1. (a) Problem 9.1.6. (5 points)

$$a(1,1,0)+b(1,0,1)+c(3,2,1)=(a+b+3c,a+2c,b+c)=(0,0,0),$$
$$\Rightarrow a=-2c,\ \ b=-c,\ \ a+b+3c=0.$$

The above linear combination of the three row vectors is a null row vector for any nonzero value of c with $a=-2c$ and $b=-c$. Therefore, the three row vectors are linearly dependent.

$$a(1,1,0)+b(1,0,1)+c(0,1,1)=(a+b,a+c,b+c)=(0,0,0),$$
$$\Rightarrow a=-b=-c,\ \ b=-c\Rightarrow b=c=-c=0\Rightarrow a=0.$$

Therefore, the above three row vectors are linearly independent.

(b) Problem 9.2.1.(ii). (10 points)

$$|V\rangle=\begin{bmatrix}1+i\\ \sqrt{3}+i\end{bmatrix}$$

$$|I\rangle=\begin{bmatrix}\frac{1+i\sqrt{3}}{4}\\ -\frac{\sqrt{3}(1+i)}{\sqrt{8}}\end{bmatrix},\quad |II\rangle=\begin{bmatrix}\frac{\sqrt{3}(1+i)}{\sqrt{8}}\\ \frac{\sqrt{3}+i}{4}\end{bmatrix}$$

$$\langle I|I\rangle=\begin{bmatrix}\frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}}\end{bmatrix}\begin{bmatrix}\frac{1+i\sqrt{3}}{4}\\ -\frac{\sqrt{3}(1+i)}{\sqrt{8}}\end{bmatrix}=\frac{1+3}{16}+\frac{3(1+1)}{8}=1$$

$$\langle II|II\rangle=\begin{bmatrix}\frac{\sqrt{3}(1-i)}{\sqrt{8}} & \frac{\sqrt{3}-i}{4}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}(1+i)}{\sqrt{8}}\\ \frac{\sqrt{3}+i}{4}\end{bmatrix}=\frac{3(1+1)}{8}+\frac{3+1}{16}=1$$

$$\langle I|II\rangle=\begin{bmatrix}\frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}}\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}(1+i)}{\sqrt{8}}\\ \frac{\sqrt{3}+i}{4}\end{bmatrix}=\frac{(1-i\sqrt{3})\sqrt{3}(1+i)-\sqrt{3}(1-i)(\sqrt{3}+i)}{4\sqrt{8}}$$
$$=\frac{-i(i+\sqrt{3})\sqrt{3}(1+i)+i\sqrt{3}(1+i)(\sqrt{3}+i)}{4\sqrt{8}}=0\Rightarrow\langle II|I\rangle=\langle I|II\rangle^*=0$$

$$v_I=\langle I|V\rangle=\begin{bmatrix}\frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{\sqrt{8}}\end{bmatrix}\begin{bmatrix}1+i\\ \sqrt{3}+i\end{bmatrix}=\frac{(1-i\sqrt{3})(1+i)}{4}-\frac{\sqrt{3}(1-i)(\sqrt{3}+i)}{\sqrt{8}}$$
$$=\frac{1+\sqrt{3}+i(1-\sqrt{3})}{4}-\frac{\sqrt{3}[1+\sqrt{3}+i(1-\sqrt{3})]}{\sqrt{8}}=\left(\frac{1}{4}-\sqrt{\frac{3}{8}}\right)[1+\sqrt{3}+i(1-\sqrt{3})]$$

$$v_{II}=\langle II|V\rangle=\begin{bmatrix}\frac{\sqrt{3}(1-i)}{\sqrt{8}} & \frac{\sqrt{3}-i}{4}\end{bmatrix}\begin{bmatrix}1+i\\ \sqrt{3}+i\end{bmatrix}=\frac{\sqrt{3}(1-i)(1+i)}{\sqrt{8}}+\frac{(\sqrt{3}-i)(\sqrt{3}+i)}{4}$$
$$=\frac{2\sqrt{3}}{\sqrt{8}}+1=1+\sqrt{\frac{3}{2}}$$

$$|v_I|^2 + |v_{II}|^2 = \left| \left(\frac{1}{4} - \sqrt{\frac{3}{8}} \right) [1 + \sqrt{3} + i(1 - \sqrt{3}\,)] \right|^2 + \left| 1 + \sqrt{\frac{3}{2}} \right|^2$$
$$= \left(\frac{1}{16} - \frac{1}{2}\sqrt{\frac{3}{8}} + \frac{3}{8} \right) [(1 + \sqrt{3}\,)^2 + (1 - \sqrt{3}\,)^2] + 1 + 2\sqrt{\frac{3}{2}} + \frac{3}{2}$$
$$= 8 \left(\frac{7}{16} - \frac{1}{4}\sqrt{\frac{3}{2}} \right) + \frac{5}{2} + 2\sqrt{\frac{3}{2}} = 6$$

(c) Problem 9.2.3. (10 points)

$$|I\rangle = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad |II\rangle = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad |III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$$
$$|1\rangle = \frac{|I\rangle}{\sqrt{\langle I|I\rangle}} = \frac{1}{3} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\langle 1|II\rangle = 0 \Rightarrow |2'\rangle = |II\rangle, \ \langle 2'|2'\rangle = 5 \Rightarrow |2\rangle = \frac{|2'\rangle}{\sqrt{\langle 2'|2'\rangle}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$
$$\langle 1|III\rangle = 0, \ \langle 2|III\rangle = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} = \frac{2 + 10}{\sqrt{5}} = \frac{12}{\sqrt{5}}$$
$$|3'\rangle = |III\rangle - |2\rangle\langle 2|III\rangle = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} - \frac{12}{\sqrt{5}} \begin{bmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 - \frac{12}{5} \\ 5 - \frac{24}{5} \end{bmatrix} = \begin{bmatrix} 0 \\ -2/5 \\ 1/5 \end{bmatrix}$$
$$\langle 3'|3'\rangle = \frac{4 + 1}{25} = \frac{1}{5} \Rightarrow |3\rangle = \frac{|3'\rangle}{\sqrt{\langle 3'|3'\rangle}} = \sqrt{5} \begin{bmatrix} 0 \\ -2/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

2. Use Tr $\sigma_i = 0$, $\sigma_i^2 = I$, and $\sigma_i \sigma_j = i \sum_k \epsilon_{ijk} \sigma_k$ to obtain the components of a general 2×2 matrix in the basis of $\{\sigma_1, \sigma_2, \sigma_3, I\}$, where σ_i represents the Pauli matrices and I is the identity matrix. (15 points)

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 + \delta I, \ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \mathrm{Tr}(I) = 2$$
$$\mathrm{Tr}(M) = a + d = \mathrm{Tr}(\alpha\sigma_1 + \beta\sigma_2 + \gamma\sigma_3 + \delta I) = \alpha\mathrm{Tr}(\sigma_1) + \beta\mathrm{Tr}(\sigma_2) + \gamma\mathrm{Tr}(\sigma_3) + \delta\mathrm{Tr}(I) = 2\delta$$
$$\delta = \frac{a + d}{2}$$
$$\mathrm{Tr}(M\sigma_1) = \mathrm{Tr}(\alpha\sigma_1^2 + \beta\sigma_2\sigma_1 + \gamma\sigma_3\sigma_1 + \delta I\sigma_1) = \alpha\mathrm{Tr}(\sigma_1^2) + \beta\mathrm{Tr}(\sigma_2\sigma_1) + \gamma\mathrm{Tr}(\sigma_3\sigma_1) + \delta\mathrm{Tr}(I\sigma_1)$$
$$= \alpha\mathrm{Tr}(I) + \beta\mathrm{Tr}(-i\sigma_3) + \gamma\mathrm{Tr}(i\sigma_2) + \delta\mathrm{Tr}(\sigma_1) = 2\alpha$$
$$\alpha = \frac{1}{2}\mathrm{Tr}(M\sigma_1) = \frac{1}{2}\mathrm{Tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = \frac{1}{2}\mathrm{Tr} \begin{bmatrix} b & a \\ d & c \end{bmatrix} = \frac{b + c}{2}$$

Similarly, we obtain

$$\beta = \frac{1}{2}\mathrm{Tr}(M\sigma_2) = \frac{1}{2}\mathrm{Tr}\left(\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right]\right) = \frac{1}{2}\mathrm{Tr}\left[\begin{array}{cc} ib & -ia \\ id & -ic \end{array}\right] = \frac{i(b-c)}{2}$$

$$\gamma = \frac{1}{2}\mathrm{Tr}(M\sigma_3) = \frac{1}{2}\mathrm{Tr}\left(\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right]\right) = \frac{1}{2}\mathrm{Tr}\left[\begin{array}{cc} a & -b \\ c & -d \end{array}\right] = \frac{a-d}{2}$$

3. Problem 9.2.5. (10 points)

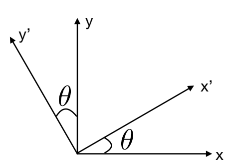
$$\begin{aligned} |V+W|^2 &= \langle V+W|V+W\rangle = \langle V|V\rangle + \langle V|W\rangle + \langle W|V\rangle + \langle W|W\rangle \\ &= |V|^2 + \langle V|W\rangle + \langle V|W\rangle^* + |W|^2 = |V|^2 + 2\mathrm{Re}\langle V|W\rangle + |W|^2 \\ &\le |V|^2 + 2|\langle V|W\rangle| + |W|^2 \le |V|^2 + 2|V||W| + |W|^2 = (|V|+|W|)^2 \\ \Rightarrow |V+W| &\le |V| + |W|. \end{aligned}$$

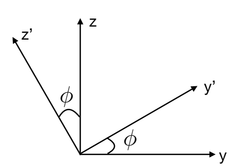
For the equality $|V+W| = |V| + |W|$ to hold, we must have $\mathrm{Re}\langle V|W\rangle = |\langle V|W\rangle|$ and $|\langle V|W\rangle| = |V||W|$. From the first condition, $\langle V|W\rangle$ is real and positive. From the second condition (see proof of the Schwarz inequality in the textbook),

$$|V\rangle = \frac{\langle W|V\rangle}{|W|^2}|W\rangle = \frac{\langle V|W\rangle^*}{|W|^2}|W\rangle = a|W\rangle, \; a = \frac{\langle V|W\rangle^*}{|W|^2}.$$

Because $\langle V|W\rangle$ is real and positive, a is also real and positive.

4. Problem 9.3.5. (20 points)





From the above figures, we have

$$R_z(\theta) = \left[\begin{array}{ccc} \vec{i}\cdot\vec{i'} & \vec{i}\cdot\vec{j'} & \vec{i}\cdot\vec{k'} \\ \vec{j}\cdot\vec{i'} & \vec{j}\cdot\vec{j'} & \vec{j}\cdot\vec{k'} \\ \vec{k}\cdot\vec{i'} & \vec{k}\cdot\vec{j'} & \vec{k}\cdot\vec{k'} \end{array}\right] = \left[\begin{array}{ccc} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{array}\right]$$

for rotation around the z-axis by an angle θ, and

$$R_x(\phi) = \left[\begin{array}{ccc} \vec{i}\cdot\vec{i'} & \vec{i}\cdot\vec{j'} & \vec{i}\cdot\vec{k'} \\ \vec{j}\cdot\vec{i'} & \vec{j}\cdot\vec{j'} & \vec{j}\cdot\vec{k'} \\ \vec{k}\cdot\vec{i'} & \vec{k}\cdot\vec{j'} & \vec{k}\cdot\vec{k'} \end{array}\right] = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{array}\right]$$

for rotation around the x-axis by an angle ϕ.

$$R_z(\theta)^T R_z(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

$$R_x(\phi)^T R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I,$$

$$R_x(\phi) R_z(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix},$$

$$\begin{aligned} &[R_x(\phi) R_z(\theta)]^T R_x(\phi) R_z(\theta) \\ &\quad = \begin{bmatrix} \cos\theta & \sin\theta\cos\phi & \sin\theta\sin\phi \\ -\sin\theta & \cos\theta\cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix} \\ &\quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I. \end{aligned}$$

In general, if $M^T M = I$ and $N^T N = I$, then we have $(NM)^T (NM) = M^T N^T N M = M^T M = I$.

5. Problem 9.5.6, but only for the proof without doing the inverse matrix part. (10 points)

For a Hermitian operator Ω with non-degenerate eigenvalues, we have

$$\Omega|\omega_i\rangle = \omega_i|\omega_i\rangle, \quad i = 1,\ 2,\ \cdots,\ n.$$

We can then expand an arbitrary vector as

$$|V\rangle = \sum_{i=1}^{n} v_i|\omega_i\rangle.$$

The characteristic polynomial satisfies $P(\omega_i) = 0$, so

$$P(\Omega)|V\rangle = \sum_{i=1}^{n} v_i P(\Omega)|\omega_i\rangle = \sum_{i=1}^{n} v_i P(\omega_i)|\omega_i\rangle = \sum_{i=1}^{n} 0|\omega_i\rangle = |0\rangle \Rightarrow P(\Omega) = 0.$$

6. Problem 9.5.10. (20 points)

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad N = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$MN = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

$$NM = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} = MN$$

$$M \Rightarrow \begin{vmatrix} 1-\omega & 0 & 1 \\ 0 & -\omega & 0 \\ 1 & 0 & 1-\omega \end{vmatrix} = -\omega(1-\omega)^2 + \omega = \omega^2(2-\omega) = 0 \Rightarrow \omega_M = 0,\ 0,\ 2.$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} a_0 + c_0 \\ 0 \\ a_0 + c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow |\omega_M = 0\rangle = \begin{bmatrix} a_0 \\ b_0 \\ -a_0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -a_2 + c_2 \\ -2b_2 \\ a_2 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow |\omega_M = 2\rangle = \begin{bmatrix} a_2 \\ 0 \\ a_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$N \Rightarrow \begin{vmatrix} 2-\omega & 1 & 1 \\ 1 & -\omega & -1 \\ 1 & -1 & 2-\omega \end{vmatrix} = (2-\omega)[-\omega(2-\omega) - 1] - (2 - \omega + 1) - 1 + \omega$$

$$= (2-\omega)(\omega - 3)(\omega + 1) = 0 \Rightarrow \omega_N = -1,\ 2,\ 3.$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} d_1 \\ e_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} 3d_1 + e_1 + f_1 \\ d_1 + e_1 - f_1 \\ d_1 - e_1 + 3f_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |\omega_N = -1\rangle = d_1 \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} d_2 \\ e_2 \\ f_2 \end{bmatrix} = \begin{bmatrix} e_2 + f_2 \\ d_2 - 2e_2 - f_2 \\ d_2 - e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |\omega_N = 2\rangle = d_2 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} d_3 \\ e_3 \\ f_3 \end{bmatrix} = \begin{bmatrix} -d_3 + e_3 + f_3 \\ d_3 - 3e_3 - f_3 \\ d_3 - e_3 - f_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow |\omega_N = 3\rangle = d_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

It is clear from the above results that M and N share a common eigenbasis

$$\{|\omega_N = -1\rangle,\ |\omega_N = 2\rangle,\ |\omega_N = 3\rangle\}.$$