HW 6

Physics 3041
Mathematical Methods for Physicists

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University of Minnesota, Twin Cities

Nasser M. Abbasi

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Contents

1 Problem 1 (9.5.11)

Problem 9.5.11. Important quantum problem. *Consider the three* spin-1 matrices:

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \tag{9.5.55}$$

which represent the components of the internal angular momentum *of some elementary particle at rest. That is to say, the particle has some angular momentum unrelated to* $\vec{r} \times \vec{p}$. *The operator* $S^2 = S_x^2 + S_y^2 + S_z^2$ *represents the total angular momentum squared. The dynamical state of the system is given by a* state vector *in the complex three dimensional space on which these spin matrices act. By this we mean that all available information on the particle is stored in this vector. According to the laws of quantum mechanics*

- *A measurement of the angular momentum along any direction will give only one of the eigenvalues of the corresponding spin operator.*
- *The probability that a given eigenvalue will result is equal to the absolute value squared of the inner product of the state vector with the corresponding eigenvector. (The state vector and all eigenvectors are all normalized.)*
- *The state of the system immediately following this measurement will be the corresponding eigenvector.*

(a) What are the possible values we can get if we measure spin along the z-axis?
(b) What are the possible values we can get if we measure spin along the x or y-axis?
(c) Say we got the largest possible value for S_x. What is the state vector immediately afterwards?
(d) If S_z is now measured what are the odds for the various outcomes? Say we got the largest value. What is the state just after the measurement? If we remeasure S_x at once, will we once again get the largest value?
(e) What are the outcomes when S^2 is measured?
(f) From the four operators S_x, S_y, S_z, S^2, what is the largest number of commuting operators we can pick at a time?
(g) A particle is in a state given by a column vector

$$|V\rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

First rescale the vector to normalize it. What are the odds for getting the three possible eigenvalues of S_z? What is the statistical or weighted average of these values? Compare this to $\langle V|S_z|V\rangle$.
(h) Repeat all this for S_x.

Figure 1: Problem statement

Solution

$$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

1.1 Part a

The first step is to find the eigenvalues of S_z. These are the possible values that can be obtained when measuring the spin along the z axis. Because S_z is a diagonal matrix, its eigenvalues are on the diagonal. Hence the eigenvalues are $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$. Because the eigenvalues are different, S_z is not degenerate. The values are

$$\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= 1 \\ \omega_3 &= -1 \end{aligned}$$

1.2 Part b

Now we need to find the eigenvalues for S_y and S_x. The factor $\frac{1}{\sqrt{2}}$ is not included in the following calculation, but added again at the end. This is to simplify the algebra.

For S_y

$$\begin{aligned} \left|S_y - \omega I\right| &= 0 \\ \begin{vmatrix} -\omega & -i & 0 \\ i & -\omega & -i \\ 0 & i & -\omega \end{vmatrix} &= 0 \\ -\omega \begin{vmatrix} -\omega & -i \\ i & -\omega \end{vmatrix} + i \begin{vmatrix} i & -i \\ 0 & -\omega \end{vmatrix} &= 0 \\ (-\omega)\left(\omega^2 + i^2\right) + i(-\omega i) &= 0 \\ (-\omega)\left(\omega^2 - 1\right) - \omega i^2 &= 0 \\ -\omega^3 + \omega + \omega &= 0 \\ -\omega^3 + 2\omega &= 0 \\ \omega\left(-\omega^2 + 2\right) &= 0 \end{aligned}$$

The eigenvalues are the roots of the above polynomial. They are

$$\begin{aligned}\omega_1 &= 0\\ \omega_2 &= \sqrt{2}\\ \omega_3 &= -\sqrt{2}\end{aligned}$$

Adding back the factor $\frac{1}{\sqrt{2}}$ which was in front of S_y by multiplying the above results with it gives

$$\begin{aligned}\omega_1 &= 0\\ \omega_2 &= 1\\ \omega_3 &= -1\end{aligned}$$

<u>For S_x</u>

$$\begin{aligned}|S_x - \omega I| &= 0\\ \begin{vmatrix}-\omega & 1 & 0\\ 1 & -\omega & 1\\ 0 & 1 & -\omega\end{vmatrix} &= 0\\ -\omega\begin{vmatrix}-\omega & 1\\ 1 & -\omega\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 0 & -\omega\end{vmatrix} &= 0\\ (-\omega)\left(\omega^2 - 1\right) - 1(-\omega) &= 0\\ -\omega^3 + \omega + \omega &= 0\\ -\omega^3 + 2\omega &= 0\\ \omega\left(2 - \omega^2\right) &= 0\end{aligned}$$

The eigenvalues are the roots of the above polynomial. They are

$$\begin{aligned}\omega_1 &= 0\\ \omega_2 &= \sqrt{2}\\ \omega_3 &= -\sqrt{2}\end{aligned}$$

Adding back the factor $\frac{1}{\sqrt{2}}$ which was in front of S_y by multiplying the above results with it gives

$$\begin{aligned}\omega_1 &= 0\\ \omega_2 &= 1\\ \omega_3 &= -1\end{aligned}$$

This table gives a summary of result found so far before going to the next part.

Spin matrix	Eigenvalues found
$S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$
$S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$	$\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$
$S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	$\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$

The above table shows that the possible values if we measure the spin along the x or y axis are $\{0, 1, -1\}$.

1.3 Part c

From part (b) and taking the largest eigenvalue of S_x as $\omega_2 = +1$, the question is asking us to find the associated eigenvector $|S_x = \omega_2\rangle$. This is found by solving

$$\begin{bmatrix} -\omega_2 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\omega_2 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\omega_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In the above $\omega_2 = 1$. Therefore

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

$R_2 = R_2 + \frac{1}{\sqrt{2}}R_1$ gives

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}$$

$R_3 = R_3 + \frac{2}{\sqrt{2}}R_2$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

The above is now in Echelon form. The system becomes

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is a free variable, and v_2, v_1 are the leading variables. Let $v_3 = s$. Second row gives $-\frac{1}{2}v_2+\frac{1}{\sqrt{2}}s = 0$ or $v_2 = \frac{2}{\sqrt{2}}s$. First row gives $-v_1+\frac{1}{\sqrt{2}}v_2 = 0$ or $v_1 = \frac{1}{\sqrt{2}}v_2$ or $v_1 = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}}s\right) = s$. Hence the solution (the eigenvector) is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ \frac{2}{\sqrt{2}}s \\ s \end{bmatrix}$$

$$= s\begin{bmatrix} 1 \\ \frac{2}{\sqrt{2}} \\ 1 \end{bmatrix}$$

Since s is a free variable, we will choose it so that the norm is 1. Therefore

$$\begin{aligned} s\sqrt{1+2+1} &= 1 \\ s\sqrt{4} &= 1 \\ s &= \frac{1}{2} \end{aligned}$$

Hence the state vector for the largest value of S_x is

$$|S_x = 1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \frac{2}{\sqrt{2}} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

1.4 Part d

$$S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We first need to find the eigenvectors $|S_z = \omega_i\rangle$ for S_z. From part (a), the eigenvalues are

$$\begin{aligned} \omega_1 &= 0 \\ \omega_2 &= 1 \\ \omega_3 &= -1 \end{aligned}$$

For $\omega_1 = 0$ the associated eigenvector is found by solving

$$\begin{bmatrix} 1-\omega_1 & 0 & 0 \\ 0 & -\omega_1 & 0 \\ 0 & 0 & -1-\omega_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_2 is a free variable, and v_1, v_3 are the leading variables. Let $v_2 = s$. Last row gives $v_3 = 0$. First row gives $v_1 = 0$. Hence the solution is

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ s \\ 0 \end{bmatrix} \\ &= s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

Choosing $s = 1$ gives

$$|S_z = \omega_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\underline{\omega_2 = 1}$ we need to solve

$$\begin{bmatrix} 1-\omega_2 & 0 & 0 \\ 0 & -\omega_2 & 0 \\ 0 & 0 & -1-\omega_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1-1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_1 is a free variable, and v_2, v_3 are the leading variables. Let $v_1 = s$. Last row gives $v_3 = 0$. Second row gives $v_2 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$$

$$= s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Choosing $s = 1$ then

$$|S_z = \omega_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\underline{\omega_3 = -1}$ the associated eigenvector is found by solving

$$\begin{bmatrix} 1-\omega_3 & 0 & 0 \\ 0 & -\omega_3 & 0 \\ 0 & 0 & -1-\omega_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is a free variable, and v_1, v_2 are the leading variables. Let $v_3 = s$. Second row gives $v_2 = 0$. First row gives $v_1 = 0$. Hence the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix}$$

$$= s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Choosing $s = 1$ gives

$$|S_z = \omega_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Summary table for S_z

eigenvalue	eigenvector
$\omega_1 = 0$	$\lvert S_z = 0\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
$\omega_2 = 1$	$\lvert S_z = 1\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
$\omega_3 = -1$	$\lvert S_z = -1\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Calculating $|\langle S_z = \omega_1|\Psi\rangle|^2$ gives the odds of $|S_z = \omega_1\rangle$. Ψ is the initial state vector. Similarly, calculating $|\langle S_x = \omega_1|\Psi\rangle|^2$ gives find the odds of $|S_x = \omega_1\rangle$ and similarly for $|S_z = \omega_3\rangle$.

Ψ is the state vector from part (c), which is

$$|\Psi\rangle = |S_x = 1\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

Hence the odds of $|S_z = 0\rangle$ is

$$|\langle S_z = \omega_1|\Psi\rangle|^2 = \left(\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^* \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \right)^2 = \left(\frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}$$

And the odds for $|S_z = 1\rangle$ is

$$|\langle S_z = \omega_2|\Psi\rangle|^2 = \left(\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \right)^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4}$$

And the odds for $|S_z = -1\rangle$ is

$$\left|\langle S_z = \omega_3|\Psi\rangle\right|^2 = \left(\begin{bmatrix}0 & 0 & 1\end{bmatrix}^* \begin{bmatrix}\frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2}\end{bmatrix}\right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

The odds for $|S_z = 0\rangle$ is 50%, the odds for $|S_z = 1\rangle$ is 25% and odds for $|S_z = -1\rangle$ is 25%. The total is 100% as expected.

Summary table of results so far S_z

eigenvalue	eigenvector	probability of this outcome
$\omega_1 = 0$	$\lvert S_z = \omega_1\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$	$P(0) = 50\%$
$\omega_2 = 1$	$\lvert S_z = \omega_2\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$	$P(1) = 25\%$
$\omega_3 = -1$	$\lvert S_z = \omega_3\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$	$P(-1) = 25\%$

The state just after the measurement is $|S_z = 1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$ since that is the state associated with the largest eigenvalue $\omega_2 = 1$. This now becomes the initial state

$$|\Psi\rangle = |S_z = 1\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

We know that $S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$ with the eigenvalues found earlier as $\omega_1 = 0, \omega_2 = 1, \omega_3 =$

-1. In part (c) we found that $|S_x = 1\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$ for S_x associated with its largest eigenvalue which is $\omega_2 = 1$. Therefore the odds of this is

$$|\langle S_x = 1|\Psi\rangle|^2 = \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}^* \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4} = 25\%$$

This says the odds of getting again the largest value (which is 1) is not likely since it is not the highest possible odd being only 25% with 3 possible values.

1.5 Part e

$$\begin{aligned} S^2 &= S_x^2 + S_y^2 + S_z^3 \\ &= \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^2 + \frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}^2 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^2 \end{aligned} \tag{1}$$

But

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence (1) becomes

$$
\begin{aligned}
S^2 &= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}
\end{aligned}
$$

Since S^2 is diagonal, then its eigenvalues are on the diagonal. They are all $\omega = 2$ with multiplicity 3. It is a degenerate matrix. Since the outcome is the eigenvalue (it is a measure of the spin angular momentum), then we see that the outcome is always 2, since that is the only possible eigenvalue.

1.6 Part f

The operators are

$$
S^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad S_x = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad S_y = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \qquad S_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}
$$

Commutator is defined as

$$
[M, N] = MN - NM
$$

If $[M, N] = 0$ then they commute. We know that S_x, S_y, S_z do not commute with each others per lecture notes. So we only need to check if S^2 commutes with S_x, S_y, S_z or not.

$$
\begin{aligned}
\left[S^2, S_x\right] &= S^2 S_x - S_x S^2 \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 2 & 0\\2 & 0 & 2\\0 & 2 & 0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 2 & 0\\2 & 0 & 2\\0 & 2 & 0\end{bmatrix} \\
&= \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}
\end{aligned}
$$

Hence S^2, S_x commute. And

$$
\begin{aligned}
\left[S^2, S_y\right] &= S^2 S_y - S_y S^2 \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}0 & -i & 0\\i & 0 & -i\\0 & i & 0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -i & 0\\i & 0 & -i\\0 & i & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -2i & 0\\2i & 0 & -2i\\0 & 2i & 0\end{bmatrix} - \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -2i & 0\\2i & 0 & -2i\\0 & 2i & 0\end{bmatrix} \\
&= \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}
\end{aligned}
$$

Hence S^2, S_y commute. And

$$\begin{aligned}
\left[S^2, S_z\right] &= S^2 S_z - S_z S^2 \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Hence S^2, S_z commute. Therefore there are three sets of commuting operators. They are $\left\{S^2, S_x\right\}, \left\{S^2, S_y\right\}, \left\{S^2, S_z\right\}$. So the maximum number of operators such that they all commute with each others is <u>two</u>.

1.7 Part g

$$|V\rangle = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

The norm is $\sqrt{1+4+9} = \sqrt{14}$. Hence the normalized state is

$$|V\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In part (c) we found the eigenvalues and associated eigenvector for S_z. Here they are again

<u>Summary table of results so far S_z</u>

eigenvalue	eigenvector
$\omega_1 = 0$	$\lvert S_z = \omega_1\rangle = \begin{bmatrix}0\\1\\0\end{bmatrix}$
$\omega_2 = 1$	$\lvert S_z = \omega_2\rangle = \begin{bmatrix}1\\0\\0\end{bmatrix}$
$\omega_3 = -1$	$\lvert S_z = \omega_3\rangle = \begin{bmatrix}0\\0\\1\end{bmatrix}$

We will now find the odds of getting $|S_z = \omega_1\rangle$ given the current state vector is $|V\rangle$ (after normalizing). The odds are

$$|\langle S_z = 0|V\rangle|^2 = \left(\frac{1}{\sqrt{14}}\begin{bmatrix}0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^2 = \left(\frac{2}{\sqrt{14}}\right)^2 = \frac{4}{14} = 28.571\%$$

And

$$|\langle S_z = +1|V\rangle|^2 = \left(\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^2 = \left(\frac{1}{\sqrt{14}}\right)^2 = \frac{1}{14} = 7.143\%$$

And

$$|\langle S_z = -1|V\rangle|^2 = \left(\frac{1}{\sqrt{14}}\begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\right)^2 = \left(\frac{3}{\sqrt{14}}\right)^2 = \frac{9}{14} = 64.285\%$$

<u>Updated summary table of results so far S_z</u>

eigenvalue	eigenvector	odd of getting this eigenvalue
$\omega_1 = 0$	$\lvert S_z = \omega_1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$P(0) = \frac{4}{14} = 28.571\%$
$\omega_2 = 1$	$\lvert S_z = \omega_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$P(1) = \frac{1}{14} = 7.143\%$
$\omega_3 = -1$	$\lvert S_z = \omega_3\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$P(-1) = \frac{9}{14} = 64.285\%$

The statistical average is

$$\begin{aligned} \omega_1\left(\frac{4}{14}\right) + \omega_2\left(\frac{1}{14}\right) + \omega_3\left(\frac{9}{14}\right) &= 0\left(\frac{4}{14}\right) + 1\left(\frac{1}{14}\right) - 1\left(\frac{9}{14}\right) \\ &= -\frac{4}{7} \\ &= -0.571\,43 \end{aligned} \tag{1}$$

The above is now compared to Now we compare $\langle V|S_z|V\rangle$

$$\langle V|(S_z|V\rangle) = \frac{1}{\sqrt{14}}\langle V|\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}^{S_z|V\rangle}\rangle \tag{2}$$

But

$$S_z|V\rangle = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$

Hence Eq. (2) becomes

$$\begin{aligned}\langle V|(S_z|V\rangle) &= \frac{1}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 2 & 3\end{bmatrix}^*\begin{bmatrix}1\\0\\-3\end{bmatrix}\right)\\ &= \frac{1}{14}(1-9)\\ &= \frac{-8}{14}\\ &= -0.571\,43 \end{aligned} \tag{3}$$

Comparing (1) and (3) shows it is the same value. This is the expectation value when measuring S_z.

1.8 Part h

Part (g) is now repeated, but using S_x. We found from the above part that

$$|V\rangle = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

From part(b), we found the eigenvalues for $S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 1 & 0\\1 & 0 & 1\\0 & 1 & 0\end{bmatrix}$ to be $\omega_1 = 0, \omega_2 = 1, \omega_3 = -1$.

But we did not find the associated eigenvectors yet in order to repeat part g as was done for S_z. So we need now to find the eigenvectors for S_x before being able to answer this part for S_x.

For $\underline{\omega_1 = 0}$

$$\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Swapping R_2, R_1

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$R_3 = R_3 - R_2$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is in echelon form. Hence the system becomes

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is free variable. Let $v_3 = s$. Second row gives $v_2 = 0$. First row gives $\frac{1}{\sqrt{2}}v_1 + \frac{1}{\sqrt{2}}s = 0$ or $v_1 = -s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{\sqrt{2}}$. Therefore

$$|S_x = \omega_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

For $\underline{\omega_2 = 1}$

$$\begin{bmatrix} -\omega_2 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\omega_2 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\omega_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + \frac{1}{\sqrt{2}} R_1$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -1 \end{bmatrix}$$

$R_3 = R_3 + \frac{2}{\sqrt{2}} R_2$

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is in echelon form. Hence the system becomes

$$\begin{bmatrix} -1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is free variable. Let $v_3 = s$. Second row gives $-\frac{1}{2} v_2 + \frac{1}{\sqrt{2}} s = 0$ or $v_2 = \frac{2}{\sqrt{2}} s$. First row gives $-v_1 + \frac{1}{\sqrt{2}}v_2 = 0$ or $v_1 = \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{2}}s\right) = s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ \frac{2}{\sqrt{2}}s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ \frac{2}{\sqrt{2}} \\ 1 \end{bmatrix} = s\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{2}$. Therefore

$$|S_x = \omega_2\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

For $\underline{\omega_3 = -1}$

$$\begin{bmatrix} -\omega_3 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\omega_3 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\omega_3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - \frac{1}{\sqrt{2}}R_1$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}$$

$R_3 = R_3 - \frac{2}{\sqrt{2}} R_2$

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is in echelon form. Hence the system becomes

$$\begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

v_3 is free variable. Let $v_3 = s$. Second row gives $\frac{1}{2} v_2 + \frac{1}{\sqrt{2}} s = 0$ or $v_2 = -\frac{2}{\sqrt{2}} s$. First row gives $v_1 + \frac{1}{\sqrt{2}} v_2 = 0$ or $v_1 = -\frac{1}{\sqrt{2}} \left(-\frac{2}{\sqrt{2}} s \right) = s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} s \\ -\frac{2}{\sqrt{2}} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ -\frac{2}{\sqrt{2}} \\ 1 \end{bmatrix} = s \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{2}$. Therefore

$$|S_x = \omega_3\rangle = \frac{1}{2} \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

Summary table of results so far S_x

eigenvalue	eigenvector
$\omega_1 = 0$	$\lvert S_x = \omega_1\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$
$\omega_2 = 1$	$\lvert S_x = \omega_2\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$
$\omega_3 = -1$	$\lvert S_x = \omega_3\rangle = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$

The odds of getting $\lvert S_x = \omega_i\rangle$ given the current state vector is $\lvert V\rangle$ are now found. Expressing $\lvert V\rangle$ in the eigenbasis of S_x gives

$$\begin{aligned} \lvert V\rangle &= c_1 \lvert S_x = \omega_1\rangle + c_2 \lvert S_x = \omega_2\rangle + c_3 \lvert S_x = \omega_3\rangle \\ &= c_1 \lvert S_x = 0\rangle + c_2 \lvert S_x = 1\rangle + c_3 \lvert S_x = -1\rangle \end{aligned} \tag{1}$$

Where

$$c_1 = \langle S_x = 0 \vert V\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^* \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} \left(\frac{-1}{\sqrt{2}} + \frac{3}{\sqrt{2}} \right) = \frac{1}{\sqrt{14}} \left(\frac{2}{\sqrt{2}} \right)$$

$$c_2 = \langle S_x = 1 \vert V\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}^* \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} \left(\frac{1}{2} + \frac{2}{\sqrt{2}} + \frac{3}{2} \right) = \frac{1}{\sqrt{14}} \left(\frac{2}{\sqrt{2}} \right)$$

$$c_3 = \langle S_x = -1 \vert V\rangle = \frac{1}{\sqrt{14}} \begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{bmatrix}^* \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}} \left(\frac{1}{2} - \frac{2}{\sqrt{2}} + \frac{3}{2} \right) = \frac{1}{\sqrt{14}} \left(2 - \sqrt{2} \right)$$

Eq. (1) becomes

$$|V\rangle = \frac{1}{\sqrt{14}}\left(\frac{2}{\sqrt{2}}\right)\begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} + \frac{1}{\sqrt{14}}\left(\sqrt{2}+2\right)\begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} + \frac{1}{\sqrt{14}}\left(2-\sqrt{2}\right)\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

The above is the <u>representation</u> of $|V\rangle$ in the eigenbasis of S_x. The odds of each eigenvalue is the square of the coefficients $|c_1|^2, |c_2|^2, |c_3|^2$ above. Therefore

$$P(0) = \left(\frac{1}{\sqrt{14}}\left(\frac{2}{\sqrt{2}}\right)\right)^2 = \frac{2}{14} = 14.286\%$$

$$P(+1) = \left(\frac{1}{\sqrt{14}}\left(\sqrt{2}+2\right)\right)^2 = \frac{1}{14}\left(6+4\sqrt{2}\right) = 83.263\%$$

$$P(-1) = \left(\frac{1}{\sqrt{14}}\left(2-\sqrt{2}\right)\right)^2 = \frac{1}{14}\left(6-4\sqrt{2}\right) = 24.51\%$$

<u>Updated summary table for S_x</u>

eigenvalue	eigenvector	Odds of getting this eigenvalue
$\omega_1 = 0$	$\|S_x = \omega_1\rangle = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$	$P(0) = \frac{2}{14} = 14.286\%$
$\omega_2 = 1$	$\|S_x = \omega_2\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$	$P(1) = \frac{1}{14}\left(6+4\sqrt{2}\right) = 83.263\%$
$\omega_3 = -1$	$\|S_x = \omega_3\rangle = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$	$P(-1) = \frac{1}{14}\left(6-4\sqrt{2}\right) = 24.51\%$

The statistical average is

$$\omega_1\left(\frac{2}{14}\right)+\omega_2\left(\frac{1}{14}\left(6+4\sqrt{2}\right)\right)+\omega_3\left(\frac{1}{14}\left(6-4\sqrt{2}\right)\right)=0\left(\frac{2}{14}\right)+1\left(\frac{1}{14}\left(6+4\sqrt{2}\right)\right)-1\left(\frac{1}{14}\left(6-4\sqrt{2}\right)\right)$$
$$=\frac{4}{7}\sqrt{2}$$
$$=0.80812 \qquad (1)$$

The above is now compared to

$$\langle V|S_x|V\rangle=\frac{1}{\sqrt{14}}\langle V|\overbrace{\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}\rangle}^{S_x|V\rangle} \qquad (2)$$

But

$$S_x|V\rangle=\begin{bmatrix}0 & \frac{1}{\sqrt{2}} & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}\sqrt{2}\\2\sqrt{2}\\\sqrt{2}\end{bmatrix}$$

Hence Eq. (2) becomes

$$\langle V|S_x|V\rangle=\frac{1}{\sqrt{14}}\left(\frac{1}{\sqrt{14}}\begin{bmatrix}1 & 2 & 3\end{bmatrix}^*\begin{bmatrix}\sqrt{2}\\2\sqrt{2}\\\sqrt{2}\end{bmatrix}\right)$$
$$=\frac{1}{14}\left(\sqrt{2}+4\sqrt{2}+3\sqrt{2}\right)$$
$$=0.80812 \qquad (3)$$

Comparing (1) and (3) shows it is the same value.This is the expectation value when measuring S_x

2 Problem 2

Prove the following results on commutators:

$$\begin{aligned}
[A, B + C] &= [A, B] + [A, C] \\
[A + B, C] &= [A, C] + [B, C] \\
[A, BC] &= B[A, C] + [A, B]C \\
[AB, C] &= A[B, C] + [A, C]B
\end{aligned}$$

Solution

2.1 Part 1

By definition of commutator, which is $[A, B] = AB - BA$, then

$$\begin{aligned}
[A, B + C] &= A(B + C) - (B + C)A \\
&= AB + AC - BA - CA \\
&= (AB - BA) + (AC - CA) \\
&= [A, B] + [A, C]
\end{aligned}$$

2.2 Part 2

By definition of commutator, which is $[A, B] = AB - BA$, then

$$\begin{aligned}
[A + B, C] &= (A + B)C - C(A + B) \\
&= AC + BC - CA - CB \\
&= (AC - CA) + (BC - CB) \\
&= [A, C] + [B, C]
\end{aligned}$$

2.3 Part 3

By definition of commutator, which is $[A, B] = AB - BA$, then

$$[A, BC] = A(BC) - (BC)A$$

Adding and subtracting BAC on the RHS gives

$$\begin{aligned}
[A, BC] &= \widehat{BAC} + ABC - BCA - \widehat{BAC} \\
&= (BAC - BCA) + (ABC - BAC) \\
&= B(AC - CA) + (AB - BA)C \\
&= B[A, C] + [A, B]C
\end{aligned}$$

2.4 Part 4

By definition of commutator, which is $[A, B] = AB - BA$, then

$$\begin{aligned}[AB, C] &= (AB)C - C(AB) \\ &= ABC - CAB\end{aligned}$$

Adding and subtracting ACB on the RHS gives

$$\begin{aligned}[AB, C] &= \overbrace{ACB} + ABC - CAB - \overbrace{ACB} \\ &= (ABC - ACB) + (ACB - CAB) \\ &= A(BC - CB) + (AC - CA)B \\ &= A[B, C] + [A, C]B\end{aligned}$$

3 Problem 3

Follow the discussion of $s_+ = s_x + is_y$ for the electron spin to derive the matrix representation of $s_- = s_x - is_y$

Solution

Experiments show that S_z has two possible values (eigenvalues) of $\frac{\hbar}{2}, -\frac{\hbar}{2}$. Using eigenbasis of S_z

$$|S_z = \frac{\hbar}{2}\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |1\rangle$$
$$|S_z = -\frac{\hbar}{2}\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |2\rangle$$

Gives

$$S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Consider $S_- = S_x - iS_y$. Then

$$\begin{aligned} [S_z, S_-] &= \left[S_z, S_x - iS_y\right] \\ &= [S_z, S_x] - i\left[S_z, S_y\right] \end{aligned} \tag{1}$$

But, using $\left[S_i, S_j\right] = i\hbar \sum_k \epsilon_{ijk} S_k$. Hence

$$[S_z, S_x] = i\hbar S_y \tag{2}$$
$$\left[S_z, S_y\right] = -i\hbar S_x \tag{3}$$

Substituting (2,3) into (1) gives

$$\begin{aligned} [S_z, S_-] &= i\hbar S_y - i(-i\hbar S_x) \\ &= i\hbar S_y + i^2(\hbar S_x) \\ &= i\hbar S_y - \hbar S_x \\ &= \hbar\left(iS_y - S_x\right) \\ &= -\hbar\left(S_x - iS_y\right) \\ &= -\hbar S_- \end{aligned}$$

Therefore we see that

$$[S_z, S_-] = S_z S_- - S_- S_z = -\hbar S_-$$

This implies

$$S_z S_- = S_- S_z - \hbar S_-$$

Therefore

$$\begin{aligned} S_z S_- |1\rangle &= (S_- S_z - \hbar S_-)|1\rangle \\ &= S_- S_z |1\rangle - \hbar S_- |1\rangle \end{aligned}$$

But $S_z|1\rangle = \frac{\hbar}{2}|1\rangle$ then the above becomes

$$\begin{aligned} S_z S_- |1\rangle &= S_- \frac{\hbar}{2}|1\rangle - \hbar S_- |1\rangle \\ &= \left(\frac{\hbar}{2} - \hbar\right) S_- |1\rangle \\ &= -\frac{\hbar}{2} S_- |1\rangle \end{aligned}$$

The above shows that $S_-|1\rangle$ is eigenvector (eigenstate) of S_z with eigenvalue $-\frac{\hbar}{2}$ which is compatible with experiments. Because $S_z|2\rangle = -\frac{\hbar}{2}|2\rangle$ then let

$$S_-|1\rangle = c|2\rangle \tag{4}$$

We now need to find c. Taking the adjoint of both sides of (4) gives

$$\langle 1|S_-^\dagger = c^* \langle 2|$$

Therefore

$$\begin{aligned} \langle 1|S_-^\dagger\, S_-|1\rangle &= c^* c \langle 2|2\rangle \\ &= |c|^2 \langle 2|2\rangle \end{aligned}$$

Since c is real. But $\langle 2|2\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$. The above becomes

$$\langle 1|S_-^\dagger\, S_-|1\rangle = |c|^2 \tag{5}$$

To find c, we need now to calculate $\langle 1|S_-^\dagger\, S_-|1\rangle$. But

$$\begin{aligned} S_-^\dagger\, S_- &= \left(S_x - iS_y\right)^\dagger \left(S_x - iS_y\right) \\ &= \left(S_x^\dagger + iS_y^\dagger\right)\left(S_x - iS_y\right) \end{aligned}$$

Since S_x, S_y are Hermitian operators then $S_x^\dagger = S_x$ and $S_y^\dagger = S_y$. The above now becomes

$$\begin{aligned} S_-^\dagger\, S_- &= \left(S_x + iS_y\right)\left(S_x - iS_y\right) \\ &= S_x^2 - iS_x S_y + iS_y S_x + S_y^2 \\ &= S_x^2 + S_y^2 - i\left(S_x S_y - S_y S_x\right) \\ &= S_x^2 + S_y^2 - i[S_x, S_y] \end{aligned}$$

Where $[S_x, S_y]$ is the commutator. But $[S_x, S_y] = i\hbar \sum_k \epsilon_{ijk} S_k$. Using $i = 1, j = 2$ for x, y, then $[S_i, S_i] = i\hbar(\epsilon_{121}S_1 + \epsilon_{122}S_2 + \epsilon_{123}S_3) = i\hbar S_3 = i\hbar S_z$. Therefore the above now becomes

$$\begin{aligned} S_-^\dagger S_- &= S_x^2 + S_y^2 - i(i\hbar S_z) \\ &= S_x^2 + S_y^2 + \hbar S_z \end{aligned} \tag{6}$$

Substituting (6) in (5) gives

$$\langle 1|\left(S_x^2 + S_y^2 + \hbar S_z\right)|1\rangle = |c|^2$$

But $S^2 = S_x^2 + S_y^2 + S_z^2$. Hence $S_x^2 + S_y^2 = S^2 - S_z^2$. Using this in the above gives

$$\langle 1|\left(S^2 - S_z^2 + \hbar S_z\right)|1\rangle = |c|^2 \tag{7}$$

But $S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Hence $S_z^2 = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\hbar^2}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\hbar^2}{4}I$. And since there is nothing special about the z direction, then $S_x^2 = S_y^2 = S_z^2$. Therefore $S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{\hbar^2}{4}I + \frac{\hbar^2}{4}I + \frac{\hbar^2}{4}I = \frac{3}{4}\hbar^2 I$. Eq. (7) now becomes

$$\langle 1|\frac{3}{4}\hbar^2 - \frac{\hbar^2}{4} + \hbar S_z|1\rangle = |c|^2$$

But $S_z|1\rangle = \frac{\hbar}{2}|1\rangle$. This is because $|1\rangle$ is an eigenvector for S_z with an eigenvalue $\frac{\hbar}{2}$. The above becomes

$$\begin{aligned} \langle 1|\frac{3}{4}\hbar^2 - \frac{\hbar^2}{4} + \hbar\frac{\hbar}{2}|1\rangle &= |c|^2 \\ \langle 1|\frac{3}{4}\hbar^2 - \frac{\hbar^2}{4} + \frac{\hbar^2}{2}|1\rangle &= |c|^2 \\ \langle 1|\hbar^2|1\rangle &= |c|^2 \\ \hbar^2\langle 1|1\rangle &= |c|^2 \\ \hbar^2 &= |c|^2 \end{aligned}$$

We pick

$$c = \hbar$$

Now that c is found, then Eq. (4) above becomes

$$S_-|1\rangle = \hbar|2\rangle \tag{8}$$

The same method is now repeated for finding $S_-|2\rangle$

$$\begin{aligned} S_z S_-|2\rangle &= (S_- S_z - \hbar S_-)|2\rangle \\ &= S_- S_z|2\rangle - \hbar S_-|2\rangle \end{aligned}$$

But $S_z|2\rangle = -\frac{\hbar}{2}|2\rangle$. The above becomes

$$\begin{aligned}S_zS_-|2\rangle &= -S_-\frac{\hbar}{2}|2\rangle - \hbar S_-|2\rangle\\ &= \left(-\frac{\hbar}{2} - \hbar\right)S_-|2\rangle\\ &= \left(-\frac{3\hbar}{2}\right)S_-|2\rangle\end{aligned}$$

The above shows that $S_-|2\rangle$ is eigenvector (eigenstate) of S_z with eigenvalue $-\frac{3\hbar}{2}$ which conflicts with experiments. This means

$$S_-|2\rangle = 0|2\rangle \tag{9}$$

is the only logical result. Therefore now we have all the information to find matrix representation of S_- using (8,9), which is

$$\begin{aligned}S_- &= \begin{bmatrix}\langle 1|S_-|1\rangle & \langle 1|S_-|2\rangle\\ \langle 2|S_-|1\rangle & \langle 2|S_-|2\rangle\end{bmatrix}\\ &= \begin{bmatrix}\langle 1|\hbar|2\rangle & \langle 1|0|2\rangle\\ \langle 2|\hbar|2\rangle & \langle 2|0|2\rangle\end{bmatrix}\\ &= \hbar\begin{bmatrix}\langle 1|2\rangle & 0\\ \langle 2|2\rangle & 0\end{bmatrix}\\ &= \hbar\begin{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} & 0\\ \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} & 0\end{bmatrix}\\ &= \hbar\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\end{aligned}$$

Therefore

$$\begin{aligned}S_- &= S_x - iS_y\\ &= \hbar\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\end{aligned}$$

Which is what we are asked to show.

4 Problem 4 (9.6.2)

Find the solutions $x_1(t), x_2(t)$ with initial conditions $x_1(0) = 0, x_2(0) = 0$ and $\dot{x}_1(0) = v_1, \dot{x}_2(0) = v_2$.

Solution

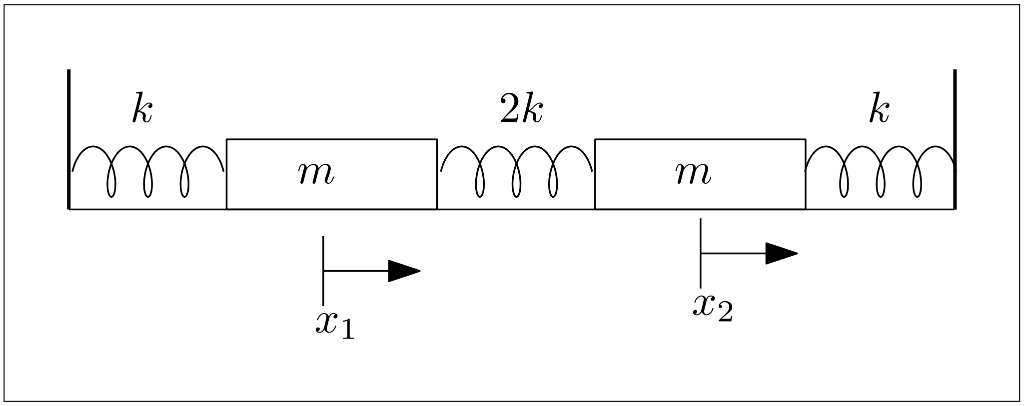

Figure 2: Coupled system to solve

The first step is to draw the free body diagram for each mass. Let us assume that first mass is at some positive distance $x_1 > 0$ so that the first string is in tension, and that $x_2 > x_1 > 0$ so that the middle spring is in tension also, and the third spring is in compression. Any other configuration will also work as well. Based on this, the free body diagrams are

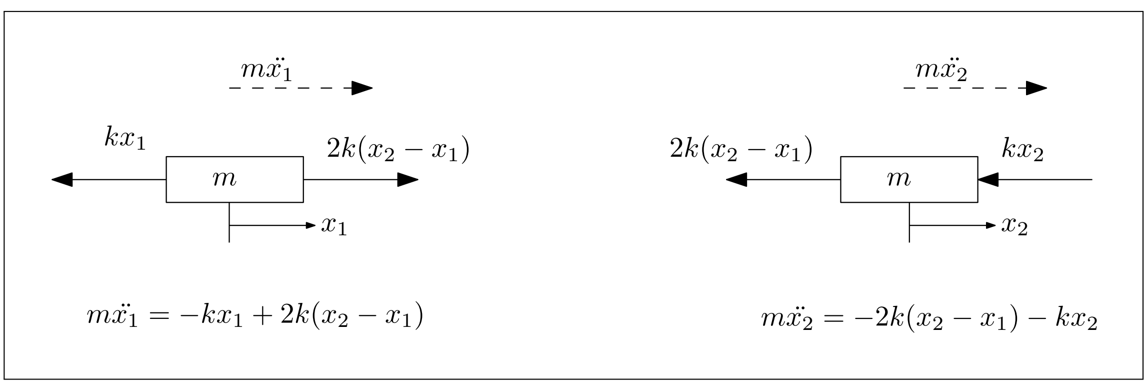

Figure 3: Free body diagram

From the free body diagram, we can now write the equation of motion based on $F = ma$ from each mass. This gives

$$m\ddot{x}_1 = -kx_1 + 2k(x_2 - x_1)$$
$$m\ddot{x}_2 = -2k(x_2 - x_1) - kx_2$$

or

$$\begin{aligned}m\ddot{x}_1 &= -kx_1 + 2kx_2 - 2kx_1\\ m\ddot{x}_2 &= -2kx_2 + 2kx_1 - kx_2\end{aligned}$$

or

$$\begin{aligned}m\ddot{x}_1 &= x_1(-k - 2k) + x_2(2k)\\ m\ddot{x}_2 &= x_1(2k) + x_2(-2k - k)\end{aligned}$$

or

$$\begin{aligned}\ddot{x}_1 &= -\frac{3k}{m}x_1 + \frac{2k}{m}x_2\\ \ddot{x}_2 &= \frac{2k}{m}x_1 - 3\frac{k}{m}x_2\end{aligned}$$

In matrix form the above becomes

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} = \frac{k}{m}\begin{bmatrix}-3 & 2\\ 2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + \frac{k}{m}\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$|\ddot{x}\rangle + M|x\rangle = |0\rangle \tag{1}$$

Where the operator M is

$$M = \frac{k}{m}\begin{bmatrix}3 & -2\\ -2 & 3\end{bmatrix} \tag{1A}$$

In (1), the state vector is $|x\rangle$ is represented using basis $|1\rangle = \begin{bmatrix}1\\ 0\end{bmatrix}$ and $|2\rangle = \begin{bmatrix}0\\ 1\end{bmatrix}$, since we can write

$$|x\rangle = x_1|1\rangle + x_2|2\rangle$$

In these basis, called the natural coordinates, we see than operator M is not diagonal. This makes solving (1) harder, since it is now a coupled system of ODE's.

We would like to decouple (1) to make solving each ODE separate and easier. To do this, we change the basis of M. The new basis are $|I\rangle, |II\rangle$. These are the eigenvectors of M. Since M is Hermitian, then its eigenvalues will be real, and its eigenvectors are orthogonal. So now we need to first find the eigenvalues of M given in (1A) by solving

$$\det(M - \omega I) = 0$$
$$\det\left(\frac{k}{m}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} - \omega\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$
$$\det\left(\frac{k}{m}\begin{bmatrix} 3-\omega & -2 \\ -2 & 3-\omega \end{bmatrix}\right) = 0$$

This gives (we remove the factor $\frac{k}{m}$ for now, then add it at the end to simplify the computation)

$$(3-\omega)(3-\omega) - 4 = 0$$
$$\omega^2 - 6\omega + 5 = 0$$
$$(\omega - 5)(\omega - 1) = 0$$

Hence the eigenvalues are (now we add back the factor $\frac{k}{m}$)

$$\omega_1 = \frac{5k}{m} \qquad \omega_1 = \frac{k}{m}$$

<u>For $\omega_1 = \frac{k}{m}$</u>

We need to solve

$$\begin{bmatrix} \frac{3k}{m} - \omega_1 & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - \omega_1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{3k}{m} - \frac{k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - \frac{k}{m} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} \frac{2k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{2k}{m} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 + R_1$

$$\begin{bmatrix} \frac{2k}{m} & -\frac{2k}{m} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_2 is free variable. Let $v_2 = s$. First row gives $\frac{2k}{m}v_1 - \frac{2k}{m}s = 0$ or $v_1 = s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{\sqrt{2}}$ then

$$|I\rangle = |M = \omega_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\omega_1 = 5\frac{k}{m}$

We need to solve

$$\begin{bmatrix} \frac{3k}{m} - \omega_2 & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - \omega_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3k}{m} - 5\frac{k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} - 5\frac{k}{m} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & -\frac{2k}{m} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 = R_2 - R_1$

$$\begin{bmatrix} -\frac{2k}{m} & -\frac{2k}{m} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence v_2 is free variable. Let $v_2 = s$. First row gives $-\frac{2k}{m}v_1 - \frac{2k}{m}s = 0$ or $v_1 = -s$. Hence solution is

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Let $s = \frac{1}{\sqrt{2}}$ then

$$|II\rangle = |M = \omega_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Summary table of results so far M

eigenvalue	eigenfrequncy	eigenvector
$\omega_1 = \frac{k}{m}$	$\sqrt{\frac{k}{m}}$	$\lvert I\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
$\omega_2 = 5\frac{k}{m}$	$\sqrt{\frac{5k}{m}}$	$\lvert II\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$

The transformation matrix Φ becomes

$$\begin{aligned}\Phi &= \begin{bmatrix}|I\rangle & |II\rangle\end{bmatrix}\\ &= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\end{aligned} \tag{2}$$

Now that we found the transformation matrix Φ we can use it to transform $|\ddot{x}\rangle + M|x\rangle = 0$ which is the natural coordinates basis $|1\rangle, |2\rangle$, to the modal coordinates based on basis $|I\rangle, |II\rangle$ as follows

$$|x\rangle = \Phi|X\rangle \tag{3}$$
$$|\ddot{x}\rangle = \Phi|\ddot{X}\rangle \tag{4}$$

Where

$$|X\rangle = X_1|I\rangle + X_2|II\rangle$$

is the state vector in the modal coordinate and $|\ddot{X}\rangle$ is the acceleration of the state vector in modal coordinates. Applying Eq. (3,4) to $|\ddot{x}\rangle = M|x\rangle$ gives the system in the modal coordinates as

$$\Phi|\ddot{X}\rangle + M\Phi|X\rangle = 0$$

Premultiplying both sides by Φ^T (since Φ is real, then transpose is same as dagger), gives

$$\Phi^T\Phi|\ddot{X}\rangle + \Phi^T M\Phi|X\rangle = 0 \tag{5}$$

But by definition of the modal transformation matrix[1]

$$\Phi^T\Phi = I \tag{6}$$

[1]This can also be shown for $\Phi = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ by working it out. $\Phi^T\Phi = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

And by definition of the transformation matrix [2]

$$\Phi^T M \Phi = \begin{bmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{bmatrix} = \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix} \tag{7}$$

Using (6,7) in (5) gives <u>the system in modal coordinates</u>

$$|\ddot{X}\rangle + \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix} |X\rangle = 0$$

$$\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{8}$$

The above is what solve, since it is now decoupled. Comparing (8) to (1) which is repeated below

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{3k}{m} & -\frac{2k}{m} \\ -\frac{2k}{m} & \frac{3k}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

Shows clearly why (8) is much simpler to solve in the modal coordinates basis $|I\rangle, |II\rangle$ since it is now decoupled, while Eq (1) which is in natural coordinates basis $|1\rangle, |2\rangle$ is coupled.

Eq (8) is now solved for $|X\rangle$, and at the end transformed back to $|x\rangle$ using Eq. (3). Eq (8) above can be written as two separate ODE's

$$\ddot{X}_1 + \frac{k}{m} X_1 = 0$$
$$\ddot{X}_2 + \frac{5k}{m} X_2 = 0$$

---

[2]This can also be shown by working it out as follows. $\Phi^T M \Phi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \frac{k}{m} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ which becomes $\Phi^T M \Phi = \frac{k}{2m} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix}$

Before solving the above, the initial conditions, given in the natural coordinates, needs to be transformed to modal coordinates. It is not clear which initial conditions we should use, since book uses

$$\begin{aligned} x_1(t=0) &= x_1(0) & \dot{x}_1(t=0) &= 0 \\ x_2(t=0) &= x_2(0) & \dot{x}_2(t=0) &= 0 \end{aligned}$$

And in the HW pdf, we are also asked to use the following initial conditions

$$\begin{aligned} x_1(t=0) &= 0 & \dot{x}_1(t=0) &= v_1 \\ x_2(t=0) &= 0 & \dot{x}_2(t=0) &= v_2 \end{aligned}$$

Should we solve it for both cases, or just the second case? I will solve the problem for both cases, since I am not sure which to pick.

4.1 Part 1

Solving using book initial conditions

$$\begin{aligned} x_1(t=0) &= x_1(0) & \dot{x}_1(0) &= 0 \\ x_2(t=0) &= x_2(0) & \dot{x}_2(0) &= 0 \end{aligned}$$

Since $|x\rangle = \Phi|X\rangle$ then the inverse is

$$|X\rangle = \Phi^{-1}|x\rangle$$

But $\Phi^{-1} = \Phi^T$ therefore

$$\begin{aligned} |X(0)\rangle &= \Phi^T|x_1(0)\rangle \\ \begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} x_1(0) + x_2(0) \\ -x_1(0) + x_2(0) \end{bmatrix} \end{aligned}$$

And

$$\begin{aligned} |\dot{X}(0)\rangle &= \Phi^T|\dot{x}_1(0)\rangle \\ \begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Since $\begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Now that we found the initial conditions in modal coordinates, we can solve Eq. (8). Here it is again

$$\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{9}$$

$$\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x_1(0) + x_2(0) \\ -x_1(0) + x_2(0) \end{bmatrix}$$

$$\begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The first equation of (9) becomes

$$\ddot{X}_1 + \frac{k}{m} X_1 = 0$$

$$X_1(0) = \frac{1}{\sqrt{2}} (x_1(0) + x_2(0))$$

$$\dot{X}_1(0) = 0$$

The solution is

$$X_1(t) = A \cos\left(\sqrt{\frac{k}{m}}\, t\right) + B \sin\left(\sqrt{\frac{k}{m}}\, t\right) \tag{10}$$

Where A, B are the constants of integrations. At $t = 0$ and from the initial conditions, the above becomes

$$\frac{1}{\sqrt{2}} (x_1(0) + x_2(0)) = A$$

Taking time derivative of (10) gives

$$\dot{X}_1 = -A \sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right) + B \sqrt{\frac{k}{m}} \cos\left(\sqrt{\frac{k}{m}}\, t\right)$$

Since $\dot{X}_1(0) = 0$ then the above becomes

$$0 = B \sqrt{\frac{k}{m}}$$

Hence $B = 0$. Therefore the solution of Eq (10) is

$$X_1 = \frac{1}{\sqrt{2}} (x_1(0) + x_2(0)) \cos\left(\sqrt{\frac{k}{m}}\, t\right) \tag{11}$$

Tthe second ODE in (9) is now solved.

$$\ddot{X}_2 + \frac{5k}{m} X_1 = 0$$
$$X_2(0) = \frac{1}{\sqrt{2}}(-x_1(0) + x_2(0))$$
$$\dot{X}_2(0) = 0$$

The solution is

$$X_2 = A\cos\left(\sqrt{\frac{5k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{12}$$

Where A, B are the constants of integrations. At $t = 0$,

$$\frac{1}{\sqrt{2}}(-x_1(0) + x_2(0)) = A$$

Taking time derivative of (12) gives

$$\dot{X}_2 = -A\sqrt{\frac{5k}{m}}\,\sin\left(\sqrt{\frac{5k}{m}}\,t\right) + B\sqrt{\frac{5k}{m}}\,\cos\left(\sqrt{\frac{5k}{m}}\,t\right)$$

At $t = 0$ the above becomes

$$0 = B\sqrt{\frac{5k}{m}}$$

Hence $B = 0$. The solution of Eq (12) becomes

$$X_2 = \frac{1}{\sqrt{2}}(-x_1(0) + x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{13}$$

Therefore the solution is

$$|X\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}}(x_1(0) + x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ \frac{1}{\sqrt{2}}(-x_1(0) + x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix} \tag{14}$$

This is the final solution. But it is in modal coordinates. This is transformed back to natural coordinates using Eq (3)

$$|x\rangle = \Phi|X\rangle$$

Therefore

$$
\begin{aligned}
|x\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) \\ \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \\ \frac{1}{\sqrt{2}}(x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{1}{\sqrt{2}}(-x_1(0)+x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} (x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) + (x_1(0)-x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \\ (x_1(0)+x_2(0))\cos\left(\sqrt{\frac{k}{m}}\,t\right) - (x_1(0)-x_2(0))\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \end{bmatrix} \qquad (15)
\end{aligned}
$$

Hence

$$
x_1(t) = \frac{x_1(0)+x_2(0)}{2}\cos\left(\sqrt{\frac{k}{m}}\,t\right) + \frac{x_1(0)-x_2(0)}{2}\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \qquad (16)
$$

$$
x_2(t) = \frac{x_1(0)+x_2(0)}{2}\cos\left(\sqrt{\frac{k}{m}}\,t\right) - \frac{x_1(0)-x_2(0)}{2}\cos\left(\sqrt{\frac{5k}{m}}\,t\right) \qquad (17)
$$

The above is the final solution in the natural coordinates. The above is repeated using the other initial conditions given in the PDF file.

4.2 Part 2

Solving using book initial conditions

$$
\begin{aligned}
x_1(t=0) &= 0 \qquad & \dot{x}_1(t=0) &= v_1 \\
x_2(t=0) &= 0 \qquad & \dot{x}_2(t=0) &= v_2
\end{aligned}
$$

Using $|x\rangle = \Phi|X\rangle$ then

$$
|X\rangle = \Phi^{-1}|x\rangle
$$

But $\Phi^{-1} = \Phi^T$ then

$$
\begin{aligned}
|X(0)\rangle &= \Phi^T |x_1(0)\rangle \\
\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
|\dot{X}(0)\rangle &= \Phi^T |\dot{x}_1(0)\rangle \\
\begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_2 \end{bmatrix}
\end{aligned}
$$

Now that we found initial conditions in modal coordinates, we can finally solve the (8). Here it is again

$$
\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} \frac{k}{m} & 0 \\ 0 & \frac{5k}{m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}
$$

$$
\begin{aligned}
\begin{bmatrix} X_1(0) \\ X_2(0) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \dot{X}_1(0) \\ \dot{X}_2(0) \end{bmatrix} &= \frac{1}{\sqrt{2}} \begin{bmatrix} v_1 + v_2 \\ -v_1 + v_2 \end{bmatrix}
\end{aligned}
$$

The first equation of (18) becomes

$$
\begin{aligned}
\ddot{X}_1 + \frac{k}{m} X_1 &= 0 \\
X_1(0) &= 0 \\
\dot{X}_1(0) &= \frac{1}{\sqrt{2}} (v_1 + v_2)
\end{aligned}
$$

The solution is (since SHM)

$$X_1 = A\cos\left(\sqrt{\frac{k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{k}{m}}\,t\right) \tag{19}$$

Where A, B are the constants of integrations. At $t = 0$,

$$0 = A$$

The solution (19) becomes

$$X_1 = B\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

Taking time derivative of the above gives

$$\dot{X}_1 = B\sqrt{\frac{k}{m}}\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

At $t = 0$ the above becomes

$$\frac{1}{\sqrt{2}}(v_1 + v_2) = B\sqrt{\frac{k}{m}}$$

$$B = \sqrt{\frac{m}{k}}\frac{1}{\sqrt{2}}(v_1 + v_2)$$

Therefore the solution of Eq (19) is

$$X_1 = \sqrt{\frac{m}{2k}}\,(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right) \tag{20}$$

The second ODE in (18) is now solved.

$$\ddot{X}_2 + \frac{5k}{m}X_1 = 0$$

$$X_2(0) = 0$$

$$\dot{X}_2(0) = \frac{1}{\sqrt{2}}(-v_1 + v_2)$$

The solution is

$$X_2 = A\cos\left(\sqrt{\frac{5k}{m}}\,t\right) + B\sin\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{21}$$

Where A, B are the constants of integrations. At $t = 0$,

$$0 = A$$

The solution (21) becomes

$$X_2 = B \sin\left(\sqrt{\frac{5k}{m}}\, t\right)$$

Taking time derivative gives

$$\dot{X}_2 = B\sqrt{\frac{5k}{m}} \cos\left(\sqrt{\frac{5k}{m}}\, t\right)$$

At $t = 0$ the above becomes

$$\frac{1}{\sqrt{2}}(-v_1 + v_2) = B\sqrt{\frac{5k}{m}}$$

$$B = \sqrt{\frac{m}{10k}}\,(-v_1 + v_2)$$

Therefore the solution of Eq (21) is

$$X_2 = \sqrt{\frac{m}{10k}}\,(-v_1 + v_2) \sin\left(\sqrt{\frac{5k}{m}}\, t\right) \tag{22}$$

Therefore the solution state vector is

$$|X\rangle = \begin{bmatrix} \sqrt{\frac{m}{2k}}\,(v_1 + v_2) \sin\left(\sqrt{\frac{k}{m}}\, t\right) \\ \sqrt{\frac{m}{10k}}\,(-v_1 + v_2) \sin\left(\sqrt{\frac{5k}{m}}\, t\right) \end{bmatrix} \tag{23}$$

This is the final solution. But it is in modal coordinates. It is now transformed back to natural coordinates using Eq (3)

$$|x\rangle = \Phi|X\rangle$$

Therefore

$$\begin{aligned}
|x\rangle &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{\frac{m}{2k}}\,(v_1 + v_2) \sin\left(\sqrt{\frac{k}{m}}\, t\right) \\ \sqrt{\frac{m}{10k}}\,(-v_1 + v_2) \sin\left(\sqrt{\frac{5k}{m}}\, t\right) \end{bmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{\frac{m}{2k}}\,(v_1 + v_2) \sin\left(\sqrt{\frac{k}{m}}\, t\right) - \sqrt{\frac{m}{10k}}\,(-v_1 + v_2) \sin\left(\sqrt{\frac{5k}{m}}\, t\right) \\ \sqrt{\frac{m}{2k}}\,(v_1 + v_2) \sin\left(\sqrt{\frac{k}{m}}\, t\right) + \sqrt{\frac{m}{10k}}\,(-v_1 + v_2) \sin\left(\sqrt{\frac{5k}{m}}\, t\right) \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{\frac{m}{4k}}\,(v_1 + v_2) \sin\left(\sqrt{\frac{k}{m}}\, t\right) - \sqrt{\frac{m}{20k}}\,(-v_1 + v_2) \sin\left(\sqrt{\frac{5k}{m}}\, t\right) \\ \sqrt{\frac{m}{4k}}\,(v_1 + v_2) \sin\left(\sqrt{\frac{k}{m}}\, t\right) + \sqrt{\frac{m}{20k}}\,(-v_1 + v_2) \sin\left(\sqrt{\frac{5k}{m}}\, t\right) \end{bmatrix} \qquad (24)
\end{aligned}$$

Hence

$$x_1(t) = \frac{1}{2}\sqrt{\frac{m}{k}}\,(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right) + \sqrt{\frac{m}{20k}}\,(v_1 - v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{25}$$

$$x_2(t) = \frac{1}{2}\sqrt{\frac{m}{k}}\,(v_1 + v_2)\sin\left(\sqrt{\frac{k}{m}}\,t\right) + \sqrt{\frac{m}{20k}}\,(-v_1 + v_2)\sin\left(\sqrt{\frac{5k}{m}}\,t\right) \tag{26}$$

The above is the final solution in the natural coordinates.