Physics 3041 (Spring 2021) Solutions to Homework Set 10

1. (10 points) Given

$$\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}\,,$$

make a 3D integral and use the transformation from Cartesian to spherical coordinates to evaluate

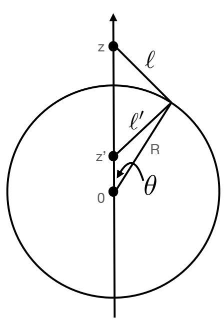
$$\int_{0}^{\infty} x^2 \exp(-x^2)dx.$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2-z^2}dxdydz = \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \int_{-\infty}^{\infty} e^{-z^2}dz = (\sqrt{\pi}\,)^3$$

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2-z^2}dxdydz &= \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} e^{-r^2}r^2\sin\theta\, drd\theta d\phi \\ &= \int_0^{\infty} r^2e^{-r^2}dr \int_0^{\pi}\sin\theta\, d\theta \int_0^{2\pi} d\phi \\ &= (\int_0^{\infty} r^2e^{-r^2}dr) \times 2 \times (2\pi) = 4\pi \int_0^{\infty} r^2e^{-r^2}dr\end{aligned}$$

$$\Rightarrow \int_0^{\infty} r^2e^{-r^2}dr = \frac{(\sqrt{\pi}\,)^3}{4\pi} = \frac{\sqrt{\pi}}{4} = \int_0^{\infty} x^2e^{-x^2}dx$$

2. Follow the lecture example of deriving the gravitational field of a thin shell and calculate the gravitational potential of such a shell over all space. (10 points)



Due to spherical symmetry, we only need to consider the potential along the z-axis. For a point at $z = r > R$, the potential is

$$\begin{aligned}\phi(r) &= -G\sigma\int_0^{\pi}\frac{2\pi R^2\sin\theta d\theta}{\ell} \\ &= -2\pi G\sigma R^2\int_0^{\pi}\frac{\sin\theta d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} \\ &= -\frac{\pi G\sigma R}{r}\int_0^{\pi}\frac{d(r^2+R^2-2rR\cos\theta)}{\sqrt{r^2+R^2-2rR\cos\theta}} \\ &= -\frac{2\pi G\sigma R}{r}\sqrt{r^2+R^2-2rR\cos\theta}\Big|_0^{\pi} \\ &= -\frac{2\pi G\sigma R}{r}[r+R-(r-R)] = -\frac{4\pi G\sigma R^2}{r} = -\frac{Gm}{r}\end{aligned}$$

Similarly, for a point at $z' = r < R$, the potential is

$$\begin{aligned}\phi(r) &= -G\sigma\int_0^{\pi}\frac{2\pi R^2\sin\theta d\theta}{\ell'} = -2\pi G\sigma R^2\int_0^{\pi}\frac{\sin\theta d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} \\ &= -\frac{\pi G\sigma R}{r}\int_0^{\pi}\frac{d(r^2+R^2-2rR\cos\theta)}{\sqrt{r^2+R^2-2rR\cos\theta}} = -\frac{2\pi G\sigma R}{r}\sqrt{r^2+R^2-2rR\cos\theta}\Big|_0^{\pi} \\ &= -\frac{2\pi G\sigma R}{r}[r+R-(R-r)] = -4\pi G\sigma R = -\frac{Gm}{R}\end{aligned}$$

3. Follow the lecture example of deriving the gas pressure and calculate the number of gas particles hitting the container per unit area per unit time. Give your answer in terms of the net number density and the average speed of these particles. (10 points)

Consider an area element ΔA perpendicular to the z-axis. The number of particles with velocity between $\vec{v}$ and $\vec{v} + d\vec{v}$ that hit ΔA during an interval Δt is

$$\Delta N = dn \cdot \Delta A \cdot (v_z \Delta t) = f(v) v_z dv_x dv_y dv_z (\Delta A \Delta t),$$

so the net number of particles hitting the container per unit area per unit time is

$$\begin{aligned}
\frac{\Delta N}{\Delta A \Delta t} &= \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) v_z dv_x dv_y dv_z \\
&= \int_0^{2\pi} \int_0^{\pi/2} \int_0^\infty f(v)(v\cos\theta) v^2 \sin\theta dv d\theta d\phi \\
&= \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^\infty f(v) v^3 dv = \pi \int_0^\infty f(v) v^3 dv
\end{aligned}$$

The average speed of the particles is

$$\begin{aligned}
\bar{v} &= \frac{\int v dn}{\int dn} = \frac{1}{n} \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) v dv_x dv_y dv_z \\
&= \frac{1}{n} \int_0^{2\pi} \int_0^\pi \int_0^\infty f(v) v^3 \sin\theta dv d\theta d\phi = \frac{4\pi}{n} \int_0^\infty f(v) v^3 dv,
\end{aligned}$$

where n is the net number density. So we obtain

$$\frac{\Delta N}{\Delta A \Delta t} = \frac{1}{4} n \bar{v}.$$

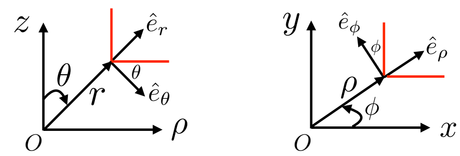
4. Derive the expressions of the quantum mechanical orbital angular momentum operators L_x, L_y, L_z in spherical coordinates. Show that

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\vec{L} \cdot \vec{L}}{\hbar^2 r^2}$$

in spherical coordinates. (40 points)

The orbital angular momentum operator is

$$\vec{L} = \vec{r} \times \vec{p} = \frac{\hbar}{i} \vec{r} \times \nabla = \frac{\hbar}{i} r \hat{e}_r \times \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right) = \frac{\hbar}{i} \left(\hat{e}_\phi \frac{\partial}{\partial \theta} - \hat{e}_\theta \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right).$$



It is clear from the above figure that

$$\hat{e}_\theta = \hat{e}_\rho \cos\theta - \hat{e}_z \sin\theta,\ \ \hat{e}_\rho = \hat{e}_x \cos\phi + \hat{e}_y \sin\phi,\ \ \hat{e}_\phi = -\hat{e}_x \sin\phi + \hat{e}_y \cos\phi.$$

So we obtain

$$\begin{aligned}
\vec{L} &= \frac{\hbar}{i}\left\{(-\hat{e}_x \sin\phi + \hat{e}_y \cos\phi)\frac{\partial}{\partial\theta} - [(\hat{e}_x\cos\phi + \hat{e}_y \sin\phi)\cos\theta - \hat{e}_z \sin\theta]\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\} \\
&= \frac{\hbar}{i}\left[-\hat{e}_x\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) + \hat{e}_y\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) + \hat{e}_z\frac{\partial}{\partial\phi}\right] \\
\Rightarrow L_x &= i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\
L_y &= i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \\
L_z &= -i\hbar\frac{\partial}{\partial\phi} \\
\vec{L}\cdot\vec{L} &= L_x^2 + L_y^2 + L_z^2 = -\hbar^2\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right)\left(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}\right) \\
&- \hbar^2\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right)\left(-\cos\phi\frac{\partial}{\partial\theta} + \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) - \hbar^2\frac{\partial}{\partial\phi}\frac{\partial}{\partial\phi} \\
&= -\hbar^2\left[\frac{\partial^2}{\partial\phi^2} + \sin^2\phi\frac{\partial^2}{\partial\theta^2} + \sin\phi\frac{\partial\cot\theta}{\partial\theta}\cos\phi\frac{\partial}{\partial\phi} + \sin\phi\cot\theta\cos\phi\frac{\partial^2}{\partial\theta\partial\phi}\right. \\
&+ \cot\theta\cos\phi\frac{\partial\sin\phi}{\partial\phi}\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\sin\phi\frac{\partial^2}{\partial\phi\partial\theta} + \cot^2\theta\cos\phi\frac{\partial\cos\phi}{\partial\phi}\frac{\partial}{\partial\phi} + \cot^2\theta\cos^2\phi\frac{\partial^2}{\partial\phi^2} \\
&+ \cos^2\phi\frac{\partial^2}{\partial\theta^2} - \cos\phi\frac{\partial\cot\theta}{\partial\theta}\sin\phi\frac{\partial}{\partial\phi} - \cos\phi\cot\theta\sin\phi\frac{\partial^2}{\partial\theta\partial\phi} \\
&\left. - \cot\theta\sin\phi\frac{\partial\cos\phi}{\partial\phi}\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\cos\phi\frac{\partial^2}{\partial\phi\partial\theta} + \cot^2\theta\sin\phi\frac{\partial\sin\phi}{\partial\phi}\frac{\partial}{\partial\phi} + \cot^2\theta\sin^2\phi\frac{\partial^2}{\partial\phi^2}\right] \\
&= -\hbar^2\left(\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2}\right) = -\hbar^2\left(\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}\right). \\
\nabla^2 &= \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \\
&= \frac{\partial^2}{\partial r^2} + \hat{e}_\theta\cdot\frac{\partial\hat{e}_r}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \hat{e}_\phi\cdot\frac{\partial\hat{e}_r}{\partial\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial r} + \hat{e}_\phi\cdot\frac{\partial\hat{e}_\theta}{\partial\phi}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2},
\end{aligned}$$

where we have used

$$d\hat{e}_r = \hat{e}_\phi\sin\theta d\phi + \hat{e}_\theta d\theta,\ \ d\hat{e}_\theta = \hat{e}_\phi\cos\theta d\phi - \hat{e}_r d\theta,\ \ d\hat{e}_\phi = -\hat{e}_r\sin\theta d\phi - \hat{e}_\theta\cos\theta d\phi.$$

5. Consider $\psi(x,t)$ for $0 \le x \le L$. Given that $\psi(0,t) = \psi(L,t) = 0$ and

$$\psi(x,0) = \begin{cases} A\sin(2\pi x/L), & 0 \le x \le L/2, \\ 0, & L/2 < x \le L, \end{cases}$$

find $\psi(x,t)$ that satisfies the following partial differential equation:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial x^2},$$

where A, L, $\hbar$, and μ are positive constants. (30 points)

$$\psi(x,t) \to X(x)T(t) \Rightarrow i\hbar X(x)\dot{T} = -\frac{\hbar^2}{2\mu}T(t)X'', \; i\hbar\frac{\dot{T}}{T(t)} = -\frac{\hbar^2}{2\mu}\frac{X''}{X(x)} = \frac{\hbar^2k^2}{2\mu}$$

$$X'' = -k^2X(x) \Rightarrow X(x) = A'\sin kx + B'\cos kx$$

$$X(0) = X(L) = 0 \Rightarrow B' = 0, \; kL = n\pi, \; k = \frac{n\pi}{L}, \; X_n(x) = A'\sin\frac{n\pi x}{L}, \; n = 1, 2, \cdots$$

$$\int_0^L [X_n(x)]^2 dx = (A')^2\int_0^L \sin^2\frac{n\pi x}{L}\,dx = \frac{(A')^2L}{2} = 1 \Rightarrow A' = \sqrt{\frac{2}{L}}, \; X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

$$\dot{T} = -\frac{i\hbar k^2}{2\mu}T(t) \Rightarrow T_n(t) = C_ne^{-in^2\pi^2\hbar t/(2\mu L^2)}$$

$$\psi(x,t) = \sum_{n=1}^{\infty}X_n(x)T_n(t) = \sum_{n=1}^{\infty}C_ne^{-in^2\pi^2\hbar t/(2\mu L^2)}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

$$\psi(x,0) = \sum_{n=1}^{\infty}C_n\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

$$C_n = \sqrt{\frac{2}{L}}\int_0^L \psi(x,0)\sin\frac{n\pi x}{L}\,dx = A\sqrt{\frac{2}{L}}\int_0^{L/2}\sin\frac{2\pi x}{L}\sin\frac{n\pi x}{L}\,dx$$

$$C_2 = A\sqrt{\frac{2}{L}}\int_0^{L/2}\sin^2\frac{2\pi x}{L}\,dx = \frac{AL}{4}\sqrt{\frac{2}{L}} = \frac{A}{2}\sqrt{\frac{L}{2}}$$

$$\sin\alpha\sin\beta = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}\left(\frac{e^{i\beta}-e^{-i\beta}}{2i}\right) = -\frac{e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)} + e^{-i(\alpha+\beta)}}{4}$$

$$= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$C_{n\neq2} = \frac{A}{2}\sqrt{\frac{2}{L}}\int_0^{L/2}\left[\cos\frac{(n-2)\pi x}{L} - \cos\frac{(n+2)\pi x}{L}\right]dx$$

$$= \frac{A}{2}\sqrt{\frac{2}{L}}\left[\frac{L}{(n-2)\pi}\sin\frac{(n-2)\pi}{2} - \frac{L}{(n+2)\pi}\sin\frac{(n+2)\pi}{2}\right]$$

$$= \frac{A}{\pi}\sqrt{\frac{L}{2}}\left[-\frac{\sin(n\pi/2)}{n-2} + \frac{\sin(n\pi/2)}{n+2}\right] = -\frac{4A\sin(n\pi/2)}{(n^2-4)\pi}\sqrt{\frac{L}{2}}$$

$$C_{2m+1} = \frac{4A(-1)^{m+1}}{(2m-1)(2m+3)\pi}\sqrt{\frac{L}{2}}, \; C_{2m+4} = 0, \; m = 0, 1, \cdots$$

$$\psi(x,t) = \frac{Ae^{-i2\pi^2\hbar t/(\mu L^2)}}{2}\sin\frac{2\pi x}{L}$$

$$+ \frac{4A}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m+1}e^{-i(2m+1)^2\pi^2\hbar t/(2\mu L^2)}}{(2m-1)(2m+3)}\sin\frac{(2m+1)\pi x}{L}$$