1. (a) Problem 5.2.3. (5 points)

$$\begin{aligned}
\frac{2+3i}{6+7i}+\frac{2}{x+iy} &= \frac{(2+3i)(6-7i)}{(6+7i)(6-7i)}+\frac{2}{x+iy}=\frac{33+4i}{85}+\frac{2}{x+iy}=2+9i,\\
\frac{2}{x+iy} &= 2+9i-\frac{33+4i}{85}=\frac{85(2+9i)-(33+4i)}{85}=\frac{137+761i}{85},\\
x+iy &= \frac{170}{137+761i}=\frac{170(137-761i)}{(137+761i)(137-761i)}=\frac{170(137-761i)}{137^2+761^2},
\end{aligned}$$

which gives

$$\begin{aligned}
x &= \frac{170\times 137}{137^2+761^2}=\frac{137}{3517},\\
y &= -\frac{170\times 761}{137^2+761^2}=-\frac{761}{3517}.
\end{aligned}$$

(b) Problem 5.2.4.(iv). (5 points)

We first show that $|z_1/z_2|=|z_1|/|z_2|$.

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \left|\frac{x_1+iy_1}{x_2+iy_2}\right|=\left|\frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}\right|=\frac{|(x_1+iy_1)(x_2-iy_2)|}{x_2^2+y_2^2}\\
&= \frac{|(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)|}{x_2^2+y_2^2}=\frac{\sqrt{(x_1x_2+y_1y_2)^2+(x_2y_1-x_1y_2)^2}}{x_2^2+y_2^2}\\
&= \frac{\sqrt{x_1^2x_2^2+y_1^2y_2^2+x_2^2y_1^2+x_1^2y_2^2}}{x_2^2+y_2^2}=\frac{\sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)}}{x_2^2+y_2^2}=\frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}}=\frac{|z_1|}{|z_2|}.
\end{aligned}$$

$$\begin{aligned}
z &= \frac{1+i\sqrt{2}}{1-i\sqrt{3}}=\frac{(1+i\sqrt{2})(1+i\sqrt{3})}{(1-i\sqrt{3})(1+i\sqrt{3})}=\frac{1-\sqrt{6}+i(\sqrt{2}+\sqrt{3})}{4}\Rightarrow \mathrm{Re}(z)=\frac{1-\sqrt{6}}{4},\\
\mathrm{Im}(z) &= \frac{\sqrt{2}+\sqrt{3}}{4},\ |z|=\frac{|1+i\sqrt{2}|}{|1-i\sqrt{3}|}=\frac{\sqrt{1+2}}{\sqrt{1+3}}=\frac{\sqrt{3}}{2}=\sqrt{[\mathrm{Re}(z)]^2+[\mathrm{Im}(z)]^2}\ ,\\
z^* &= \frac{1-\sqrt{6}-i(\sqrt{2}+\sqrt{3})}{4},\ \frac{1}{z}=\frac{z^*}{|z|^2}=\frac{1-\sqrt{6}-i(\sqrt{2}+\sqrt{3})}{4(\sqrt{3}/2)^2}=\frac{1-\sqrt{6}-i(\sqrt{2}+\sqrt{3})}{3}.
\end{aligned}$$

(c) Problem 5.2.5. (10 points)

We first show that $(z_1z_2)^*=z_1^*z_2^*$.

$$\begin{aligned}
(z_1z_2)^* &= [(x_1+iy_1)(x_2+iy_2)]^*=[x_1x_2-y_1y_2+i(x_1y_2+y_1x_2)]^*=x_1x_2-y_1y_2-i(x_1y_2+y_1x_2)\\
z_1^*z_2^* &= (x_1+iy_1)^*(x_2+iy_2)^*=(x_1-iy_1)(x_2-iy_2)=x_1x_2-y_1y_2-i(x_1y_2+y_1x_2)=(z_1z_2)^*
\end{aligned}$$

It is straightforward to generalize the above result to $(z^m)^*=(z^*)^m$ for integers of $m\geq 2$.

Now if z satifies

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0,$$

where the coefficients a_0, a_1, $\cdots$, a_n are real, then taking complex conjugation of both sides of the equation gives

$$a_0 + a_1 z^* + a_2 (z^2)^* + \cdots + a_n (z^n)^* = 0 \Rightarrow a_0 + a_1 z^* + a_2 (z^*)^2 + \cdots + a_n (z^*)^n = 0.$$

Therefore, both z and z^* are the roots of the above polynomial equation, which means the roots are either real ($z = z^*$) or pairs of complex conjugates.

(d) Problem 5.3.2. (20 points)

$$\begin{aligned}
\text{(i) } z_1 &= \frac{1+i}{\sqrt{2}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\pi/4} \Rightarrow z_1^* = e^{-i\pi/4},\ |z_1| = 1, \\
z_2 &= \sqrt{3} - i = 2 \times \frac{\sqrt{3}-i}{2} = 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = 2e^{-i\pi/6} \Rightarrow z_2^* = 2e^{i\pi/6},\ |z_2| = 2, \\
z_1 z_2 &= e^{i\pi/4} \times 2e^{-i\pi/6} = 2e^{i\pi/12},\ \frac{z_1}{z_2} = \frac{e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{1}{2}e^{5i\pi/12},\ \left(\frac{z_1}{z_2}\right)^* = \frac{1}{2}e^{-5i\pi/12}.
\end{aligned}$$

$$\begin{aligned}
\text{(ii) } z_1 &= \frac{3+4i}{3-4i} = \frac{5e^{i\tan^{-1}(4/3)}}{5e^{-i\tan^{-1}(4/3)}} = e^{2i\tan^{-1}(4/3)} \Rightarrow z_1^* = e^{-2i\tan^{-1}(4/3)},\ |z_1| = 1, \\
z_2 &= \left[\frac{1+2i}{1-3i}\right]^2 = \left[\frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}\right]^2 = \left[\frac{1-6+5i}{10}\right]^2 = \left[\frac{-1+i}{2}\right]^2 = \frac{1-1-2i}{4} = -\frac{i}{2} \\
&= \frac{e^{-i\pi/2}}{2} \Rightarrow z_2^* = \frac{e^{i\pi/2}}{2},\ |z_2| = \frac{1}{2}, \\
z_1 z_2 &= e^{2i\tan^{-1}(4/3)} \times \frac{e^{-i\pi/2}}{2} = \frac{e^{i[2\tan^{-1}(4/3)-(\pi/2)]}}{2},\ \frac{z_1}{z_2} = \frac{e^{2i\tan^{-1}(4/3)}}{(1/2)e^{-i\pi/2}} = 2e^{i[2\tan^{-1}(4/3)+(\pi/2)]}, \\
\left(\frac{z_1}{z_2}\right)^* &= 2e^{-i[2\tan^{-1}(4/3)+(\pi/2)]}.
\end{aligned}$$

2. (a) Problem 5.3.5. (10 points)

$$S = e^{i\theta} + e^{3i\theta} + \cdots + e^{(2n-1)i\theta},\ e^{2i\theta}S = e^{3i\theta} + \cdots + e^{(2n-1)i\theta} + e^{(2n+1)i\theta}$$

$$(1 - e^{2i\theta})S = e^{i\theta} - e^{(2n+1)i\theta}$$

$$S = \frac{e^{i\theta} - e^{(2n+1)i\theta}}{1 - e^{2i\theta}} = \frac{1 - e^{2ni\theta}}{e^{-i\theta} - e^{i\theta}} = \frac{i(1 - e^{2ni\theta})}{2\sin\theta} = \frac{i(1-\cos 2n\theta) + \sin 2n\theta}{2\sin\theta}$$

$$\begin{aligned}
\text{Re}(S) &= \cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta = \frac{\sin 2n\theta}{2\sin\theta}, \\
\text{Im}(S) &= \sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{1-\cos 2n\theta}{2\sin\theta} = \frac{\sin^2 n\theta}{\sin\theta}.
\end{aligned}$$

(b) Problem 5.3.6. (10 points)

$$\begin{aligned}
\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\
&= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4 \\
&= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + 4i(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta), \\
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta, \\
\sin 4\theta &= 4(\cos^3\theta\sin\theta - \cos\theta\sin^3\theta).
\end{aligned}$$

$$\begin{aligned}
e^{i(A+B)} &= \cos(A+B) + i\sin(A+B) \\
&= e^{iA}e^{iB} = (\cos A + i\sin A)(\cos B + i\sin B) \\
&= \cos A\cos B - \sin A\sin B + i(\sin A\cos B + \cos A\sin B), \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B, \\
\sin(A+B) &= \sin A\cos B + \cos A\sin B.
\end{aligned}$$

(c) Find $\int_0^\infty xe^{-ax}\cos kx dx$ using Euler's formula. (10 points)

$$\begin{aligned}
\int_0^\infty xe^{-ax}\cos kx dx &= \int_0^\infty (xe^{-ax})\frac{e^{ikx}+e^{-ikx}}{2}dx = \int_0^\infty x \times \frac{e^{-(a-ik)x}+e^{-(a+ik)x}}{2}dx \\
&= \frac{1}{2}\left[\frac{1}{(a-ik)^2} + \frac{1}{(a+ik)^2}\right] = \frac{(a+ik)^2+(a-ik)^2}{2(a-ik)^2(a+ik)^2} = \frac{a^2-k^2}{(a^2+k^2)^2},
\end{aligned}$$

where we have made the substitutions $z = (a \pm ik)x$ and used $\int_0^\infty ze^{-z}dz = 1$ for $a > 0$.

3. Given the intensity pattern for the N-slit interference with separation d between adjacent slits, show that the pattern becomes that for the single-slit diffraction with slit width a when d goes to zero but with a fixed value of $Nd = a$. (10 points)

From the lectures, the intensity pattern for the N-slit interference is described by

$$\bar{I}(\theta) = \bar{I}(0)\left[\frac{\sin(N\pi d\sin\theta/\lambda)}{N\sin(\pi d\sin\theta/\lambda)}\right]^2.$$

In the limit of $d \to 0$, $\sin(\pi d\sin\theta/\lambda) \to \pi d\sin\theta/\lambda$, so we have

$$\bar{I}(\theta) \to \bar{I}(0)\left[\frac{\sin(N\pi d\sin\theta/\lambda)}{N\pi d\sin\theta/\lambda}\right]^2 = \bar{I}(0)\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2,$$

where we have used $Nd = a$. The above limiting result is the intensity pattern for the single-slit diffraction.

4. (1) Find the roots z_n $(n = 1,\, 2,\, \cdots,\, N)$ of the complex equation $z^N = 1$. (5 points)

$$z^N = 1 = e^{i2k\pi},\ \ k = 0,\ 1,\ 2,\ \cdots \Rightarrow z = e^{i2k\pi/N}.$$

So we can take $z_n = e^{i2(n-1)\pi/N}$. Note that values of $n \geq N+1$ do not give new roots as $e^{i2k\pi} = 1$.

(2) Find $S_N = \sum_{n=1}^{N} z_n$ and give a geometric interpretation of the result. (10 points)

Let $\phi = 2\pi/N$. So $z_n = e^{i(n-1)\phi}$.

$$S_N = 1 + e^{i\phi} + e^{i2\phi} + \cdots + e^{i(N-1)\phi},\ \ e^{i\phi} S_N = e^{i\phi} + e^{i2\phi} + \cdots + e^{i(N-1)\phi} + e^{iN\phi}$$

$$(1 - e^{i\phi}) S_N = 1 - e^{iN\phi} \Rightarrow S_N = \frac{1 - e^{iN\phi}}{1 - e^{i\phi}} = \frac{1 - e^{i2\pi}}{1 - e^{i2\pi/N}} = 0.$$

Recall that the complex number $e^{i\phi}$ corresponds to a unit vector making an angle ϕ with respect to the x-axis and that counterclockwise rotation of a vector by an angle ϕ corresponds to multiplication by $e^{i\phi}$. So S_N represents the sum of N unit vectors that form the sides of a regular polygon. This vectorial sum vanishes because the vectors form a closed figure.

(3) Note that $1 - z^N = (1 - z)(1 + z + z^2 + \cdots + z^{N-1})$. Relate this result and the roots z_n to the conditions for destructive interference among N slits. (5 points)

The net electric field at a point on the observational screen for N-slit interference can be represented by a sum $1 + z + z^2 + \cdots + z^{N-1}$, where $z = e^{i\phi'}$ with $\phi' = 2\pi d \sin\theta/\lambda$ being the phase difference between the contributions from adjacent slits. When $\phi' = 2k\pi/N$ or $d \sin\theta = k\lambda/N$ $(k = 1,\, 2,\, \cdots,\, N-1)$, z becomes the roots z_n $(n \geq 2)$ and the sum vanishes because $1 - z^N = 0$ but $1 - z \neq 0$. For values of $k \geq N + 1$ that are not integer multiples of N, the same roots are repeated due to $e^{i2\pi} = 1$.