Physics 3041 (Spring 2021) Solutions to Homework Set 4

1. (a) Problem 8.1.1. (5 points)

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow R_{\theta'} R_\theta = \begin{bmatrix} \cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta' \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos\theta'\cos\theta - \sin\theta'\sin\theta & \cos\theta'\sin\theta + \sin\theta'\cos\theta \\ -\sin\theta'\cos\theta - \cos\theta'\sin\theta & -\sin\theta'\sin\theta + \cos\theta'\cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} \cos(\theta+\theta') & \sin(\theta+\theta') \\ -\sin(\theta+\theta') & \cos(\theta+\theta') \end{bmatrix} = R_{\theta+\theta'}.$$

(b) Problem 8.1.2, and find the expression of θ in terms of the relative velocity. (10 points)

From the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1-v^2}} = x\cosh\theta - t\sinh\theta,$$
$$t' = \frac{t - vx}{\sqrt{1-v^2}} = t\cosh\theta - x\sinh\theta,$$

we obtain

$$\cosh\theta = \frac{1}{\sqrt{1-v^2}},\ \sinh\theta = \frac{v}{\sqrt{1-v^2}} \Rightarrow \frac{\sinh\theta}{\cosh\theta} = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}} = \frac{\eta - \eta^{-1}}{\eta + \eta^{-1}} = \frac{\eta^2 - 1}{\eta^2 + 1} = v,$$
$$\Rightarrow \eta^2 - 1 = (\eta^2+1)v,\ \eta = e^\theta = \sqrt{\frac{1+v}{1-v}},$$
$$\Rightarrow \theta = \ln\sqrt{\frac{1+v}{1-v}} = \frac{1}{2}\ln\frac{1+v}{1-v},$$

where v is in units of the speed of light c.

$$\begin{bmatrix} x'' \\ t'' \end{bmatrix} = \begin{bmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{bmatrix} \begin{bmatrix} x' \\ t' \end{bmatrix} = \begin{bmatrix} \cosh\theta' & -\sinh\theta' \\ -\sinh\theta' & \cosh\theta' \end{bmatrix} \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$
$$= \begin{bmatrix} \cosh\theta'\cosh\theta + \sinh\theta'\sinh\theta & -\cosh\theta'\sinh\theta - \sinh\theta'\cosh\theta \\ -\sinh\theta'\cosh\theta - \cosh\theta'\sinh\theta & \sinh\theta'\sinh\theta + \cosh\theta'\cosh\theta \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}$$
$$= \begin{bmatrix} \cosh(\theta+\theta') & -\sinh(\theta+\theta') \\ -\sinh(\theta+\theta') & \cosh(\theta+\theta') \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix},$$

where we have used

$$\cosh\theta'\cosh\theta + \sinh\theta'\sinh\theta = \frac{(e^{\theta'} + e^{-\theta'})(e^\theta + e^{-\theta})}{4} + \frac{(e^{\theta'} - e^{-\theta'})(e^\theta - e^{-\theta})}{4}$$
$$= \frac{e^{\theta'+\theta} + e^{\theta'-\theta} + e^{-\theta'+\theta} + e^{-\theta'-\theta}}{4} + \frac{e^{\theta'+\theta} - e^{\theta'-\theta} - e^{-\theta'+\theta} + e^{-\theta'-\theta}}{4}$$
$$= \frac{e^{\theta'+\theta} + e^{-\theta'-\theta}}{2} = \cosh(\theta+\theta'),$$
$$\sinh\theta'\cosh\theta + \cosh\theta'\sinh\theta = \frac{(e^{\theta'} - e^{-\theta'})(e^\theta + e^{-\theta})}{4} + \frac{(e^{\theta'} + e^{-\theta'})(e^\theta - e^{-\theta})}{4}$$
$$= \frac{e^{\theta'+\theta} + e^{\theta'-\theta} - e^{-\theta'+\theta} - e^{-\theta'-\theta}}{4} + \frac{e^{\theta'+\theta} - e^{\theta'-\theta} + e^{-\theta'+\theta} - e^{-\theta'-\theta}}{4}$$
$$= \frac{e^{\theta'+\theta} - e^{-\theta'-\theta}}{2} = \sinh(\theta+\theta').$$

(c) Problem 8.2.4, for Lorentz transformation only. (5 points)

For the Lorentz transformation,

$$L_\theta = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}, \; |L_\theta| = \cosh^2\theta - \sinh^2\theta = \frac{(e^\theta + e^{-\theta})^2}{4} - \frac{(e^\theta - e^{-\theta})^2}{4} = 1,$$

$$L_{\theta,C} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \Rightarrow L_\theta^{-1} = \frac{L_{\theta,C}^T}{|L_\theta|} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} = L_{-\theta}.$$

The above result makes sense as the inverse Lorentz transformation corresponds to changing the sign of the relative velocity $v \to -v$, which in turn changes the sign of the rapidity $\theta = \frac{1}{2}\ln\frac{1+v}{1-v} \to -\theta = \frac{1}{2}\ln\frac{1-v}{1+v}$.

2. (a) Problem 8.3.4, but using Cramer's rule to solve the first set of equations only. (5 points)

$$\begin{aligned} 3x - y - z &= 2 \\ x - 2y - 3z &= 0 \\ 4x + y + 2z &= 4 \end{aligned}$$

$$\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} = 3 \times (-1) + 14 - 9 = 2$$

$$x = \frac{\begin{vmatrix} 2 & -1 & -1 \\ 0 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}} = \frac{1}{2}\left(2\begin{vmatrix} -2 & -3 \\ 1 & 2 \end{vmatrix} + 4\begin{vmatrix} -1 & -1 \\ -2 & -3 \end{vmatrix}\right) = \frac{-2+4}{2} = 1$$

$$y = \frac{\begin{vmatrix} 3 & 2 & -1 \\ 1 & 0 & -3 \\ 4 & 4 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}} = \frac{1}{2}\left(-2\begin{vmatrix} 1 & -3 \\ 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ 1 & -3 \end{vmatrix}\right) = \frac{-2 \times 14 - 4 \times (-8)}{2} = 2$$

$$z = \frac{\begin{vmatrix} 3 & -1 & 2 \\ 1 & -2 & 0 \\ 4 & 1 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & -1 \\ 1 & -2 & -3 \\ 4 & 1 & 2 \end{vmatrix}} = \frac{1}{2}\left(2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix}\right) = \frac{2 \times 9 + 4 \times (-5)}{2} = -1$$

(b) Problem 8.3.5. (5 points)

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \Rightarrow M_C = \begin{bmatrix} 2 & 2 & -3 \\ 4 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix}, \; |M| = 1 \times 2 + 2 \times 2 + 3 \times (-3) = -3,$$

$$M^{-1} = \frac{M_C^T}{|M|} = -\frac{1}{3} \begin{bmatrix} 2 & 4 & -3 \\ 2 & -11 & 6 \\ -3 & 6 & -3 \end{bmatrix} = \begin{bmatrix} -2/3 & -4/3 & 1 \\ -2/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{bmatrix},$$

$$M^{-1}M = \begin{bmatrix} -2/3 & -4/3 & 1 \\ -2/3 & 11/3 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. (a) Problem 8.4.3. (5 points)

$$(MN)_{ij} = \sum_k M_{ik} N_{kj}$$

$$\Rightarrow (MN)^\dagger_{ij} = (MN)^*_{ji} = \sum_k M^*_{jk} N^*_{ki} = \sum_k M^\dagger_{kj} N^\dagger_{ik} = \sum_k N^\dagger_{ik} M^\dagger_{kj} = (N^\dagger M^\dagger)_{ij}$$

$$\Rightarrow (MN)^\dagger = N^\dagger M^\dagger$$

For $M^\dagger = M$ and $N^\dagger = N$, we have

$$(MN)^\dagger = N^\dagger M^\dagger = NM.$$

If $NM = MN$, i.e., M and N commute, then $(MN)^\dagger = MN$, i.e., MN is Hermitian. Otherwise, MN is not Hermitian.

(b) Problem 8.4.19, proving the first result only. (5 points)

$$\begin{aligned} \text{Tr } MN &= \sum_i (MN)_{ii} = \sum_i \sum_k M_{ik} N_{ki} \\ &= \sum_i \sum_k N_{ki} M_{ik} = \sum_k \sum_i N_{ki} M_{ik} \\ &= \sum_k (NM)_{kk} = \text{Tr } NM \end{aligned}$$

(c) Problem 8.4.20. (10 points)

From the given properties of the Dirac matrices, $M_i^2 = I$, and for $i \neq j$,

$$M_i M_j + M_j M_i = 0 \Rightarrow M_i^2 M_j + M_i M_j M_i = M_j + M_i M_j M_i = 0,$$

where we have multiplied both sides of the first equality by M_i. Taking trace of the two sides of the last equality, we obtain

$$\begin{aligned} \text{Tr } (M_j + M_i M_j M_i) &= \text{Tr } M_j + \text{Tr } M_i M_j M_i = \text{Tr } M_j + \text{Tr } M_j M_i^2 \\ &= \text{Tr } M_j + \text{Tr } M_j = 2\text{Tr } M_j = 0 \Rightarrow \text{Tr } M_j = 0, \end{aligned}$$

where we have used $\text{Tr } ABC = \text{Tr } BCA$.

4. (a) Problem 8.4.5. (10 points)

$$U = \begin{bmatrix} \frac{1+i\sqrt{3}}{4} & \frac{\sqrt{3}(1+i)}{2\sqrt{2}} \\ -\frac{\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4} \end{bmatrix} \Rightarrow U^\dagger = \begin{bmatrix} \frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1-i)}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{4} \end{bmatrix}$$

$$UU^\dagger = \begin{bmatrix} \frac{1+i\sqrt{3}}{4} & \frac{\sqrt{3}(1+i)}{2\sqrt{2}} \\ -\frac{\sqrt{3}(1+i)}{2\sqrt{2}} & \frac{i+\sqrt{3}}{4} \end{bmatrix} \begin{bmatrix} \frac{1-i\sqrt{3}}{4} & -\frac{\sqrt{3}(1-i)}{2\sqrt{2}} \\ \frac{\sqrt{3}(1-i)}{2\sqrt{2}} & \frac{-i+\sqrt{3}}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As $(U^\dagger)_{ij} = [(U^T)_{ij}]^*$ and $|U| = |U^T|$, we have $|U^\dagger| = |U|^*$. Let $|U| = re^{i\theta}$, so $|U^\dagger| = re^{-i\theta}$.

$$|UU^\dagger| = |U||U^\dagger| = r^2 = |I| = 1 \Rightarrow r = 1 \Rightarrow |U| = e^{i\theta}$$

Note here $|\ |$ means the determinant of a matrix, **NOT** the modulus of a complex number. For the above example,

$$|U| = \left(\frac{1+i\sqrt{3}}{4}\right)\frac{i+\sqrt{3}}{4} - \frac{\sqrt{3}(1+i)}{2\sqrt{2}}\left[-\frac{\sqrt{3}(1+i)}{2\sqrt{2}}\right] = i\left(\frac{-i+\sqrt{3}}{4}\right)\frac{i+\sqrt{3}}{4} + \frac{3(1+i)^2}{8}$$
$$= \frac{i}{4} + \frac{3i}{4} = i = e^{i\pi/2}$$

$$|U^\dagger| = \left(\frac{1-i\sqrt{3}}{4}\right)\frac{-i+\sqrt{3}}{4} - \frac{\sqrt{3}(1-i)}{2\sqrt{2}}\left[-\frac{\sqrt{3}(1-i)}{2\sqrt{2}}\right] = -i\left(\frac{i+\sqrt{3}}{4}\right)\frac{-i+\sqrt{3}}{4} + \frac{3(1-i)^2}{8}$$
$$= -\frac{i}{4} - \frac{3i}{4} = -i = e^{-i\pi/2} = |U|^*$$

(b) Problem 8.4.8. (10 points)

$$L = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow L^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

$$F(L) = e^{\theta L} = \sum_{n=0}^{\infty} \frac{(\theta L)^n}{n!} = I - \frac{\theta^2}{2!}I + \frac{\theta^4}{4!}I - \cdots + \left(\theta L - \frac{\theta^3}{3!}L + \frac{\theta^5}{5!}L - \cdots\right)$$
$$= I\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + L\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) = I\cos\theta + L\sin\theta$$
$$= \begin{bmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(c) Problem 8.4.10. (5 points)

$$e^{iH} = I + iH + \frac{(iH)^2}{2!} + \frac{(iH)^3}{3!} + \cdots + \frac{(iH)^n}{n!} + \cdots, \quad H^\dagger = H$$
$$\left(e^{iH}\right)^\dagger = I - iH^\dagger + \frac{(-iH^\dagger)^2}{2!} + \frac{(-iH^\dagger)^3}{3!} + \cdots + \frac{(-iH^\dagger)^n}{n!} + \cdots$$
$$= I - iH + \frac{(-iH)^2}{2!} + \frac{(-iH)^3}{3!} + \cdots + \frac{(-iH)^n}{n!} + \cdots = e^{-iH}$$
$$e^{iH}\left(e^{iH}\right)^\dagger = e^{iH}e^{-iH} = I \Rightarrow e^{iH} \text{ is unitary}$$

5. Problem 8.4.17. (5 points)

$$
\begin{aligned}
(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) &= \left(\sum_{i=1}^{3}\sigma_i a_i\right)\left(\sum_{j=1}^{3}\sigma_j b_j\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \sigma_i \sigma_j \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \left(I\delta_{ij} + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k\right) \\
&= I\sum_{i=1}^{3} a_i b_i + i\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk} a_i b_j \left(\hat{e}_k\cdot\vec{\sigma}\right) \\
&= \vec{a}\cdot\vec{b}I + i\left(\vec{a}\times\vec{b}\right)\cdot\vec{\sigma} = \vec{a}\cdot\vec{b}I + i\vec{\sigma}\cdot\left(\vec{a}\times\vec{b}\right),
\end{aligned}
$$

where we have used

$$
\vec{a}\times\vec{b} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk} a_i b_j \hat{e}_k.
$$

6. (a) Consider a horizontal spring-mass system. The spring has a spring constant k and is fixed at one end. The other end is attached to a block of mass m that can move without friction on a horizontal surface. The spring is stretched a length a beyond its rest length and let go. Without solving the problem using Newton's second law, find the angular frequency of oscillations and show that it is independent of a. (5 points)

Using units to indicate dimensions, we have

$$
\begin{aligned}
[k] &= \text{N/meter} = \text{kg}\cdot(\text{meter/s}^2)/\text{meter} = \text{kg/s}^2, \\
[m] &= \text{kg},\ \ [a] = \text{meter}.
\end{aligned}
$$

$$
\begin{aligned}
[\omega] &= 1/\text{s} = [k]^\alpha [m]^\beta [a]^\gamma = \text{kg}^{\alpha+\beta}\text{meter}^\gamma/\text{s}^{2\alpha} \\
\alpha+\beta = 0,\ 2\alpha &= 1, \gamma = 0 \Rightarrow \alpha = 1/2,\ \beta = -1/2,\ \gamma = 0 \\
[\omega] &= [k]^{1/2}[m]^{-1/2} = \left[\sqrt{k/m}\right].
\end{aligned}
$$

Therefore, the angular frequency ω is independent of a.

(b) Derive the Planck mass, length, and time in terms of Planck's constant $\hbar$, Newton's constant G, and speed of light c. Evaluate these quantities in SI units. (10 points)

Using units to indicate dimensions, we have

$$
\begin{aligned}
[\hbar] &= \text{J}\cdot\text{s} = \text{kg}\cdot(\text{m/s})^2\cdot\text{s} = \text{kg}\cdot\text{m}^2/\text{s}, \\
[G] &= \text{N}\cdot\text{m}^2/\text{kg}^2 = \text{kg}\cdot(\text{m/s}^2)\cdot\text{m}^2/\text{kg}^2 = \text{m}^3/(\text{kg}\cdot\text{s}^2), \\
[c] &= \text{m/s}.
\end{aligned}
$$

$$
\begin{aligned}
[M_{\text{Pl}}] &= \text{kg} = [\hbar]^\alpha [G]^\beta [c]^\gamma = \text{kg}^{\alpha-\beta}\cdot\text{m}^{2\alpha+3\beta+\gamma}/\text{s}^{\alpha+2\beta+\gamma}, \\
\alpha-\beta = 1,\ 2\alpha+3\beta+\gamma &= 0,\ \alpha+2\beta+\gamma = 0 \Rightarrow \alpha = 1/2,\ \beta = -1/2,\ \gamma = 1/2.
\end{aligned}
$$

So the Planck mass is

$$M_{\rm Pl} = \left(\frac{\hbar c}{G}\right)^{1/2} = 2.18 \times 10^{-8}\ \text{kg}.$$

With $[M_{\rm Pl}c^2] = \text{J}$, it is straightforward to obtain the Planck time and length

$$T_{\rm Pl} = \frac{\hbar}{M_{\rm Pl}c^2} = \left(\frac{\hbar G}{c^5}\right)^{1/2} = 5.39 \times 10^{-44}\ \text{s},$$
$$L_{\rm Pl} = cT_{\rm Pl} = \left(\frac{\hbar G}{c^3}\right)^{1/2} = 1.62 \times 10^{-35}\ \text{m}.$$

(c) Identify the relevant physical quantities and use dimensional analysis to find the characteristic length for a black hole of mass M. (5 points)

The relevant physical quantities are Newton's constant G, the speed of light c, and the black-hole mass M, the first two of which are fundamental to general relativity and the last of which specifies the macroscopic property of the black hole.

Using units to denote the dimenions, we have

$$[G] = \text{N} \cdot \text{m}^2/\text{kg}^2 = \text{kg} \cdot (\text{m/s}^2) \cdot \text{m}^2/\text{kg}^2 = \text{m}^3/(\text{s}^2 \cdot \text{kg}),$$
$$[c] = \text{m/s},\ \ [M] = \text{kg}.$$

$$[\text{length}] = \text{m} = [G]^\alpha [c]^\beta [M]^\gamma = \text{m}^{3\alpha+\beta} \cdot \text{s}^{-2\alpha-\beta} \cdot \text{kg}^{-\alpha+\gamma},$$
$$3\alpha + \beta = 1,\ -2\alpha - \beta = 0,\ -\alpha + \gamma = 0 \Rightarrow \alpha = 1,\ \beta = -2,\ \gamma = 1.$$

So we obtain

$$[\text{length}] = \frac{GM}{c^2}.$$