+1)} a_n \tag{1}$$

For $n = 2$

$$\begin{aligned} a_4 &= \frac{n(n-1) + 2n - L(L+1)}{(n+2)(n+1)} a_2 \\ &= \frac{2(2-1) + 4 - L(L+1)}{(4)(3)} a_2 \\ &= \frac{6 - L(L+1)}{12} a_2 \end{aligned}$$

But $a_2 = \frac{-L(L+1)}{2} a_0$ hence the above becomes

$$a_4 = \frac{6 - L(L+1)}{12} \left(\frac{-L(L+1)}{2} a_0 \right)$$

For $n = 3$

$$\begin{aligned} a_5 &= \frac{n(n-1) + 2n - L(L+1)}{(n+2)(n+1)} a_3 \\ &= \frac{3(3-1) + 6 - L(L+1)}{(3+2)(3+1)} a_3 \\ &= \frac{12 - L(L+1)}{20} a_3 \end{aligned}$$

But $a_3 = \frac{2-L(L+1)}{6} a_1$, hence the above becomes

$$a_5 = \frac{12 - L(L+1)}{20} \left(\frac{2 - L(L+1)}{6} a_1 \right)$$

And so on. The solution becomes

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\ &= a_0 + a_1 x - \frac{L(L+1)}{2} a_0 x^2 + \frac{2 - L(L+1)}{6} a_1 x^3 - \left(\frac{6 - L(L+1)}{12} \right) \left(\frac{L(L+1)}{2} \right) a_0 x^4 + \left(\frac{12 - L(L+1)}{20} \right) \left(\frac{2 - L(L+1)}{6} \right) a_1 x^5 + \cdots \\ &= a_0 \left(1 - \frac{L(L+1)}{2} x^2 - \left(\frac{6 - L(L+1)}{12} \right) \left(\frac{L(L+1)}{2} \right) x^4 + \cdots \right) + a_1 \left(x + \frac{2 - L(L+1)}{6} x^3 + \left(\frac{12 - L(L+1)}{20} \right) \left(\frac{2 - L(L+1)}{6} \right) x^5 + \cdots \right) \end{aligned}$$

Or

$$y(x) = a_0 y_0(x) + a_1 y_1(x)$$

Where

$$y_0(x) = 1 - \frac{L(L+1)}{2}x^2 - \left(\frac{6 - L(L+1)}{12}\right)\left(\frac{L(L+1)}{2}\right)x^4 + \cdots$$
$$y_1(x) = x + \frac{2 - L(L+1)}{6}x^3 + \left(\frac{12 - L(L+1)}{20}\right)\left(\frac{2 - L(L+1)}{6}\right)x^5 + \cdots$$

Where y_0, y_1 are two linearly independent solutions. The even Legendre polynomials are obtained from $y_0(x)$ for integer $L = 0, 2, 4, \cdots$ and the odd Legendre polynomials are obtained from $y_1(x)$ for integer $L = 1, 3, 5, \cdots$.

For $L = 0$

$$y(x) = a_0(1)$$

Since all higher terms vanish. Choosing $a_0 = 1$ then

$$P_0(x) = 1$$

For $L = 2$

$$\begin{aligned} y(x) &= a_0\left(1 - \frac{L(L+1)}{2}x^2 - \left(\frac{6 - L(L+1)}{12}\right)\left(\frac{L(L+1)}{2}\right)x^4 + \cdots\right) \\ &= a_0\left(1 - \frac{2(2+1)}{2}x^2 - \left(\frac{6 - 2(2+1)}{12}\right)\left(\frac{2(2+1)}{2}\right)x^4 + \cdots\right) \\ &= a_0\left(1 - 3x^2\right) \end{aligned}$$

Since all higher terms vanish. Choosing $a_0 = -\frac{1}{2}$ then

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

For $L = 1$

Since L is odd, then $y_1(x)$ is used now.

$$\begin{aligned} y(x) &= a_1\left(x + \frac{2 - L(L+1)}{6}x^3 + \left(\frac{12 - L(L+1)}{20}\right)\left(\frac{2 - L(L+1)}{6}\right)x^5 + \cdots\right) \\ &= a_1\left(x + \frac{2 - (1+1)}{6}x^3 + \left(\frac{12 - (1+1)}{20}\right)\left(\frac{2 - (1+1)}{6}\right)x^5 + \cdots\right) \\ &= a_1 x \end{aligned}$$

Since all higher terms vanish. Choosing $a_1 = 1$ then

$$P_1(x) = x$$

For $L = 3$

$$
\begin{aligned}
y(x) &= a_1\left(x + \frac{2 - L(L+1)}{6}x^3 + \left(\frac{12 - L(L+1)}{20}\right)\left(\frac{2 - L(L+1)}{6}\right)x^5 + \cdots\right) \\
&= a_1\left(x + \frac{2 - 3(3+1)}{6}x^3 + \left(\frac{12 - 3(3+1)}{20}\right)\left(\frac{2 - 3(3+1)}{6}\right)x^5 + \cdots\right) \\
&= a_1\left(x - \frac{5}{3}x^3\right)
\end{aligned}
$$

Since all higher terms vanish. Choosing $a_1 = -\frac{3}{2}$ then

$$
\begin{aligned}
P_3(x) &= -\frac{3}{2}\left(x - \frac{5}{3}x^3\right) \\
&= \frac{1}{2}\left(5x^3 - 3x\right)
\end{aligned}
$$

Summary

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= x \\
P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right) \\
P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right)
\end{aligned}
$$

2.2 Part 2

To show any two are orthogonal over $-1 \leq x \leq 1$. Selecting $P_0(x)$ and $P_1(x)$, then

$$
\begin{aligned}
\int_{-1}^{1} P_0(x)P_1(x)dx &= \int_{-1}^{1} xdx \\
&= \frac{1}{2}\left[x^2\right]_{-1}^{1} \\
&= \frac{1}{2}(1 - 1) \\
&= 0
\end{aligned}
$$

Hence $P_0(x)$ and $P_1(x)$ are orthogonal to each others. Verified.

3 Problem 3 (10.4.5)

Problem 10.4.5. *The functions* $1,\ x,\ x^2, \cdots$ *are linearly independent—there is no way, for example, to express* x^3 *in terms of sums of other powers. Use the Gram–Schmidt procedure to extract from this set the first four Legendre polynomials (up to normalization) known to be orthonormal in the interval* $-1 \le x \le 1$.

Figure 3: Problem statement

Solution

Let

$$\{|x\rangle\} = \left\{1, x, x^2, x^3, \cdots\right\}$$

Where $|x_1\rangle = 1, |x_2\rangle = x, |x_3\rangle = x^2$ and so on. Let

$$\begin{aligned} P_0 &= |x_1\rangle \\ &= 1 \end{aligned}$$

Normalizing gives

$$P_0 = \frac{P_0}{\|P_0\|} = \frac{1}{\sqrt{\int_{-1}^{1} dx}} = \sqrt{\frac{1}{2}}$$

And

$$\begin{aligned} P_1 &= |x_2\rangle - P_0\langle P_0|x_2\rangle \\ &= x - \sqrt{\frac{1}{2}}\langle\sqrt{\frac{1}{2}}|x_2\rangle \\ &= x - \frac{1}{2}\int_{-1}^{1} x dx \\ &= x - 0 \\ &= x \end{aligned}$$

Normalizing gives

$$P_1 = \frac{P_1}{\|P_1\|} = \frac{x}{\sqrt{\int_{-1}^{1} x^2 dx}} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x$$

And

$$
\begin{aligned}
P_2 &= |x_3\rangle - (P_0\langle P_0|x_3\rangle + P_1\langle P_1|x_3\rangle) \\
&= x^2 - \left(\sqrt{\frac{1}{2}}\,\langle\sqrt{\frac{1}{2}}\,|x_3\rangle + \sqrt{\frac{3}{2}}\,x\langle\sqrt{\frac{3}{2}}\,x|x_3\rangle\right) \\
&= x^2 - \left(\frac{1}{2}\int_{-1}^{1} x^2 dx + \frac{3}{2}x\int_{-1}^{1} xx^2 dx\right) \\
&= x^2 - \left(\frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} + \frac{3}{2}x\int_{-1}^{1} x^3 dx\right) \\
&= x^2 - \left(\frac{1}{2}\frac{1}{3}\left[1-(-1)^3\right] + 0\right) \\
&= x^2 - \left(\frac{1}{2}\frac{1}{3}(2)\right) \\
&= x^2 - \frac{1}{3}
\end{aligned}
$$

Normalizing

$$
\begin{aligned}
P_2 &= \frac{P_2}{||P_2||} \\
&= \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)\left(x^2 - \frac{1}{3}\right)dx}} \\
&= \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} \\
&= \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right) \\
&= \sqrt{\frac{5}{8}}\,3\left(x^2 - \frac{1}{3}\right) \\
&= \sqrt{\frac{5}{8}}\left(3x^2 - 1\right)
\end{aligned}
$$

And

$$
\begin{aligned}
P_3 &= |x_4\rangle - (P_0\langle P_0|x_4\rangle + P_1\langle P_1|x_4\rangle + P_2\langle P_2|x_4\rangle) \\
&= x^3 - \left(\sqrt{\frac{1}{2}}\langle\sqrt{\frac{1}{2}}|x_4\rangle + \sqrt{\frac{3}{2}}x\langle\sqrt{\frac{3}{2}}x|x_4\rangle + \sqrt{\frac{5}{8}}\left(3x^2-1\right)\langle\sqrt{\frac{5}{8}}\left(3x^2-1\right)|x_4\rangle\right) \\
&= x^3 - \left(\frac{1}{2}\int_{-1}^{1} x^3 dx + \frac{3}{2}x\int_{-1}^{1} xx^3 dx + \frac{5}{8}\left(3x^2-1\right)\int_{-1}^{1}\left(3x^2-1\right)x^3 dx\right) \\
&= x^3 - \left(\frac{1}{2}\int_{-1}^{1} x^3 dx + \frac{3}{2}x\int_{-1}^{1} x^4 dx + \frac{5}{8}\left(3x^2-1\right)\int_{-1}^{1}\left(3x^5-x^3\right)dx\right) \\
&= x^3 - \left(\frac{1}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} + \frac{3}{2}x\left[\frac{x^5}{5}\right]_{-1}^{1} + \frac{5}{8}\left(3x^2-1\right)(0)\right) \\
&= x^3 - \left(\frac{1}{8}\left[x^4\right]_{-1}^{1} + \frac{3}{10}x\left[x^5\right]_{-1}^{1}\right) \\
&= x^3 - \left(\frac{1}{8}\left[1-(-1)^4\right] + \frac{3}{10}x\left[1-(-1)^5\right]\right) \\
&= x^3 - \left(\frac{1}{8}[0] + \frac{3}{10}x[2]\right) \\
&= x^3 - \frac{3}{5}x
\end{aligned}
$$

Normalizing

$$
\begin{aligned}
P_3 &= \frac{P_3}{\|P_3\|} \\
&= \frac{x^3 - \frac{3}{5}x}{\sqrt{\int_{-1}^{1}\left(x^3-\frac{3}{5}x\right)\left(x^3-\frac{3}{5}x\right)dx}} \\
&= \frac{x^3 - \frac{3}{5}x}{\sqrt{\frac{8}{175}}} \\
&= \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right) \\
&= \sqrt{\frac{(25)(7)}{8}}\left(x^3 - \frac{3}{5}x\right) \\
&= \sqrt{\frac{7}{8}}\left(5x^3 - 3x\right)
\end{aligned}
$$

These are the first 4 Legenrdre polynomials. The scaling is different from the last problem due to difference in method used to normalize them. The following table shows the final result and difference in scaling.

P_n	Problem 10.4.5 result	Problem 10.4.4 result
$P_0(x)$	$\sqrt{\frac{1}{2}}$	1
$P_1(x)$	$\sqrt{\frac{3}{2}}\,x$	x
$P_2(x)$	$\sqrt{\frac{5}{8}}\left(3x^2-1\right)$	$\frac{1}{2}\left(3x^2-1\right)$
$P_3(x)$	$\sqrt{\frac{7}{8}}\left(5x^3-3x\right)$	$\frac{1}{2}\left(5x^3-3x\right)$

4 Problem 4 (10.4.10)

Problem 10.4.10. *Solve* Laguerre's Equation *which enters the solution of the hydrogen atom problem in quantum mechanics*

$$xy'' + (1-x)y' + my = 0 \qquad (10.4.65)$$

by the power series method. Show that there is a repeated root and focus on the solution which is regular at the origin. Show that this reduces to a polynomial when m is an integer. These are the Laguerre polynomials L_m. *Find the first four polynomials choosing $c_0 = 1$. Show that L_1 and L_2 are orthogonal in the interval $0 \leq x \leq \infty$ with a weight function e^{-x}. (Recall the gamma function.)*

Figure 4: Problem statement

Solution

Since the ODE is singular at $x = 0$ then Frobenius series is used. Let

$$\begin{aligned} y &= x^s \sum_{n=0}^{\infty} c_n x^n \\ &= \sum_{n=0}^{\infty} c_n x^{n+s} \qquad c_0 \neq 0 \end{aligned}$$

Hence

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} (n+s) c_n x^{n+s-1} \\ y'' &= \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n x^{n+s-2} \end{aligned}$$

Substituting this in the ODE (10.4.65) gives

$$\begin{aligned} x \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n x^{n+s-2} + (1-x) \sum_{n=0}^{\infty} (n+s) c_n x^{n+s-1} + m \sum_{n=0}^{\infty} c_n x^{n+s} &= 0 \\ \sum_{n=0}^{\infty} (n+s)(n+s-1) c_n x^{n+s-1} + \sum_{n=0}^{\infty} (n+s) c_n x^{n+s-1} - \sum_{n=0}^{\infty} (n+s) c_n x^{n+s} + m \sum_{n=0}^{\infty} c_n x^{n+s} &= 0 \\ \sum_{n=0}^{\infty} ((n+s)(n+s-1) + (n+s)) c_n x^{n+s-1} + \sum_{n=0}^{\infty} (m - (n+s)) c_n x^{n+s} &= 0 \end{aligned}$$

To make all power on x the same, the second sum is rewritten by shifting the index. This gives

$$\sum_{n=0}^{\infty} ((n+s)(n+s-1) + (n+s)) c_n x^{n+s-1} + \sum_{n=1}^{\infty} (m - (n-1+s)) c_{n-1} x^{n+s-1} = 0$$

For $n = 0$

$$
\begin{aligned}
((n+s)(n+s-1)+(n+s))c_n x^{n+s-1} &= 0\\
((n+s)(n+s-1)+(n+s))c_0 &= 0
\end{aligned}
$$

But by definition $c_0 \neq 0$. Therefore the indicial equation is

$$(n+s)(n+s-1)+(n+s) = 0$$

But $n = 0$. This becomes

$$
\begin{aligned}
s(s-1)+s &= 0\\
s^2-s+s &= 0\\
s^2 &= 0
\end{aligned}
$$

Hence root is $s = 0$ (repeated root). Since there is a repeated root, then this is degenerate case. First solution $y_1(x)$ is the assumed form but with $s = 0$. This means

$$
\begin{aligned}
y_1(x) &= x^0 \sum_{n=0}^{\infty} c_n x^n\\
&= \sum_{n=0}^{\infty} c_n x^n
\end{aligned}
$$

And the second solution is

$$
\begin{aligned}
y_2(x) &= y_1 \ln x + x^s \sum_{n=0}^{\infty} b_n x^n\\
&= y_1 \ln x + \sum_{n=0}^{\infty} b_n x^n
\end{aligned}
$$

But this solution $y_2(x)$ is not bounded at $x = 0$ due to $\ln x$ blowing up at origin. The regular solution is only $y_1(x)$. So $y_1(x)$ will be used from now on and not $y_2(x)$. Therefore

$$
\begin{aligned}
y_1'(x) &= \sum_{n=0}^{\infty} n c_n x^{n-1}\\
y_1''(x) &= \sum_{n=0}^{\infty} n(n-1) c_n x^{n-2}
\end{aligned}
$$

Substituting the above in ODE (10.4.65) gives

$$
\begin{aligned}
x \sum_{n=0}^{\infty} n(n-1)c_n x^{n-2} + (1-x)\sum_{n=0}^{\infty} n c_n x^{n-1} + m \sum_{n=0}^{\infty} c_n x^n &= 0\\
\sum_{n=0}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} m c_n x^n &= 0\\
\sum_{n=0}^{\infty}(n(n-1)+n)c_n x^{n-1} + \sum_{n=0}^{\infty}(m-n)c_n x^n &= 0
\end{aligned}
$$

To make powers on x the same, the index of the first sum is shifted to give

$$\sum_{n=-1}^{\infty}((n+1)n+(n+1))c_{n+1}x^{n}+\sum_{n=0}^{\infty}(m-n)c_{n}x^{n}=0$$

But when $n=-1$ the first sum is zero. So the first sum index can start $n=0$ which gives

$$\sum_{n=0}^{\infty}((n+1)n+(n+1))c_{n+1}x^{n}+\sum_{n=0}^{\infty}(m-n)c_{n}x^{n}=0$$

Now the sums are combined to give

$$\sum_{n=0}^{\infty}[((n+1)n+(n+1))c_{n+1}+(m-n)c_{n}]x^{n}=0$$

Hence recursive relation is

$$\begin{aligned}((n+1)n+(n+1))c_{n+1}+(m-n)c_{n}&=0\\ c_{n+1}&=\frac{n-m}{((n+1)n+(n+1))}c_{n}\\ &=\frac{n-m}{n^{2}+2n+1}c_{n}\end{aligned}$$

For $n=0$

$$c_{1}=-mc_{0}$$

For $n=1$

$$\begin{aligned}c_{2}&=\frac{1-m}{1+2+1}c_{1}\\ &=\frac{1-m}{4}c_{1}\\ &=\frac{1-m}{4}(-mc_{0})\\ &=\frac{m^{2}-m}{4}c_{0}\end{aligned}$$

For $n=2$

$$\begin{aligned}c_{3}&=\frac{2-m}{2^{2}+4+1}c_{2}\\ &=\frac{2-m}{9}c_{2}\\ &=\frac{2-m}{9}\left(\frac{m^{2}-m}{4}c_{0}\right)\\ &=\frac{(2-m)\left(m^{2}-m\right)}{36}c_{0}\\ &=\frac{-m^{3}+3m^{2}-2m}{36}c_{0}\end{aligned}$$

For $n = 3$

$$
\begin{aligned}
c_4 &= \frac{n-m}{n^2+2n+1}c_3 \\
&= \frac{3-m}{9+6+1}c_3 \\
&= \frac{3-m}{16}\left(\frac{-m^3+3m^2-2m}{36}c_0\right) \\
&= \frac{(3-m)\left(-m^3+3m^2-2m\right)}{(16)(36)}c_0 \\
&= \frac{m^4-6m^3+11m^2-6m}{576}c_0
\end{aligned}
$$

And so on. The solution becomes

$$
\begin{aligned}
y_1(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots \\
&= c_0 - m c_0 x + \frac{m^2-m}{4}c_0 x^2 + \frac{-m^3+3m^2-2m}{36}c_0 x^3 + \frac{m^4-6m^3+11m^2-6m}{576}c_0 x^4 + \cdots \\
&= c_0\left(1 - mx + \frac{m^2-m}{4}x^2 + \frac{-m^3+3m^2-2m}{36}x^3 + \frac{m^4-6m^3+11m^2-6m}{576}x^4 + \cdots\right)
\end{aligned}
$$

Setting $c_0 = 1$, the solution is

$$
y_1(x) = 1 - mx + \frac{m^2-m}{4}x^2 + \frac{-m^3+3m^2-2m}{36}x^3 + \frac{m^4-6m^3+11m^2-6m}{576}x^4 + \cdots
$$

For integer m these are polynomials given by

For $m = 0$

$$
L_0(x) = 1
$$

Since rest of terms are zero.

For $m = 1$

$$
L_1(x) = 1 - x
$$

Since rest of terms are zero.

For $m = 2$

$$
L_2(x) = 1 - 2x + \frac{1}{2}x^2
$$

Since rest of terms are zero.

For $m = 3$

$$
\begin{aligned}
L_3(x) &= 1 - 3x + \frac{3^2-3}{4}x^2 + \frac{-3^3+3\left(3^2\right)-6}{36}x^3 \\
&= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3
\end{aligned}
$$

Since rest of terms are zero. Hence

$$
\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= 1 - x \\
L_2(x) &= 1 - 2x + \frac{1}{2}x^2 \\
L_3(x) &= 1 - 3x + \frac{3}{2}x^2 - \frac{1}{6}x^3
\end{aligned}
$$

Or

$$
\begin{aligned}
L_0(x) &= 1 \\
L_1(x) &= 1 - x \\
L_2(x) &= \frac{1}{2}\left(2 - 4x + x^2\right) \\
L_3(x) &= \frac{1}{6}\left(6 - 18x + 9x^2 - x^3\right)
\end{aligned}
$$

The following shows that $L_1(x), L_2(x)$ are orthogonal on $0 \leq x \leq \infty$ with weight e^{-x}

$$
\begin{aligned}
\int_0^\infty L_1(x)L_2(x)e^{-x}dx &= \int_0^\infty (1-x)\left(\frac{1}{2}\left(2 - 4x + x^2\right)\right)e^{-x}dx \\
&= \int_0^\infty \left(-\frac{1}{2}x^3 + \frac{5}{2}x^2 - 3x + 1\right)e^{-x}dx \\
&= -\frac{1}{2}\int_0^\infty x^3e^{-x}dx + \frac{5}{2}\int_0^\infty x^2e^{-x}dx - 3\int_0^\infty xe^{-x}dx + \int_0^\infty e^{-x}dx
\end{aligned}
$$

To evaluate these integrals the following relation will be used

$$
\int_0^\infty x^n e^{-x} = n!
$$

Therefore

$$
\begin{aligned}
\int_0^\infty x^3e^{-x}dx &= 3! = 6 \\
\int_0^\infty x^2e^{-x}dx &= 2! = 2 \\
\int_0^\infty xe^{-x}dx &= 1! = 1
\end{aligned}
$$

And

$$
\int_0^\infty e^{-x}dx = -\left[e^{-x}\right]_0^\infty = -(0-1) = 1
$$

Using these results gives

$$\int_0^{\infty} L_1(x)L_2(x)e^{-x}dx = -\frac{1}{2}(6) + \frac{5}{2}(2) - 3(1) + 1$$
$$= 0$$

This shows that $L_1(x), L_2(x)$ are orthogonal on $0 \leq x \leq \infty$ with weight e^{-x}. This complete the solution.