HW 3

Physics 3041
Mathematical Methods for Physicists

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Nasser M. Abbasi

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Contents

1 Problem 1(a) (problem 5.2.3)

Solve for x and y given

$$\frac{2+3i}{6+7i}+\frac{2}{x+iy}=2+9i$$

<u>solution</u>

Let $z=x+iy$ be the complex number to solve for. The above becomes

$$\begin{aligned}
\frac{2}{z} &= 2+9i-\frac{2+3i}{6+7i}\\
&= 2+9i-\frac{(2+3i)(6-7i)}{(6+7i)(6-7i)}\\
&= 2+9i-\frac{12-14i+18i+21}{36+49}\\
&= 2+9i-\frac{33+4i}{85}\\
&= \frac{85(2+9i)-33-4i}{85}\\
&= \frac{170+765i-33-4i}{85}\\
&= \frac{137+761i}{85}
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{2}{z} &= \frac{137+761i}{85}\\
z &= \frac{170}{137+761i}\\
&= \frac{170(137-761i)}{(137+761i)(137-761i)}\\
&= \frac{23290-129370i}{597\,890}\\
&= \frac{23290}{597890}-\frac{129370}{597890}i\\
&= \frac{137}{3517}-\frac{761}{3517}i
\end{aligned}$$

But $z=x+iy$. Hence

$$x+iy=\frac{137}{3517}-\frac{761}{3517}i$$

Comparing real and imaginary parts shows that

$$\begin{aligned}
x &= \frac{137}{3517}\\
y &= -\frac{761}{3517}
\end{aligned}$$

2 Problem 1(b) (problem 5.2.4(iv))

Find the real part, imaginary part, modulus, complex conjugate and inverse of the following (iv) $\frac{1+\sqrt{2}i}{1-\sqrt{3}i}$

solution

$$
\begin{aligned}
z &= \frac{1+\sqrt{2}i}{1-\sqrt{3}i} \\
&= \frac{\left(1+\sqrt{2}i\right)\left(1+\sqrt{3}i\right)}{\left(1-\sqrt{3}i\right)\left(1+\sqrt{3}i\right)} \\
&= \frac{1+\sqrt{3}i+\sqrt{2}i-\sqrt{2}\sqrt{3}}{4} \\
&= \frac{1-\sqrt{6}}{4} + i\frac{\sqrt{3}+\sqrt{2}}{4}
\end{aligned}
$$

Hence the real part is $\frac{1-\sqrt{6}}{4}$ and the imaginary part is $\frac{\sqrt{3}+\sqrt{2}}{4}$. Therefore we can now write

$$
\begin{aligned}
z &= x + iy \\
&= \left(\frac{1-\sqrt{6}}{4}\right) + i\left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)
\end{aligned}
$$

The modulus is

$$
\begin{aligned}
|z| &= \sqrt{x^2+y^2} \\
&= \sqrt{\left(\frac{1-\sqrt{6}}{4}\right)^2 + \left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)^2} \\
&= \sqrt{\frac{7}{16} - \frac{1}{8}\sqrt{6} + \frac{1}{8}\sqrt{6} + \frac{5}{16}} \\
&= \sqrt{\frac{3}{4}}
\end{aligned}
$$

The complex conjugate of z is z^*. Hence

$$
\begin{aligned}
z^* &= x - iy \\
&= \left(\frac{1-\sqrt{6}}{4}\right) - i\left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)
\end{aligned}
$$

The inverse is

$$
\begin{aligned}
\frac{1}{z} &= \frac{z^*}{zz^*} \\
&= \frac{z^*}{|z|^2} \\
&= \frac{\left(\frac{1-\sqrt{6}}{4}\right) - i\left(\frac{\sqrt{3}+\sqrt{2}}{4}\right)}{\frac{3}{4}} \\
&= \frac{1-\sqrt{6}}{3} - i\frac{\sqrt{3}+\sqrt{2}}{3}
\end{aligned}
$$

3 Problem 1(c) (problem 5.2.5)

Show that a polynomial with real coefficients has only real roots or complex roots that come in complex conjugate pairs.

solution

Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

be polynomial in z where a_i are all real. We just need to show now that if λ is a root, then its complex conjugate λ^* must also a root. If the root happened to be real, then its complex conjugate is itself. Hence nothing to do in this case. We only need to worry about the case when the root is complex and show that its complex conjugate must also be root.

Assuming λ is a root, then by definition of a root we have

$$\begin{aligned} p(\lambda) &= 0 \\ &= a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \\ &= \sum_{k=0}^{n} a_k \lambda^k \end{aligned} \tag{1}$$

Therefore, replacing λ by λ^* on both sides of (1) gives

$$p(\lambda^*) = \sum_{k=0}^{n} a_k (\lambda^*)^k$$

But $(\lambda^*)^k = \left(\lambda^k\right)^*$ from complex numbers properties (equation 5.2.20 in book). The above becomes

$$p(\lambda^*) = \sum_{k=0}^{n} a_k \left(\lambda^k\right)^*$$

Since a_k are real coefficients, then $a_k^* = a_k$ and the above can be written as

$$p(\lambda^*) = \sum_{k=0}^{n} \left(a_k \lambda^k\right)^*$$

Using property that $A^*B^* = (AB)^*$ where $A = a_k, B = \lambda^k$ in the above. Now we can move the complex conjugate outside the sum, using property that $A^* + B^* = (A + B)^*$. Hence the above becomes

$$p(\lambda^*) = \left(\sum_{k=0}^{n} a_k \lambda^k\right)^*$$

But from (1), we know that $\sum_{k=0}^{n} a_k \lambda^k = 0$, this is because λ is assumed to be a root. Therefore the above gives

$$\begin{aligned} p(\lambda^*) &= 0^* \\ &= 0 \end{aligned}$$

The above shows that λ^* is also a root if λ is a root. Therefore, the root can be either real, or complex. If the root is complex, its complex conjugate is also a root. A real root is just special case of complex root. QED.

4 Problem 1(d) (problem 5.3.2)

For the following pairs of numbers, give their polar form, their complex conjugate, moduli, product, the quotient $\frac{z_1}{z_2}$, and the complex conjugate of the quotient

$$z_1 = \frac{1+i}{\sqrt{2}} \qquad z_2 = \sqrt{3} - i$$
$$z_1 = \frac{3+4i}{3-4i} \qquad z_2 = \left(\frac{1+2i}{1-3i}\right)^2$$

solution

4.1 First pair

$$z_1 = \frac{1+i}{\sqrt{2}} \qquad z_2 = \sqrt{3} - i$$

The polar form of z is $re^{i\theta}$ where $r = |z|$ and $\theta = \arctan\left(\frac{y}{x}\right)$ when $z = x + iy$. The first step is to write $z = x + iy$

For z_1

$$\begin{aligned} z_1 &= \frac{1+i}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \end{aligned}$$

Hence $x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}$. Therefore $|z_1| = \sqrt{x^2 + y^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$. And $\theta = \arctan(1) = 45^0$. Therefore in polar

$$\begin{aligned} z_1 &= re^{i\theta} \\ &= e^{i\left(45^0\right)} \\ &= e^{i\frac{\pi}{4}} \end{aligned}$$

For z_2

$$z_2 = \sqrt{3} - i$$

Hence $x = \sqrt{3}, y = -1$. Therefore $|z_1| = \sqrt{x^2 + y^2} = \sqrt{3+1} = 2$. And $\theta = \arctan\left(\frac{-1}{\sqrt{3}}\right) = -30^0$. Therefore in polar

$$\begin{aligned} z_2 &= re^{i\theta} \\ &= 2e^{i\left(-30^0\right)} \\ &= 2e^{-i\frac{\pi}{6}} \end{aligned}$$

The complex conjugate is

$$\begin{aligned} z_1^* &= re^{-i\theta} \\ &= e^{-i\frac{\pi}{4}} \end{aligned}$$

And

$$\begin{aligned} z_2^* &= re^{-i\theta} \\ &= 2e^{i\frac{\pi}{6}} \end{aligned}$$

And moduli is

$$
\begin{aligned}
|z_1| &= r \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
|z_2| &= r \\
&= 2
\end{aligned}
$$

And product

$$
\begin{aligned}
z_1 z_2 &= \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right) \\
&= r_1 r_2 e^{i(\theta_1+\theta_2)}
\end{aligned}
$$

But $r_1 = 1, r_2 = 2, \theta_1 = 45^0, \theta_2 = -30^0$. The above becomes

$$
\begin{aligned}
z_1 z_2 &= 2e^{i\left(45^0-30^0\right)} \\
&= 2e^{i\left(15^0\right)} \\
&= 2e^{i\frac{\pi}{12}}
\end{aligned}
$$

And the quotient $\frac{z_1}{z_2}$ is

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} \\
&= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}
\end{aligned}
$$

But $r_1 = 1, r_2 = 2, \theta_1 = 45^0, \theta_2 = -30^0$. The above becomes

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{1}{2} e^{i\left(45^0+30^0\right)} \\
&= \frac{1}{2} e^{i\left(75^0\right)} \\
&= \frac{1}{2} e^{i\frac{5\pi}{12}}
\end{aligned}
$$

And the complex conjugate of the quotient is

$$
\begin{aligned}
\left(\frac{z_1}{z_2}\right)^* &= \left(\frac{1}{2} e^{i\frac{5\pi}{12}}\right)^* \\
&= \frac{1}{2} e^{-i\frac{5\pi}{12}}
\end{aligned}
$$

4.2 Second pair

$$
z_1 = \frac{3+4i}{3-4i} \qquad z_2 = \left(\frac{1+2i}{1-3i}\right)^2
$$

The polar form of z is $re^{i\theta}$ where $r = |z|$ and $\theta = \arctan\left(\frac{y}{x}\right)$ where $z = x + iy$. Hence

For z_1

$$\begin{aligned}
z_1 &= \frac{3+4i}{3-4i} \\
&= \frac{\sqrt{3^2+4^2}\, e^{i\arctan\left(\frac{4}{3}\right)}}{\sqrt{3^2+4^2}\, e^{i\arctan\left(-\frac{4}{3}\right)}} \\
&= \frac{e^{i\arctan\left(\frac{4}{3}\right)}}{e^{-i\arctan\left(\frac{4}{3}\right)}} \\
&= e^{i\arctan\left(\frac{4}{3}\right)+\arctan\left(\frac{4}{3}\right)} \\
&= e^{i\left(2\arctan\left(\frac{4}{3}\right)\right)} \\
&= e^{i\left(106.26^0\right)}
\end{aligned}$$

For z_2

$$\begin{aligned}
z_2 &= \left(\frac{1+2i}{1-3i}\right)^2 \\
&= \left(\frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}\right)^2 \\
&= \left(\frac{-5+5i}{10}\right)^2 \\
&= \frac{25-25-50i}{100} \\
&= \frac{-1}{2}i
\end{aligned}$$

Hence $x=0, y=-\frac{1}{2}$. Therefore $|z_1| = \sqrt{x^2+y^2} = \sqrt{0+\frac{1}{4}} = \frac{1}{2}$. And $\theta = \arctan(-\infty) = -90^0$. Therefore in polar

$$\begin{aligned}
z_2 &= re^{i\theta} \\
&= \frac{1}{2}e^{i\left(-90^0\right)} \\
&= \frac{1}{2}e^{-i\frac{\pi}{2}}
\end{aligned}$$

The complex conjugate is

$$\begin{aligned}
z_1^* &= r_1 e^{-i\theta_1} \\
&= e^{-i\left(2\arctan\left(\frac{4}{3}\right)\right)} \\
&= e^{i\left(-106.26^0\right)}
\end{aligned}$$

And

$$\begin{aligned}
z_2^* &= r_2 e^{-i\theta_2} \\
&= \frac{1}{2}e^{i\frac{\pi}{2}}
\end{aligned}$$

And moduli is

$$\begin{aligned}
|z_1| &= r_1 \\
&= 1
\end{aligned}$$

And

$$|z_2| = r_2$$
$$= \frac{1}{2}$$

And product

$$z_1 z_2 = \left(r_1 e^{i\theta_1}\right)\left(r_2 e^{i\theta_2}\right)$$
$$= r_1 r_2 e^{i(\theta_1+\theta_2)}$$

But $r_1 = 1, r_2 = \frac{1}{2}, \theta_1 = 106.26^0, \theta_2 = -90^0$. The above becomes

$$z_1 z_2 = \frac{1}{2} e^{i\left(106.26^0 - 90^0\right)}$$
$$= \frac{1}{2} e^{i\left(16.26^0\right)}$$

And the quotient $\frac{z_1}{z_2}$ is

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$$
$$= \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

But $r_1 = 1, r_2 = \frac{1}{2}, \theta_1 = 106.26^0, \theta_2 = -90^0$. The above becomes

$$\frac{z_1}{z_2} = 2e^{i\left(106.26^0+90^0\right)}$$
$$= 2e^{i\left(196.26^0\right)}$$

And the complex conjugate of the quotient is

$$\left(\frac{z_1}{z_2}\right)^* = \left(2e^{i\left(196.26^0\right)}\right)^*$$
$$= 2e^{-i\left(196.26^0\right)}$$

5 Problem 2(a) (problem 5.3.5)

Consider series

$$e^{i\theta}+e^{3i\theta}+\cdots+e^{(2n-1)i\theta}$$

Sum this geometric series, take the real and imaginary parts of both sides and show that

$$\cos\theta+\cos(3\theta)+\cdots+\cos((2n-1)\theta)=\frac{\sin(2n\theta)}{2\sin\theta}$$

And that a similar sum with sines adds up to $\frac{\sin^2(n\theta)}{\sin\theta}$

solution

Let

$$S=e^{i\theta}+e^{3i\theta}+\cdots+e^{(2n-1)i\theta} \tag{1}$$

Then

$$\begin{aligned}
e^{2i\theta}S &= e^{2i\theta}\left(e^{i\theta}+e^{3i\theta}+\cdots+e^{(2n-1)i\theta}\right)\\
&= e^{i3\theta}+e^{5i\theta}+\cdots+e^{(2n-1)i\theta+2i\theta}\\
&= e^{i3\theta}+e^{5i\theta}+\cdots+e^{(2n+1)i\theta} \qquad (2)
\end{aligned}$$

Hence (2-1) gives

$$\begin{aligned}
e^{2i\theta}S-S &= e^{(2n+1)i\theta}-e^{i\theta}\\
S\left(e^{2i\theta}-1\right) &= e^{(2n+1)i\theta}-e^{i\theta}\\
S &= \frac{e^{(2n+1)i\theta}-e^{i\theta}}{e^{2i\theta}-1}
\end{aligned}$$

Hence

$$\begin{aligned}
S &= \frac{e^{i\theta}\left(e^{i2n\theta}-1\right)}{e^{2i\theta}-1}\\
&= e^{i\theta}\frac{\left(e^{in\theta}\left(e^{in\theta}-e^{-in\theta}\right)\right)}{e^{i\theta}\left(e^{i\theta}-e^{-i\theta}\right)}\\
&= \frac{e^{in\theta}\left(e^{in\theta}-e^{-in\theta}\right)}{\left(e^{i\theta}-e^{-i\theta}\right)}\\
&= e^{in\theta}\frac{\left(e^{in\theta}-e^{-in\theta}\right)}{\left(e^{i\theta}-e^{-i\theta}\right)}\\
&= e^{in\theta}\frac{\sin n\theta}{\sin\theta}\\
&= \cos(n\theta+i\sin n\theta)\frac{\sin n\theta}{\sin\theta}\\
&= \frac{\cos(n\theta)\sin(n\theta)}{\sin\theta}+i\frac{\sin^2(n\theta)}{\sin\theta}
\end{aligned}$$

But $\cos(n\theta)\sin(n\theta)=\frac{1}{2}\sin(2n\theta)$. Therefore the above becomes

$$S=\frac{\sin(2n\theta)}{2\sin\theta}+i\frac{\sin^2(n\theta)}{\sin\theta}$$

Hence

$$\operatorname{Re}(S)=\frac{\sin(2n\theta)}{2\sin\theta} \tag{3}$$

$$\operatorname{Im}(S)=\frac{\sin^2(n\theta)}{\sin\theta} \tag{4}$$

Now we look at the LHS. Since $S = e^{i\theta} + e^{3i\theta} + \cdots + e^{(2n-1)i\theta}$, then

$$\begin{aligned} S &= (\cos\theta + i\sin\theta) + (\cos 3\theta + i\sin 3\theta) + \cdots + (\cos(2n-1)\theta + i\sin(2n-1)\theta) \\ &= (\cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta) + i(\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta) \end{aligned} \tag{5}$$

Comparing (5) and (3,4) shows that

$$\begin{aligned} \cos\theta + \cos 3\theta + \cdots + \cos(2n-1)\theta &= \mathrm{Re}(S) \\ &= \frac{\sin(2n\theta)}{2\sin\theta} \end{aligned}$$

And

$$\begin{aligned} \sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta &= \mathrm{Im}(S) \\ &= \frac{\sin^2(n\theta)}{\sin\theta} \end{aligned}$$

Which is the result we are asked to show.

6 Problem 2(b) (problem 5.3.6)

(1) Consider De Moivre's theorem, which states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. This follows from taking the *nth* power of both sides of Euler's theorem. Find the formula for $\cos 4\theta$ and $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(2) Given $e^{iA}e^{iB} = e^{i(A+B)}$ deduce $\cos(A+B)$ and $\sin(A+B)$

solution

6.1 Part 1

Let $n = 4$, therefore, using De Moivre's theorem gives

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta \tag{1}$$

We now expand the LHS of the above directly as follows

$$(\cos\theta + i\sin\theta)^4 = (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^2 \tag{2}$$

But

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta$$

Substituting the above into (2) gives

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= \left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\\
&= \cos^2\theta\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\\
&- \sin^2\theta\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)\\
&+ 2i\cos\theta\sin\theta\left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)
\end{aligned}$$

Expanding the RHS above more, then the above becomes

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= \left(\cos^4\theta - \cos^2\theta\sin^2\theta + 2i\cos^3\theta\sin\theta\right)\\
&- \left(\sin^2\theta\cos^2\theta - \sin^4\theta + 2i\cos\theta\sin^3\theta\right)\\
&+ \left(2i\cos^3\theta\sin\theta - 2i\cos\theta\sin^3\theta - 4\cos^2\theta\sin^2\theta\right)
\end{aligned}$$

Simplifying gives

$$\begin{aligned}
(\cos\theta + i\sin\theta)^4 &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + 4i\cos^3\theta\sin\theta + \sin^4\theta - 4i\cos\theta\sin^3\theta\\
&= \left(\cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta\right) + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)
\end{aligned} \tag{3}$$

Comparing the real and imaginary parts of (3) with the real and imaginary parts of (1) shows that

$$\begin{aligned}
\cos 4\theta &= \cos^4\theta + \sin^4\theta - 6\cos^2\theta\sin^2\theta\\
\sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta
\end{aligned}$$

6.2 Part 2

Given

$$e^{iA}e^{iB} = e^{i(A+B)}$$

Applying Euler's formula $e^{ix} = \cos x + i\sin x$, on both sides of the above results in

$$\begin{aligned}
(\cos A + i\sin A)(\cos B + i\sin B) &= \cos(A+B) + i\sin(A+B)\\
\cos A\cos B + i\cos A\sin B + i\sin A\cos B - \sin B\sin A &= \cos(A+B) + i\sin(A+B)\\
(\cos A\cos B - \sin B\sin A) + i(\cos A\sin B + \sin A\cos B) &= \cos(A+B) + i\sin(A+B)
\end{aligned}$$

Comparing the real parts and the imaginary parts in the above shows that

$$\cos A\cos B - \sin B\sin A = \cos(A+B)$$

And

$$\cos A\sin B + \sin A\cos B = \sin(A+B)$$

7 Problem 2(c)

Find $\int_0^\infty xe^{-ax}\cos(kx)dx$ using Euler's formula.

solution

Let

$$I=\int_0^\infty xe^{-ax}\cos(kx)dx$$

Then, we replace $\cos(kx)$ by e^{ikx}, evaluate the integral, and then take the real part of the result. Therefore

$$\begin{aligned}I&=\operatorname{Re}\left(\int_0^\infty xe^{-ax}e^{ikx}dx\right)\\&=\operatorname{Re}\left(\int_0^\infty xe^{x(-a+ik)}dx\right)\end{aligned}$$

Integration by parts. Let $u=x,du=dx$ and $dv=e^{x(-a+ik)},v=\frac{e^{x(-a+ik)}}{-a+ik}$. The above now becomes

$$\begin{aligned}I&=\operatorname{Re}\left(uv|_0^\infty-\int_0^\infty vdu\right)\\&=\operatorname{Re}\left(\frac{1}{-a+ik}xe^{x(-a+ik)}\Big|_0^\infty-\int_0^\infty\frac{e^{x(-a+ik)}}{-a+ik}dx\right)\end{aligned}\tag{1}$$

But

$$xe^{x(-a+ik)}\Big|_0^\infty=0$$

With the assumption that $\operatorname{Re}(a)>0$. To see this more clearly, let us write $e^{x(-a+ik)}=e^{-ax}e^{ikx}$. e^{ikx} is bounded since it is a complex exponential. So the contribution comes from e^{-ax}. Hence when $a>0$, and $x\to\infty$ then the exponential will go to zero, and the whole term $xe^{x(-a+ik)}\to0$, even though $x\to\infty$, since exponential subdues any polynomial order. When $x=0$, it is clear that $xe^{x(-a+ik)}=0$. Therefore (1) now simplifies to

$$\begin{aligned}I&=\operatorname{Re}\left(-\int_0^\infty\frac{e^{x(-a+ik)}}{-a+ik}dx\right)\\&=\operatorname{Re}\left(-\frac{1}{-a+ik}\int_0^\infty e^{x(-a+ik)}dx\right)\\&=\operatorname{Re}\left(-\frac{1}{-a+ik}\frac{e^{x(-a+ik)}}{-a+ik}\Big|_0^\infty\right)\\&=\operatorname{Re}\left(-\frac{1}{(-a+ik)^2}e^{x(-a+ik)}\Big|_0^\infty\right)\end{aligned}$$

But $e^{x(-a+ik)}\Big|_0^\infty = 0 - 1 = -1$. The above becomes

$$
\begin{aligned}
I &= \operatorname{Re}\left(\frac{1}{(-a+ik)^2}\right) \\
&= \operatorname{Re}\left(\frac{1}{a^2-k^2-2aik}\right) \\
&= \operatorname{Re}\left(\frac{\left(a^2-k^2+2aik\right)}{\left(a^2-k^2-2aik\right)\left(a^2-k^2+2aik\right)}\right) \\
&= \operatorname{Re}\left(\frac{a^2-k^2+2aik}{\left(a^2-k^2\right)^2+4a^2k^2}\right) \\
&= \operatorname{Re}\left(\frac{a^2-k^2}{\left(a^2-k^2\right)^2+4a^2k^2} + i\frac{2ak}{\left(a^2-k^2\right)^2+4a^2k^2}\right) \\
&= \frac{a^2-k^2}{\left(a^2-k^2\right)^2+4a^2k^2}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_0^\infty xe^{-ax}\cos(kx)dx &= \frac{a^2-k^2}{\left(a^2-k^2\right)^2+4a^2k^2} \\
&= \frac{a^2-k^2}{a^4+k^4-2a^2k^2+4a^2k^2} \\
&= \frac{a^2-k^2}{a^4+k^4+2a^2k^2} \\
&= \frac{a^2-k^2}{\left(a^2+k^2\right)^2} \qquad a>0
\end{aligned}
$$

8 Problem 3

Given the intensity pattern for the N-slit interference with separation d between adjacent slits, show that the pattern becomes that for the single-slit diffraction with slit width a when d goes to zero but with a fixed value of $Nd = a$. (10 points)

Solution

Short version: In this version, The result for N slit $\bar{I}_N(\theta)$ will be used as given in lecture notes without deriving it again, and will also use the single slit $\bar{I}_1(\theta)$ from the lecture notes, then show that $\bar{I}_N(\theta)$ becomes $\bar{I}_1(\theta)$ as $d \to 0$ but with $Nd = a$.

Here $\bar{I}_N(\theta)$ is the average intensity for N slits at location on the screen at angle θ and similarly $\bar{I}_1(\theta)$ is the average intensity for one slit at same location on the screen at angle θ. From lecture notes (lecture 3, pages 6,7) we have the expressions for $\bar{I}_N(\theta), \bar{I}_1(\theta)$ given as

$$\bar{I}_N(\theta) = \bar{I}(0)\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2 \tag{1}$$

$$\bar{I}_1(\theta) = \bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin\theta}{\lambda}\right)}{\frac{\pi a\sin\theta}{\lambda}}\right)^2 \tag{2}$$

Now we need to show that (1) gives same result as (2) when d goes to zero in the limit, but with a fixed value of $Nd = a$. Replacing $Nd = a$ in the numerator of (1) and taking the limit gives

$$\begin{aligned}\lim_{d\to 0}\bar{I}_N(\theta) &= \bar{I}(0)\left(\lim_{d\to 0}\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right)}\right)^2 \\ &= \bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{N\lim_{d\to 0}\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right)}\right)^2\end{aligned} \tag{3}$$

But

$$\lim_{d\to 0}\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right) \approx \frac{\pi d\sin(\theta)}{\lambda} + \cdots \tag{4}$$

In the above we used that $\lim_{d\to 0}\sin\left(\frac{\pi d\sin(\theta)}{\lambda}\right) \approx \frac{\pi d\sin(\theta)}{\lambda}$. This comes from Taylor series expansion of sin function, for small angle approximation by keeping only the linear term in the Taylor series expansion since $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$.

Substituting (4) back into (3) gives

$$\lim_{d\to 0}\bar{I}_N(\theta) = \bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{N\frac{\pi d\sin(\theta)}{\lambda}}\right)^2$$

But $Nd = a$. The above simplifies to

$$\lim_{d\to 0}\bar{I}_N(\theta) = \bar{I}(0)\left(\frac{\sin\left(\frac{\pi a\sin(\theta)}{\lambda}\right)}{\frac{\pi a\sin(\theta)}{\lambda}}\right)^2 \tag{5}$$

Comparing (5) with (2) shows that are the same. Hence

$$\lim_{d\to 0}\bar{I}_N(\theta) = \bar{I}_1(\theta)$$

Which is what we are asked to show.

8.1 Appendix

Here, the derivation of

$$\bar{I}_N(\theta) = \bar{I}(0)\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2 \tag{1}$$

is given. First, let us consider a slit located at y_n relative to the origin as show in the diagram below

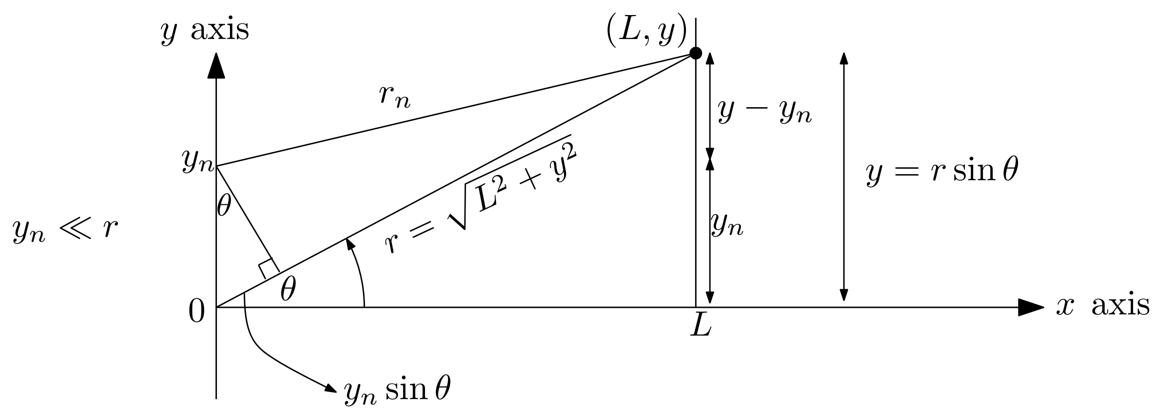

Figure 1: Geometry for slit at location y_n

Therefore

$$\begin{aligned}
r_n &= \sqrt{L^2 + \left(y - y_n\right)^2} \\
&= \sqrt{L^2 + \left(y^2 + y_n^2 - 2yy_n\right)} \\
&= \sqrt{L^2 + y^2 + y_n^2 - 2yy_n} \\
&= \sqrt{\left(L^2 + y^2\right)\left(1 + \frac{y_n^2 - 2yy_n}{\left(L^2 + y^2\right)}\right)} \\
&= \sqrt{\left(L^2 + y^2\right)}\sqrt{1 + \frac{y_n^2 - 2yy_n}{\left(L^2 + y^2\right)}} \\
&= \sqrt{\left(L^2 + y^2\right)}\sqrt{1 - \frac{2yy_n}{\left(L^2 + y^2\right)} + \frac{y_n^2}{\left(L^2 + y^2\right)}}
\end{aligned}$$

Since y_n is very small compared to $\left(L^2 + y^2\right)$ and it is also of order 2, then we can ignore the term $\frac{y_n^2}{\left(L^2+y^2\right)}$ above, giving

$$\begin{aligned}
r_n &\approx \sqrt{\left(L^2 + y^2\right)}\sqrt{1 - \frac{2yy_n}{\left(L^2 + y^2\right)}} \\
&= \sqrt{\left(L^2 + y^2\right)}\left(1 - \frac{2yy_n}{\left(L^2 + y^2\right)}\right)^{\frac{1}{2}}
\end{aligned}$$

But since y_n is very small compared to $\left(L^2 + y^2\right)$, then the term $\frac{2yy_n}{\left(L^2+y^2\right)}$ is very small. So we can use $(1+x)^p = 1 + px$ and ignore higher order terms. Hence the above becomes

$$\begin{aligned} r_n &\approx \sqrt{\left(L^2+y^2\right)}\left(1+\frac{1}{2}\left(\frac{-2yy_n}{\left(L^2+y^2\right)}\right)\right) \\ &= \sqrt{\left(L^2+y^2\right)} - \frac{yy_n}{\sqrt{\left(L^2+y^2\right)}} \\ &= r - y_n\frac{y}{r} \end{aligned}$$

But $\frac{y}{r} = \sin\theta$, therefore

$$r_n = r - y_n \sin\theta \tag{2}$$

The electric field E_n measured at point $\left(L, y\right)$ due to slit at y_n is

$$E_n = E_0 \sin(kr_n - \omega t)$$

Where k is the wave number $k = \frac{2\pi}{\lambda}$. Therefore for N slits, the total E is

$$\begin{aligned} E &= \sum_{n=1}^{N} E_n \\ &= \sum_{n=1}^{N} E_0 \sin(kr_n - \omega t) \\ &= E_0\left(\operatorname{Im}\sum_{n=1}^{N} e^{i(kr_n-\omega t)}\right) \\ &= E_0\left(\operatorname{Im}\sum_{n=1}^{N} e^{i(kr_N-\omega t)}e^{ik(r_n-r_N)}\right) \\ &= E_0\left(\operatorname{Im}\left(e^{i(kr_N-\omega t)}\sum_{n=1}^{N} e^{ik(r_n-r_N)}\right)\right) \end{aligned} \tag{3}$$

But

$$\begin{aligned} r_n - r_N &= \left(r - y_n\sin\theta\right) - \left(r - y_N\sin\theta\right) \\ &= r - y_n\sin\theta - r + y_N\sin\theta \\ &= \left(y_N - y_n\right)\sin\theta \\ &= (Nd - nd)\sin\theta \\ &= (N-n)d\sin\theta \end{aligned} \tag{4}$$

Substituting (4) in (3) gives

$$E = E_0\left(\operatorname{Im}\left(e^{i(kr_N-\omega t)}\sum_{n=1}^{N} e^{ik(N-n)d\sin\theta}\right)\right)$$

Let $m = N - n$. When $n = 1$ then $m = N - 1$. When $n = N$ then $m = 0$. The above now becomes

$$\begin{aligned} E &= E_0\left(\operatorname{Im}\left(e^{i(kr_N-\omega t)}\sum_{m=N-1}^{0} e^{ikmd\sin\theta}\right)\right) \\ &= E_0\left(\operatorname{Im}\left(e^{i(kr_N-\omega t)}\sum_{m=0}^{N-1} e^{ikmd\sin\theta}\right)\right) \\ &= E_0\left(\operatorname{Im}\left(e^{i(kr_N-\omega t)}\sum_{n=0}^{N-1} e^{iknd\sin\theta}\right)\right) \end{aligned}$$

Let $\phi = kd\sin\theta$. The above becomes

$$E = E_0\left(\operatorname{Im}\left(e^{i(kr_N-\omega t)}\sum_{n=0}^{N-1} e^{in\phi}\right)\right) \tag{5}$$

But

$$\begin{aligned}
\sum_{n=0}^{N-1} e^{in\phi} &= \frac{1-e^{iN\phi}}{1-e^{i\phi}} \\
&= \frac{e^{i\frac{N}{2}\phi}\left(e^{-i\frac{N}{2}\phi}-e^{i\frac{N}{2}\phi}\right)}{e^{i\frac{\phi}{2}}\left(e^{-i\frac{\phi}{2}}-e^{i\frac{\phi}{2}}\right)} \\
&= \frac{e^{i\frac{N}{2}\phi}}{e^{i\frac{\phi}{2}}}\frac{\left(e^{-i\frac{N}{2}\phi}-e^{i\frac{N}{2}\phi}\right)}{\left(e^{-i\frac{\phi}{2}}-e^{i\frac{\phi}{2}}\right)} \\
&= \frac{-e^{i\frac{N}{2}\phi}}{-e^{i\frac{\phi}{2}}}\frac{\left(e^{i\frac{N}{2}\phi}-e^{-i\frac{N}{2}\phi}\right)}{\left(e^{i\frac{\phi}{2}}-e^{-i\frac{\phi}{2}}\right)} \\
&= \frac{e^{i\frac{N}{2}\phi}}{e^{i\frac{\phi}{2}}}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)} \\
&= e^{i\frac{(N-1)\phi}{2}}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)} \qquad (6)
\end{aligned}$$

Substituting (6) in (5) gives

$$\begin{aligned}
E &= E_0\left(\mathrm{Im}\left(e^{i(kr_N-\omega t)}e^{i\frac{(N-1)\phi}{2}}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\right)\right) \\
&= E_0\left(\mathrm{Im}\left(e^{i\left(kr_N-\omega t+\frac{(N-1)\phi}{2}\right)}\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\right)\right) \qquad (7)
\end{aligned}$$

Let

$$t_0 = kr_N + \frac{(N-1)\phi}{2}$$

Substituting this in (7) gives

$$\begin{aligned}
E &= E_0\left(\mathrm{Im}\left(e^{i\omega(t_0-t)}\right)\right) \\
&= E_0\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\left(\mathrm{Im}\, e^{i\omega(t_0-t)}\right) \\
&= E_0\frac{\sin\left(\frac{N}{2}\phi\right)}{\sin\left(\frac{\phi}{2}\right)}\sin(\omega(t_0-t)) \qquad (8)
\end{aligned}$$

The electric field intensity is

$$\begin{aligned}
I &= c\varepsilon_0 E^2 \\
&= c\varepsilon_0 E_0^2\frac{\sin^2\left(\frac{N}{2}\phi\right)}{\sin^2\left(\frac{\phi}{2}\right)}\sin^2(\omega(t_0-t))
\end{aligned}$$

The time (period) averaged intensity is therefore

$$\begin{aligned}
I_{av} &= \frac{1}{T}\int_0^T I dt \\
&= \frac{1}{T}c\varepsilon_0 E_0^2 \frac{\sin^2\left(\frac{N}{2}\phi\right)}{\sin^2\left(\frac{\phi}{2}\right)} \int_0^T \sin^2(\omega(t_0 - t))dt \\
&= \frac{1}{2}c\varepsilon_0 E_0^2 \frac{\sin^2\left(\frac{N}{2}\phi\right)}{\sin^2\left(\frac{\phi}{2}\right)}
\end{aligned}$$

But $\phi = kd\sin\theta$ and $k = \frac{2\pi}{\lambda}$, then the above becomes

$$\begin{aligned}
I(\theta)_{av} &= \frac{1}{2}c\varepsilon_0 E_0^2 \frac{\sin^2\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin^2\left(\frac{\pi d\sin\theta}{\lambda}\right)} \\
&= \frac{1}{2}c\varepsilon_0 E_0^2 \left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2
\end{aligned}$$

At $\theta = 0$, we have

$$\begin{aligned}
I(0)_{av} &= \lim_{\theta\to 0}\frac{1}{2}c\varepsilon_0 E_0^2 \frac{\sin^2\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin^2\left(\frac{\pi d\sin\theta}{\lambda}\right)} \\
&= N^2\frac{1}{2}c\varepsilon_0 E_0^2
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{I(\theta)_{av}}{I(0)_{av}} &= \frac{\frac{1}{2}c\varepsilon_0 E_0^2 \left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2}{N^2\frac{1}{2}c\varepsilon_0 E_0^2} \\
&= \frac{1}{N^2}\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2 \\
&= \left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2
\end{aligned}$$

Therefore

$$I(\theta)_{av} = I(0)_{av}\left(\frac{\sin\left(\frac{N\pi d\sin\theta}{\lambda}\right)}{N\sin\left(\frac{\pi d\sin\theta}{\lambda}\right)}\right)^2$$

Which is the formula used in the earlier derivation.

9 Problem 4

(1) Find the roots $z_n (n = 1, 2, \cdots, N)$ of the complex equation $z^N = 1$. (2) Find $S_N = \sum_{n=1}^{N} z_n$ and give a geometric interpretation of the result. (3) Note that $1 - z^N = (1-z)\left(1 + z + z^2 + \cdots + z^{l}\right.$
Relate this result and the roots z_n to the conditions for destructive interference among N slits.

solution

9.1 Part 1

$$Z^N = 1$$
$$Z = 1^{\frac{1}{N}}$$

But $1 = e^{i(2\pi)}$ and the above becomes

$$
\begin{aligned}
Z &= \left(e^{i(2\pi)}\right)^{\frac{1}{N}} \\
Z_n &= (\cos(2\pi + (2\pi)n) + i\sin(2\pi + (2\pi)n))^{\frac{1}{N}} \qquad n = 0, 1, 2, \cdots, N-1
\end{aligned}
$$

Since cos and sin are periodic with period 2π. Using De Moivre's theorem the above becomes

$$
\begin{aligned}
Z_n &= \left(\cos\left(\frac{2\pi}{N} + \frac{n}{N}(2\pi)\right) + i\sin\left(\frac{2\pi}{N} + \frac{n}{N}(2\pi)\right)\right) \\
&= e^{i\left(\frac{2\pi}{N} + \frac{n}{N}(2\pi)\right)} \\
&= e^{i\left(\frac{2\pi(n+1)}{N}\right)} \qquad n = 0, 1, 2, \cdots, N-1
\end{aligned}
$$

Which is the same as

$$Z_n = e^{i\left(\frac{2\pi n}{N}\right)} \qquad n = 1, 2, \cdots, N$$

For an example, let $N = 3$. Therefore we have 3 roots, given by $n = 1, 2, 3$. They are

$$
\begin{aligned}
Z_1 &= e^{i\left(\frac{2\pi}{3}\right)} = e^{i(120^0)} \\
Z_2 &= e^{i\left(\frac{2\pi(2)}{3}\right)} = e^{i\left(\frac{4\pi}{3}\right)} = e^{i\left(240^0\right)} \\
Z_3 &= e^{i\left(\frac{2\pi(3)}{3}\right)} = e^{i(2\pi)} = e^{i360^0}
\end{aligned}
$$

The roots are 120^0 degrees apart on the unit circle. First root has phase 0^0 (or 360^0), second at 120^0 and the third at 240^0. There are only 3 unique roots, since after that, they repeat. Here is a diagram showing the roots for $N = 3$ for illustration. The root with phase 0^0 is the real root 1 since $e^{i0^0} = 1$, the other two roots are complex valued, and complex conjugate of each others.

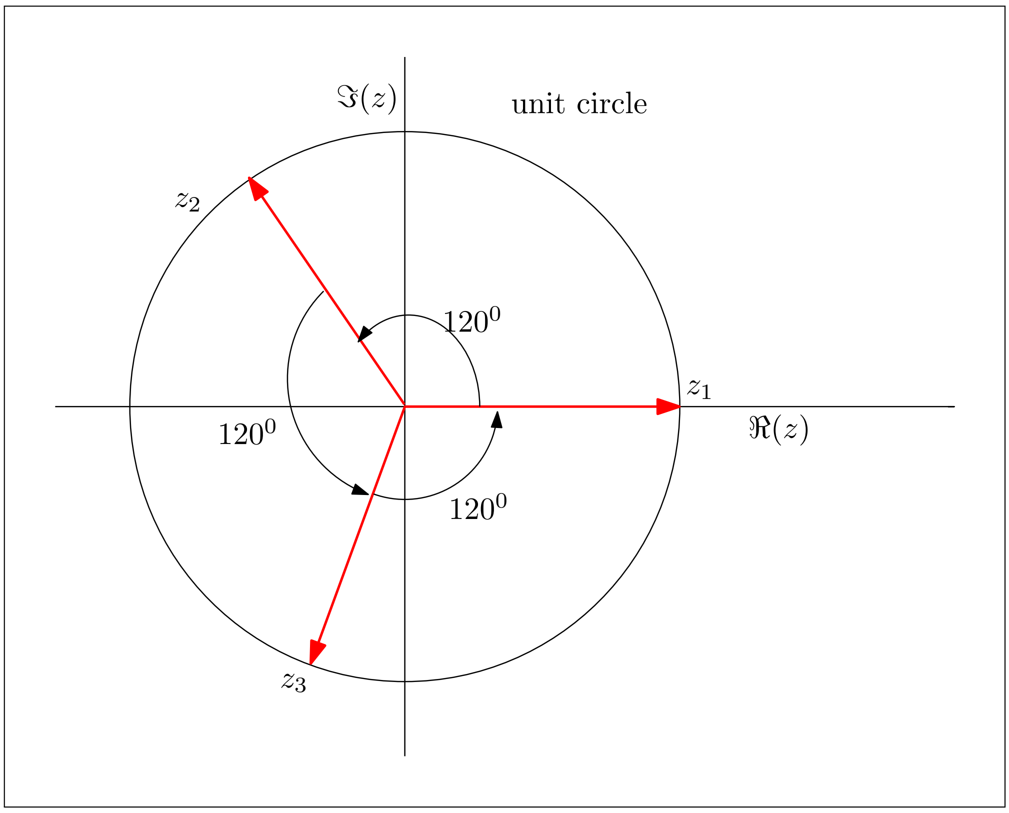

Figure 2: Roots of z^N for case of $N = 3$

There are only 3 unique roots, since after moving around the unit circle once, the roots repeat.

9.2 Part 2

$$S_N = \sum_{n=1}^{N} z_n \tag{1}$$

It is assumed that z_n above are all the roots of Z^N from part(a), even though the problem did not say that. Hence all roots have same modulus. But differ by the phase as found in part 1.

Let $z = x + iy = e^{i\theta}$ where $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\frac{y}{x}\right)$. The above becomes

$$\begin{aligned} S_N &= z_1 + z_2 + \cdots + z_N \\ &= re^{i\theta_1} + re^{i\theta_2} + \cdots + re^{i\theta_N} \\ &= r\left(e^{i\theta_1} + e^{i\theta_2} + \cdots + e^{i\theta_N}\right) \end{aligned}$$

But $r = 1$, hence

$$S_N = e^{i\theta_1} + e^{i\theta_2} + \cdots + e^{i\theta_N}$$

From part 1, we found that

$$\theta_n = \frac{2\pi n}{N} \qquad n = 1, 2, 3, \cdots, N \tag{2}$$

Using (2) in (1), now the sum can be written as

$$S_N = \sum_{n=1}^{N} e^{i\frac{2\pi n}{N}} \tag{3}$$

If $N = 1$, then the sum is just $e^{i2\pi} = 1$. But if $N > 1$ then to find the partial sum, let

$$S_N = e^{i\frac{2\pi}{N}} + e^{i\left(\frac{4\pi}{N}\right)} + e^{i\frac{6\pi}{N}} + \cdots + e^{i\frac{N(2\pi)}{N}} \tag{4}$$

$$e^{i\frac{2\pi}{N}} S_N = e^{i\frac{4\pi}{N}} + e^{i\frac{6\pi}{N}} + \cdots + e^{i\frac{N(2\pi)+2\pi}{N}} \tag{5}$$

(4-5) gives

$$S_N - e^{i\frac{2\pi}{N}} S_N = e^{i\frac{2\pi}{N}} - e^{i\frac{N(2\pi)+2\pi}{N}}$$
$$S_N\left(1 - e^{i\frac{2\pi}{N}}\right) = e^{i\frac{2\pi}{N}} - e^{i\frac{N(2\pi)+2\pi}{N}}$$
$$S_N = \frac{e^{i\frac{2\pi}{N}} - e^{i\frac{N(2\pi)+2\pi}{N}}}{1 - e^{i\frac{2\pi}{N}}}$$

But $e^{i\frac{(N+1)(2\pi)}{N}} = e^{i\frac{N(2\pi)+2\pi}{N}} = e^{i2\pi} e^{i\frac{2\pi}{N}} = e^{i\frac{2\pi}{N}}$. The above becomes

$$S_N = \frac{e^{i\frac{2\pi}{N}} - e^{i\frac{2\pi}{N}}}{1 - e^{i\frac{2\pi}{N}}} = 0$$

Therefore, the final result is

$$S_N = \begin{cases} 1 & N = 1 \\ 0 & N > 1 \end{cases}$$

For geometric interpretation. Each root z_n is a unit vector, where the angle between each root is the same. it is $\frac{2\pi}{N}$. Looking at each root as a vector in the complex plane, these vectors originate from the origin and end up at the unit circle, each with phase which is $\frac{2\pi}{N}$ more than the vector just to the right of it as we go anticlockwise around the circle. The first vector starts with phase 0.

The sum $\sum_{n=1}^{N} z_n$ is therefore the a vector sum of these N root. The easiest way to see that this sum is zero geometrically, is to add these vectors, by putting each vector tail, at the tip of the previous vector. To illustrate this, we will look at the case of $N = 3$ where the angle between each vector is 120^0. This is because $\frac{2\pi}{3} = 120^0$. Using this method to add the roots gives this

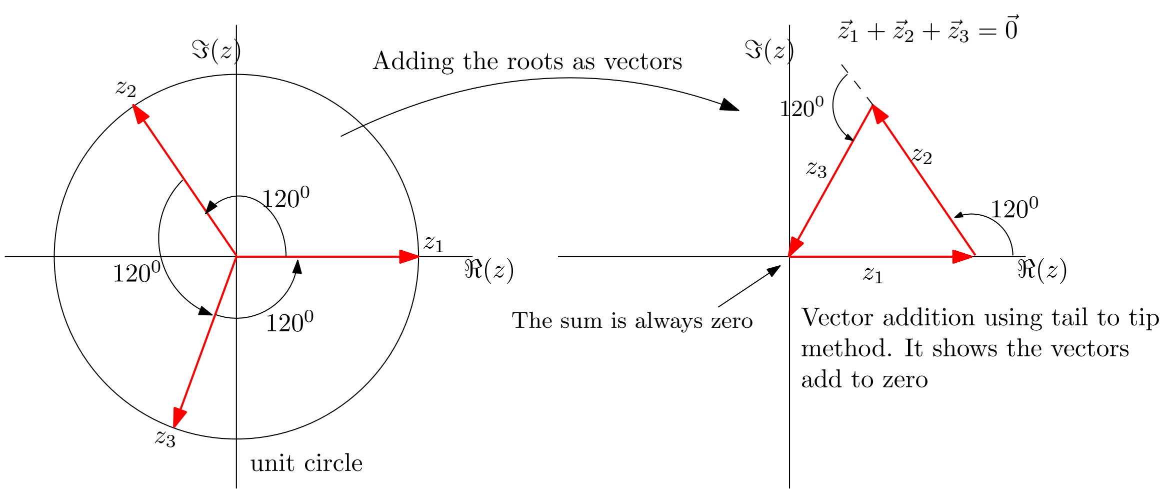

Figure 3: Geometric interpretation of adding the roots. Example for $N = 3$

The above generalizes for any N. If the vector sum using the tail to tip method gives a closed shape which in this case ends up back at the origin, then the vector sum is zero.

9.3 Part 3

Looking at the Electric field E at an observation point at angle θ we obtain the following diagram

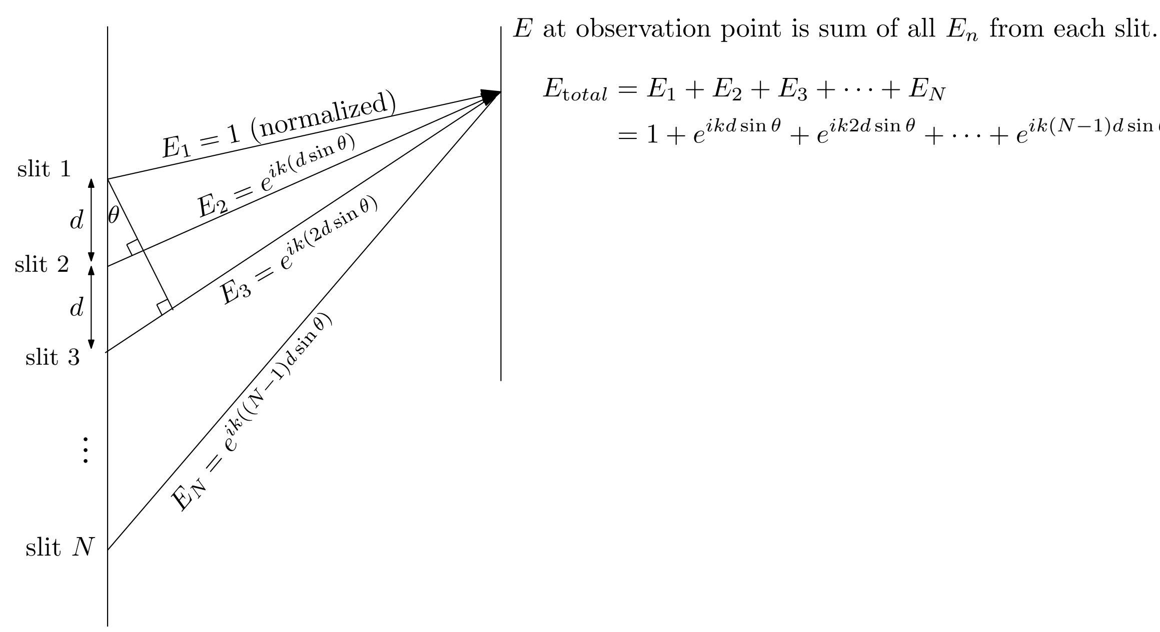

Figure 4: Contribution of E from each slit

In the above, the E contribution from slit 1 was normalized to be $E_1 = 1$. Therefore, the contribution of E_2 from the second slit will have a phase shift relative to the first slit. This is given by $d\sin\theta$ as seen in the diagram. For each addition slit, the phase will increase by $d\sin\theta$. Hence the E_3 will have phase of $2d\sin\theta$ and so on until the last slit N which will have phase shift of $(N-1)d\sin\theta$.

Therefore we see that electric field at the observation point is the sum of all E_n from each slit, and given by

$$\begin{aligned} E &= E_1 + E_2 + \cdots + E_N \\ &= 1 + e^{ikd\sin\theta} + e^{ik(2d\sin\theta)} + e^{ik(3d\sin\theta)} + \cdots + e^{ik((N-1)d\sin\theta)} \end{aligned} \tag{1}$$

Now, from lecture notes, we are given the conditions for minima (i.e. destructive interference) as

$$d\sin\theta = \frac{k}{N}\lambda \qquad k = \pm1, \pm2, \cdots \tag{2}$$

Substituting (2) into (1) gives

$$E = 1 + e^{ik\left(\frac{k}{N}\lambda\right)} + e^{ik(2\left(\frac{k}{N}\lambda\right))} + e^{ik(3\left(\frac{k}{N}\lambda\right))} + \cdots + e^{ik((N-1)\left(\frac{k}{N}\lambda\right))}$$

Replacing the first k in each term by $\frac{2\pi}{\lambda}$ since k is wave number, then the above becomes

$$\begin{aligned} E &= 1 + e^{i\left(\frac{2\pi}{\lambda}\right)\left(\frac{k}{N}\lambda\right)} + e^{i\left(\frac{2\pi}{\lambda}\right)(2\left(\frac{k}{N}\lambda\right))} + e^{i\left(\frac{2\pi}{\lambda}\right)(3\left(\frac{k}{N}\lambda\right))} + \cdots + e^{i\left(\frac{2\pi}{\lambda}\right)((N-1)\left(\frac{k}{N}\lambda\right))} \\ &= 1 + e^{i\frac{2\pi k}{N}} + e^{i2\left(\frac{2\pi k}{N}\right)} + e^{i3\left(\frac{2\pi k}{N}\right)} + \cdots + e^{i(N-1)\left(\frac{2\pi k}{N}\right)} \end{aligned}$$

Let $\phi = \frac{2\pi}{N}$. The above becomes

$$E = 1 + e^{ik\phi} + e^{i2k\phi} + e^{i3k\phi} + \cdots + e^{ik(N-1)\phi} \tag{3}$$

Comparing the above to the result obtain in part 1 we found that the sum of the roots for $Z^n = 1$ to be

$$\begin{aligned} S_N &= z_0 + z_1 + \cdots + z_{N-1} \\ &= e^{i\theta_0} + e^{i\theta_1} + \cdots + e^{i\theta_{N-1}} \end{aligned} \tag{4}$$

Where

$$\theta_n = \frac{2\pi n}{N} \qquad n = 0, 1, 2, \cdots, N-1 \tag{5}$$

Hence (4) becomes

$$S_N = 1 + e^{i\frac{2\pi}{N}} + e^{i2\frac{2\pi}{N}} + \cdots + e^{i(N-1)\frac{2\pi}{N}} \tag{6}$$

Therefore, for each different k Eq(3) is the same as (6). So (3) can be written as

$$E = 1 + z + z^2 + \cdots + z^{N-1} \tag{7}$$

Where now $z = e^{ik\phi}$ with $\phi = \frac{2\pi}{N}$. But we know that

$$\left(1 - z^N\right) = (1 - z)\left(1 + z + z^2 + \cdots + z^{N-1}\right) \tag{8}$$

But $\left(1 - z^N\right) = 0$ since 1 is root of z^N. Hence the above becomes

$$0 = (1 - z)\left(1 + z + z^2 + \cdots + z^{N-1}\right)$$

Since $z \neq 1$ (unless $\frac{2\pi}{N}k$ happened to be exact multiple of 2π), then we conclude that $1 + z + z^2 + \cdots + z^{N-1}$ must be zero. This implies that

$$\begin{aligned} E &= 1 + e^{ik\phi} + e^{i2k\phi} + e^{i3k\phi} + \cdots + e^{ik(N-1)\phi} \\ &= 0 \end{aligned}$$

Under the condition of destructive interference. This says the total Electric field from the N slits will vanish at the observation point when destructive interference condition is applied. Which is what we are asked to show.