HW 10

Physics 3041
Mathematical Methods for Physicists

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Contents

1 Problem 1

Given

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Make a 3D integral and use the transformation from Cartesian to spherical coordinates to evaluate $\int_0^{\infty} x^2 e^{-x^2} dx$.

Solution

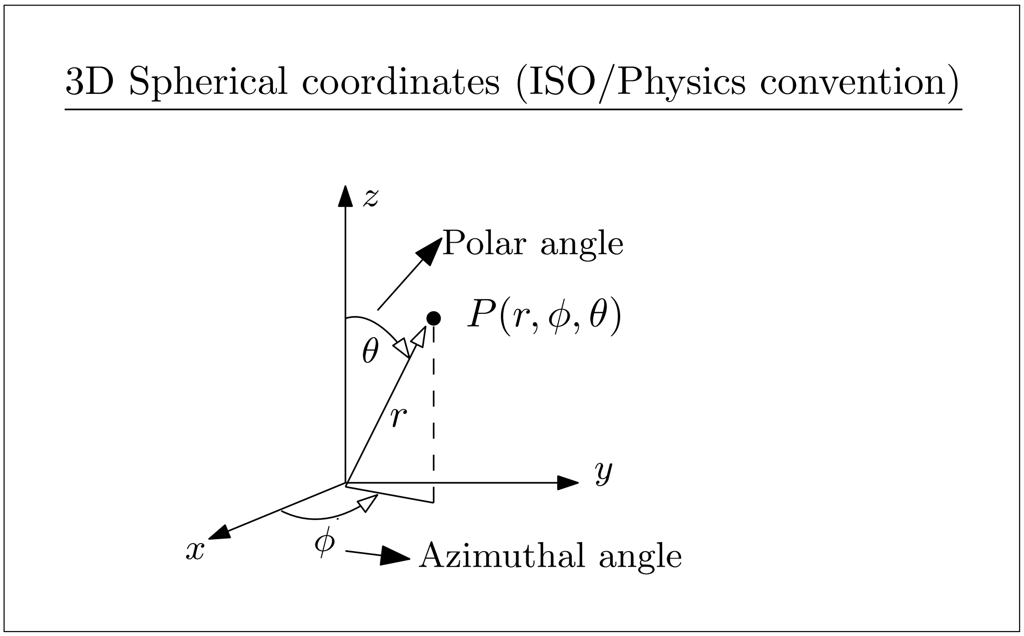

Figure 1: Spherical coordinates

The relation between the Cartesian and spherical coordinates is

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{1}$$

The 3D integral in Cartesian coordinates is

$$\int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} \int_{z=-\infty}^{z=\infty} e^{-x^2-y^2-z^2} dxdydz = \left(\sqrt{\pi}\right)^3$$

But $x^2 + y^2 + z^2 = r^2$ in spherical coordinates. The above now simplifies to

$$\int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} \int_{z=-\infty}^{z=\infty} e^{-r^2} dxdydz = \pi^{\frac{3}{2}}$$

Changing integration from Cartesian to spherical and changing the limits accordingly. the above becomes

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{\phi=2\pi} e^{-r^2}\, J\, drd\theta d\phi = \pi^{\frac{3}{2}} \tag{2}$$

The Jacobian J is

$$J = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} & \frac{dx}{d\phi} \\ \frac{dy}{dr} & \frac{dy}{d\theta} & \frac{dy}{d\phi} \\ \frac{dz}{dr} & \frac{dz}{d\theta} & \frac{dz}{d\phi} \end{vmatrix} \tag{3}$$

The relation between Cartesian and spherical in (1) shows that

$$\begin{aligned} \frac{dx}{dr} &= \sin\theta\cos\phi \\ \frac{dx}{d\theta} &= r\cos\theta\cos\phi \\ \frac{dx}{d\phi} &= -r\sin\theta\sin\phi \\ \frac{dy}{dr} &= \sin\theta\sin\phi \\ \frac{dy}{d\theta} &= r\cos\theta\sin\phi \\ \frac{dy}{d\phi} &= r\sin\theta\cos\phi \\ \frac{dz}{dr} &= \cos\theta \\ \frac{dz}{d\theta} &= -r\sin\theta \\ \frac{dz}{d\phi} &= 0 \end{aligned}$$

Substituting the above in (3) gives

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \tag{3}$$

Expanding along the last row to find the determinant (since last row has most number of zeros in it) gives the determinant as

$$
\begin{aligned}
J &= \cos\theta \begin{vmatrix} r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ r\cos\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} + r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} \\
&= \cos\theta\Big(\big(r\cos\theta\cos\phi\big)\big(r\sin\theta\cos\phi\big) + \big(r\sin\theta\sin\phi\big)\big(r\cos\theta\sin\phi\big)\Big) + r\sin\theta\Big(\big(\sin\theta\cos\phi\big)\big(r\sin\theta\cos\phi\big) \\
&= \cos\theta\big(r^2\cos\theta\sin\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\big) + r\sin\theta\big(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\big) \\
&= r^2\sin\theta\cos^2\theta\big(\cos^2\phi + \sin^2\phi\big) + r^2\sin^3\theta\big(\cos^2\phi + \sin^2\phi\big) \\
&= r^2\sin\theta\cos^2\theta + r^2\sin^3\theta \\
&= r^2\sin\theta\big(\cos^2\theta + \sin^2\theta\big)
\end{aligned}
$$

Therefore

$$\boxed{J = r^2\sin\theta}$$

Substituting the Jacobian in integral (2) gives

$$
\begin{aligned}
\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{\phi=2\pi} e^{-r^2}\big(r^2\sin\theta\big)drd\theta d\phi &= \pi^{\frac{3}{2}} \\
\int_{\phi=0}^{\phi=2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \pi^{\frac{3}{2}} \\
2\pi \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \pi^{\frac{3}{2}} \\
-2\pi[\cos\theta]_0^{\pi} \int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \pi^{\frac{3}{2}} \\
-2\pi[(-1) - 1] \int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \pi^{\frac{3}{2}} \\
-2\pi[-2] \int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \pi^{\frac{3}{2}} \\
4\pi \int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \pi^{\frac{3}{2}} \\
\int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \frac{\pi^{\frac{3}{2}}}{4\pi} \\
\int_{r=0}^{\infty} r^2 e^{-r^2} dr &= \frac{1}{4}\sqrt{\pi}
\end{aligned}
$$

Since r is just an integration variable, changing it to x gives

$$\int_0^{\infty} x^2 e^{-x^2} dx = \frac{1}{4}\sqrt{\pi}$$

Which is what we asked to show.

2 Problem 2

Follow the lecture example of deriving the gravitational field of a thin shell and calculate the gravitational *potential* of such a shell over all space

Solution

2.1 Field outside the shell

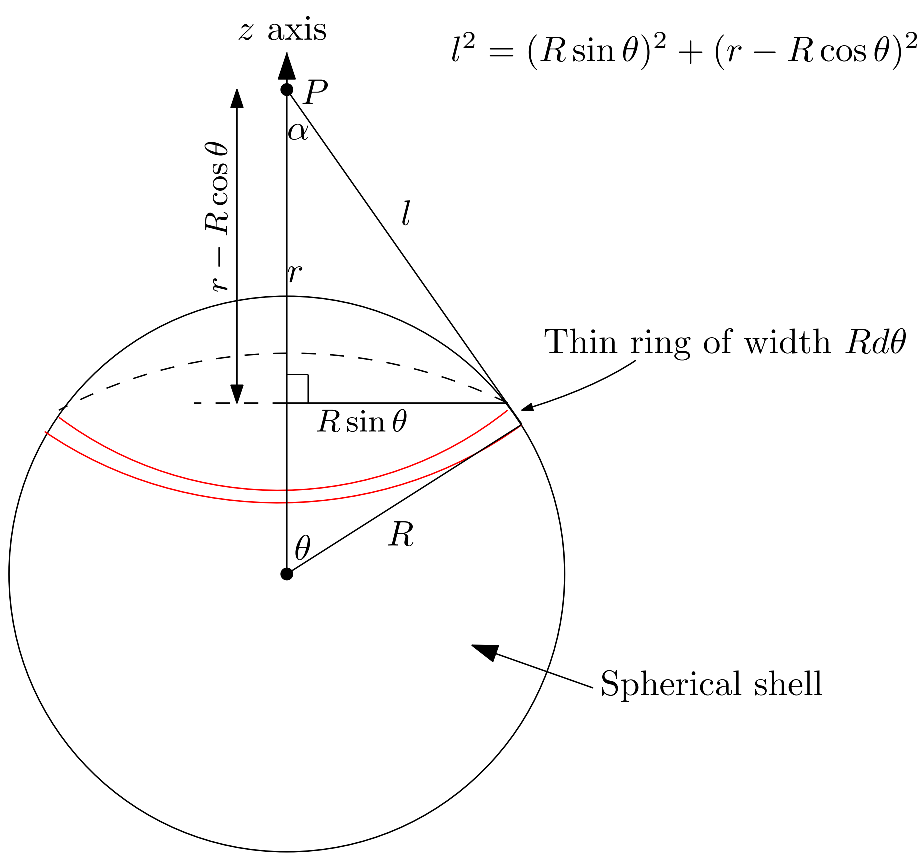

Figure 2: Problem setup

The gravitational field at point p as shown in the diagram will be determined. The point p is at distance r from the center of the shell. Due to symmetry any radial direction can be used as z axis.

A small ring is considered as shown. The field due to this at point p is due to vertical contribution only, since horizontal contribution cancel out. This means field due to this ring is given by

$$dg = G\frac{dm}{l^2}\cos\alpha \tag{1}$$

Where dm is the mass of the ring. But $dm = \sigma dA$, where dA is the surface area of the ring between θ and $\theta + d\theta$.

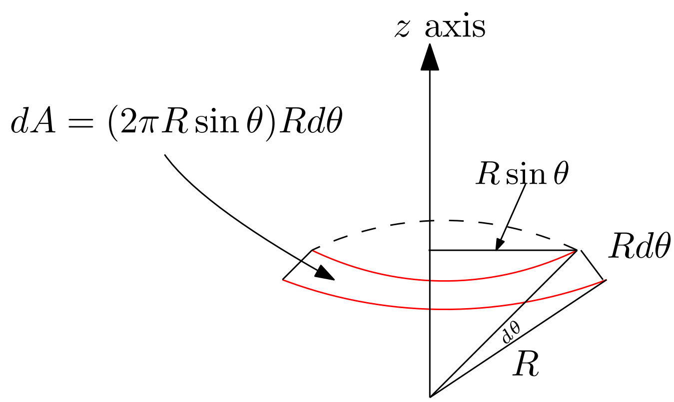

Figure 3: Surface area of ring

Hence

$$dA = (2\pi R\sin\theta)Rd\theta$$

Where $2\pi R\sin\theta$ is the circumference. Hence (1) becomes

$$\begin{aligned} dg &= G\frac{\sigma dA}{l^2}\cos\alpha \\ &= G\frac{\sigma(2\pi R\sin\theta)Rd\theta}{l^2}\cos\alpha \end{aligned} \tag{2}$$

Where σ is the surface mass density of the shell. But from the above diagram

$$\cos\alpha = \frac{r - R\cos\theta}{l}$$

Using this in (2) gives

$$\begin{aligned} dg &= G\frac{\sigma(2\pi R\sin\theta)Rd\theta}{l^2}\left(\frac{r - R\cos\theta}{l}\right) \\ &= G\sigma\left(2\pi R^2\sin\theta\right)(r - R\cos\theta)\frac{1}{l^3}d\theta \end{aligned} \tag{3}$$

l is now found from Pythagoras theorem (another option would have been to use the cosine angle rule)

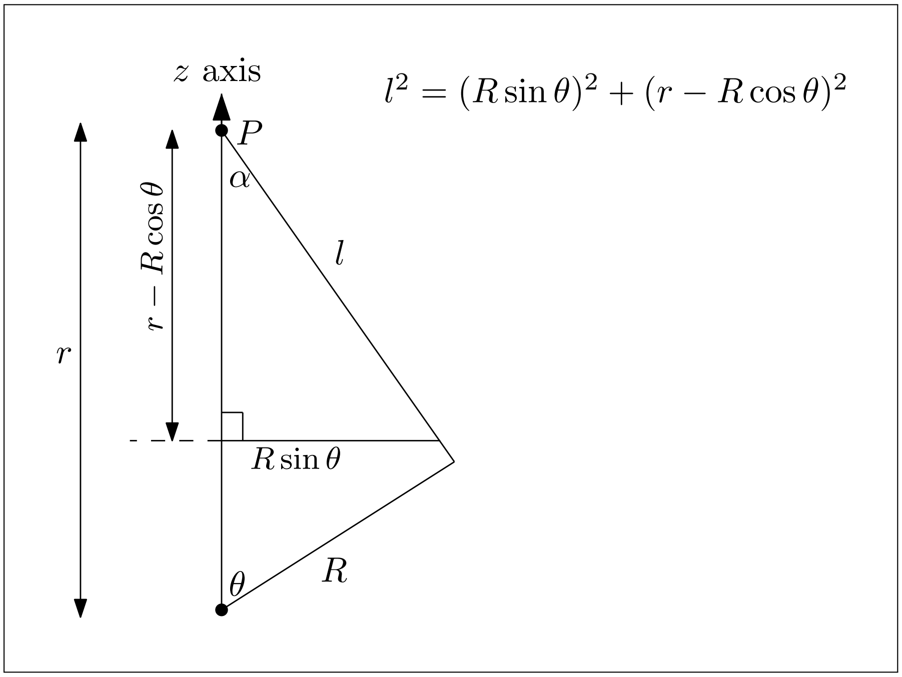

Figure 4: Finding l

$$
\begin{aligned}
l^2 &= (r - R\cos\theta)^2 + (R\sin\theta)^2 \\
&= r^2 + R^2\cos^2\theta - 2rR\cos\theta + R^2\sin^2\theta \\
&= r^2 + R^2 - 2rR\cos\theta
\end{aligned}
$$

Therefore

$$
\boxed{l = \sqrt{r^2 + R^2 - 2rR\cos\theta}}
$$

Substituting this in (3) gives

$$
dg = G\sigma\frac{\left(2\pi R^2\sin\theta\right)(r - R\cos\theta)}{\left(r^2 + R^2 - 2rR\cos\theta\right)^{\frac{3}{2}}}d\theta
$$

The above is the field at point p due to the small ring shown. To find the contribution from all of the shell, we need to integrate the above, which gives

$$
\begin{aligned}
g &= \int_{\theta=0}^{\theta=\pi} G\sigma\frac{\left(2\pi R^2\sin\theta\right)(r - R\cos\theta)}{\left(r^2 + R^2 - 2rR\cos\theta\right)^{\frac{3}{2}}}d\theta \\
&= G\sigma\left(2\pi R^2\right)\int_0^{\pi}\frac{\sin\theta(r - R\cos\theta)}{\left(r^2 + R^2 - 2rR\cos\theta\right)^{\frac{3}{2}}}d\theta \qquad (4)
\end{aligned}
$$

Let $u = \cos\theta$, then $du = -\sin\theta d\theta$. When $\theta = 0, u = 1$ and when $\theta = \pi, u = -1$. Hence the integral (4) becomes

$$\begin{aligned}
g &= G\sigma\left(2\pi R^2\right)\int_1^{-1}\frac{\sin\theta(r-Ru)}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}\frac{du}{-\sin\theta}\\
&= G\sigma\left(2\pi R^2\right)\int_{-1}^{1}\frac{r-Ru}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du\\
&= G\sigma\left(2\pi R^2\right)\left(\int_{-1}^{1}\frac{r}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du-\int_{-1}^{1}\frac{Ru}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du\right)\\
&= G\sigma\left(2\pi R^2\right)\left(r\int_{-1}^{1}\frac{1}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du-R\int_{-1}^{1}\frac{u}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du\right)\\
&= G\sigma\left(2\pi R^2\right)(rI_1-RI_2) \qquad (5)
\end{aligned}$$

Where

$$I_1 = \int_{-1}^{1}\frac{1}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du \qquad (6)$$

$$I_2 = \int_{-1}^{1}\frac{u}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du \qquad (7)$$

To evaluate I_1. Let

$$v^2 = r^2+R^2-2rRu$$

Hence

$$\begin{aligned}
\frac{d}{dv}\left(v^2\right) &= \frac{d}{du}\left(r^2+R^2-2rRu\right)\\
2vdv &= -2rRdu
\end{aligned}$$

Therefore

$$\begin{aligned}
du &= \frac{2v}{-2rR}dv\\
&= \frac{-v}{rR}dv
\end{aligned}$$

When $u=-1, v=\sqrt{r^2+R^2+2rR}$ and when $u=1, v=\sqrt{r^2+R^2-2rR}$. Hence I_1 becomes

$$
\begin{aligned}
I_1 &= \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^3}\left(\frac{-v}{rR}dv\right)\\
&= -\frac{1}{rR}\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^2}dv\\
&= -\frac{1}{rR}\left(\frac{-1}{v}\right)_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}\\
&= \frac{1}{rR}\left(\frac{1}{v}\right)_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}\\
&= \frac{1}{rR}\left(\frac{1}{\sqrt{r^2+R^2-2rR}}-\frac{1}{\sqrt{r^2+R^2+2rR}}\right)\\
&= \frac{1}{rR}\left(\frac{\sqrt{r^2+R^2+2rR}-\sqrt{r^2+R^2-2rR}}{\sqrt{r^2+R^2-2rR}\sqrt{r^2+R^2+2rR}}\right)
\end{aligned}
$$

Since $r>R$, the above can be written as

$$
\begin{aligned}
I_1 &= \frac{1}{rR}\left(\frac{\sqrt{(r+R)^2}-\sqrt{(r-R)^2}}{\sqrt{R^4-2R^2r^2+r^4}}\right)\\
&= \frac{1}{rR}\left(\frac{(r+R)-(r-R)}{\sqrt{\left(r^2-R^2\right)^2}}\right)\\
&= \frac{1}{rR}\left(\frac{2R}{\left(r^2-R^2\right)}\right)\\
&= \frac{1}{r}\frac{2}{\left(r^2-R^2\right)}
\end{aligned}
\tag{8}
$$

Now that I_1 is found, similar calculation is made to evaluate I_2 from (7)

$$
I_2=\int_{-1}^{1}\frac{u}{\left(r^2+R^2-2rRu\right)^{\frac{3}{2}}}du
$$

Similar to I_1, Let

$$
v^2=r^2+R^2-2rRu
$$

Hence

$$
u=\frac{v^2-r^2-R^2}{-2rR}
$$

Hence I_2 becomes

$$
\begin{aligned}
I_2 &= \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{\left(\frac{v^2-r^2-R^2}{-2rR}\right)}{v^3}\left(\frac{-v}{rR}dv\right)\\
&= \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2-r^2-R^2}{v^3(-2rR)}\left(\frac{-v}{rR}dv\right)\\
&= \left(\frac{1}{-2rR}\right)\left(\frac{1}{-rR}\right)\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2-r^2-R^2}{v^3}v dv\\
&= \frac{1}{2r^2R^2}\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2-r^2-R^2}{v^2}dv\\
&= \frac{1}{2r^2R^2}\left(\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{v^2}{v^2}dv - \int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{r^2+R^2}{v^2}dv\right)\\
&= \frac{1}{2r^2R^2}\left(\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} dv - \left(r^2+R^2\right)\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^2}dv\right) \qquad (9)
\end{aligned}
$$

The first integral in above is, and since $r > R$

$$
\begin{aligned}
\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} dv &= \sqrt{r^2+R^2-2rR} - \sqrt{r^2+R^2+2rR}\\
&= \sqrt{(r-R)^2} - \sqrt{(r+R)^2}\\
&= (r-R) - (r+R)\\
&= -2R \qquad (10)
\end{aligned}
$$

The second integral in (9) is

$$
\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^2}dv = -\left[\frac{1}{v}\right]_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}}
$$

As was done for I_1, the above simplifies to

$$
\int_{\sqrt{r^2+R^2+2rR}}^{\sqrt{r^2+R^2-2rR}} \frac{1}{v^2}dv = -\frac{2R}{\left(r^2-R^2\right)} \qquad (11)
$$

Substituting (10,11) back in (9) gives I_2

$$
\begin{aligned}
I_2 &= \frac{1}{2r^2R^2}\left(-2R - \left(r^2 + R^2\right)\left(-\frac{2R}{\left(r^2 - R^2\right)}\right)\right) \\
&= \frac{1}{2r^2R^2}\left(-2R + 2R\frac{r^2 + R^2}{\left(r^2 - R^2\right)}\right) \\
&= \frac{2R}{2r^2R^2}\left(-1 + \frac{r^2 + R^2}{\left(r^2 - R^2\right)}\right) \\
&= \frac{2}{2r^2R}\left(\frac{-\left(r^2 - R^2\right) + \left(r^2 + R^2\right)}{r^2 - R^2}\right) \\
&= \frac{2}{2r^2R}\left(\frac{-r^2 + R^2 + r^2 + R^2}{r^2 - R^2}\right) \\
&= \frac{1}{r^2}\frac{2R}{r^2 - R^2} \qquad (12)
\end{aligned}
$$

Now that I_1 and I_2 are found in (8) and (12), then substituting these in (5) gives

$$
\begin{aligned}
g &= G\sigma\left(2\pi R^2\right)(rI_1 - RI_2) \\
&= G\sigma\left(2\pi R^2\right)\left(r\left(\frac{1}{r}\frac{2}{\left(r^2 - R^2\right)}\right) - R\left(\frac{1}{r^2}\frac{2R}{r^2 - R^2}\right)\right) \\
&= G\sigma\left(2\pi R^2\right)\left(\frac{2}{\left(r^2 - R^2\right)} - \frac{1}{r^2}\frac{2R^2}{r^2 - R^2}\right) \\
&= G\sigma\left(2\pi R^2\right)\left(\frac{2r^2 - 2R^2}{r^2\left(r^2 - R^2\right)}\right) \\
&= \frac{G\sigma\left(2\pi R^2\right)}{r^2}\left(\frac{2\left(r^2 - R^2\right)}{\left(r^2 - R^2\right)}\right) \\
&= \frac{G\sigma\left(4\pi R^2\right)}{r^2}\left(\frac{\left(r^2 - R^2\right)}{\left(r^2 - R^2\right)}\right) \\
&= \frac{G\sigma\left(4\pi R^2\right)}{r^2}
\end{aligned}
$$

But $\sigma\left(4\pi R^2\right) = M$, which is the mass of the shell. Hence the above becomes

$$
\boxed{g = \frac{GM}{r^2}}
$$

This is the field strength at distance r from the center of the shell, where $r > R$. This shows that the field strength is the same as if the total mass of the shell was concentrated at a point in its center.

Now we need to obtain the potential energy of a particle of mass m located at distance r from the center of the shell. Taking potential energy of m to be zero at $r = \infty$, the potential energy is the work needed to move m from ∞ to distance r from center of shell. But work done is $U = -\int_{\infty}^{r} F dr'$ where F is the wight of m which is mg. Hence

$$U = -\int_{\infty}^{r} -mg dr'$$

The minus sign inside the integral is because the weight acts down, which is in the negative direction. The minus sign outside the integral is because work is done being done to increase the U of the mass. The rule is that, if work increases the potential energy of m, then it is negative. Since U is zero at infinity, then this work is negative. Therefore the above becomes

$$\begin{aligned} U &= \int_{\infty}^{r} mg dr' \\ &= \int_{\infty}^{r} \frac{GMm}{r'^2} dr' \\ &= -\left[\frac{GMm}{r'}\right]_{\infty}^{r} \\ &= -\left[\frac{GMm}{r} - 0\right] \end{aligned}$$

Therefore the gravitational potential energy of mass m at distance r from center of shell is

$$\boxed{U = -\frac{GMm}{r}}$$

2.2 Field inside the shell

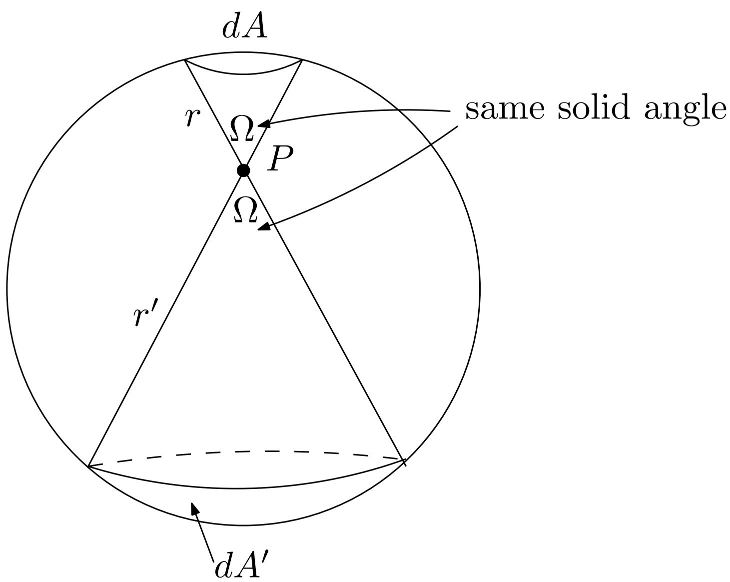

Figure 5: Problem setup

Let P be any arbitrary location inside the shell. Then the field at P due to contribution from dA only is

$$dg_A = G\frac{\sigma dA}{r^2}$$

And the field at P due to contribution from dA' only is

$$dg_{A'} = G\frac{\sigma dA'}{(r')^2}$$

The mass due to dA is pulling P upwards and the mass due to dA' is pulling P down. If we can show that these fields are of equal strength, then this shows the net gravitational field will be zero at P.

But $\frac{dA}{r^2} = \Omega$ where Ω is the solid angle made by the area dA as shown above. By symmetry, this is the same solid angle made by dA'. Therefore

$$\frac{dA}{r^2} = \frac{dA'}{(r')^2}$$

Therefore the net gravitational field is zero at P. Since P is arbitrary point. Then any point inside the shell will have zero net gravitational field.

Potential energy of a particle of mass m inside the shell is the same as the potential energy at surface of the shell, this is because $g = 0$ inside the shell.

Using the same derivation of potential energy in part 1 above gives

$$
\begin{aligned}
U &= \int_{\infty}^{R} mg\,dr' \\
&= \int_{\infty}^{R} \frac{GMm}{r'^2}\,dr' \\
&= -\left[\frac{GMm}{r'}\right]_{\infty}^{R} \\
&= -\left[\frac{GMm}{R} - 0\right]
\end{aligned}
$$

Therefore the gravitational potential energy of mass *m* anywhere inside the shell is

$$\boxed{U = -\frac{GMm}{R}}$$

3 Problem 3

Follow the lecture example of deriving the gas pressure and calculate the number of gas particles hitting the container per unit area per unit time. Give your answer in terms of the net number density and the average speed of these particles.

Solution

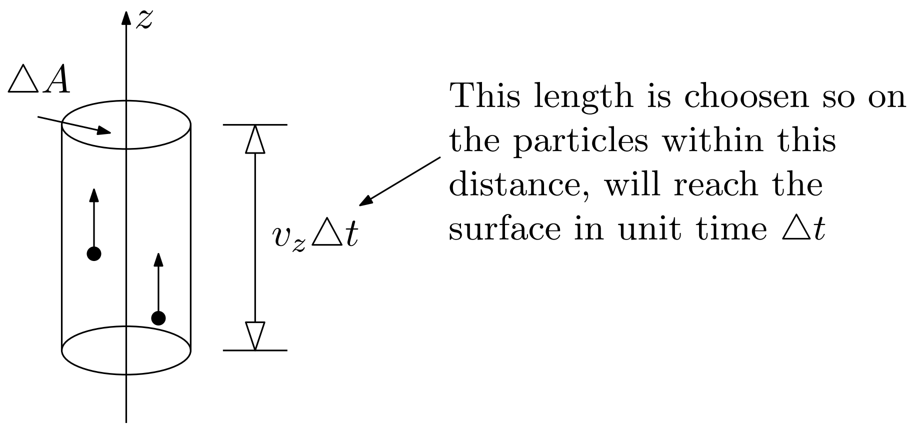

Figure 6: Problem setup

In the above diagram v_z is the average speed of particles in the z direction within Δt time from hitting ΔA. The number of particles per unit volume with velocity $\vec{v}$ and $\vec{v} + d\vec{v}$ is given by

$$dn = f(v) dv_x dv_y dv_z$$

Where v above is the magnitude (speed) of $\vec{v}$. During interval Δt, the number of particles hitting the wall is dN which is therefore given by

$$dN = dn(\Delta V) \tag{1}$$

Where dV is the unit volume shown in the diagram. But

$$dV = (v_z \Delta t)\Delta A$$

Therefore (1) becomes

$$\begin{aligned} dN &= dn(v_z \Delta t)\Delta A \\ &= f(v) dv_x dv_y dv_z (v_z \Delta t)\Delta A \end{aligned}$$

The above is the number of particles hitting ΔA of the wall in interval Δt.

4 Problem 4

Derive the expressions of the orbital angular momentum operators L_x, L_y, L_z in spherical coordinates. Show that

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2}$$

<u>Solution</u>

$$\vec{L} = \vec{r}\times\vec{p}$$

Where $\vec{L}$ is vector whose components are the orbital angular momentum operators L_x, L_y, L_z and $\vec{r}$ is a vector whose components are the position operators and $\vec{p}$ is a vector whose components are the momentum operators and $\times$ is the vector cross product. In Cartesian coordinates, $\hat{e}_x, \hat{e}_y, \hat{e}_z$ are the orthonormal basis. Hence

$$\begin{aligned}\vec{L} &= \begin{vmatrix}\hat{e}_x & \hat{e}_y & \hat{e}_z\\ x & y & z\\ p_x & p_y & p_z\end{vmatrix}\\ &= \hat{e}_x\left(yp_z - zp_y\right) - \hat{e}_y\left(xp_z - zp_x\right) + \hat{e}_z\left(xp_y - yp_x\right)\end{aligned}$$

Hence the corresponding components of $\vec{L} = \left\{L_x, L_y, L_z\right\}$ are

$$\begin{aligned}L_x &= yp_z - zp_y\\ L_y &= zp_x - xp_z\\ L_z &= xp_y - yp_x\end{aligned}\tag{1}$$

But in Quantum mechanics, the operators p_x, p_y, p_z are

$$\begin{aligned}p_x &= -i\hbar\left(\frac{\partial}{\partial x}\right)\\ p_y &= -i\hbar\left(\frac{\partial}{\partial y}\right)\\ p_z &= -i\hbar\left(\frac{\partial}{\partial z}\right)\end{aligned}$$

Hence (1) becomes

$$\begin{aligned}L_x &= y\left(-i\hbar\left(\frac{\partial}{\partial z}\right)\right) - z\left(-i\hbar\left(\frac{\partial}{\partial y}\right)\right)\\ L_y &= z\left(-i\hbar\left(\frac{\partial}{\partial x}\right)\right) - x\left(-i\hbar\left(\frac{\partial}{\partial z}\right)\right)\\ L_z &= x\left(-i\hbar\left(\frac{\partial}{\partial y}\right)\right) - y\left(-i\hbar\left(\frac{\partial}{\partial x}\right)\right)\end{aligned}$$

Or

$$\begin{aligned} L_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{aligned} \tag{1A}$$

Hence in Cartesian coordinates

$$\vec{L} = \begin{bmatrix} -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{bmatrix}$$

Now the above is converted to spherical coordinates. The relation between the Cartesian and spherical coordinates is

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{2}$$

We also need expression for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. But by chain rule

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{dr}{dx} + \frac{\partial}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial}{\partial \phi}\frac{d\phi}{dx} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{dr}{dy} + \frac{\partial}{\partial \theta}\frac{d\theta}{dy} + \frac{\partial}{\partial \phi}\frac{d\phi}{dy} \\ \frac{\partial}{\partial z} &= \frac{\partial}{\partial r}\frac{dr}{dz} + \frac{\partial}{\partial \theta}\frac{d\theta}{dz} + \frac{\partial}{\partial \phi}\frac{d\phi}{dz} \end{aligned}$$

To evaluate the above, we need to do the reverse of (2), which is to relate r, θ, ϕ to x, y, z. From the geometry we see that

$$r = \sqrt{x^2 + y^2 + z^2} \tag{3}$$

$$\cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \tag{4}$$

$$\tan\phi = \frac{y}{x} \tag{5}$$

Therefore, from (3)

$$dr = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2+z^2}}dx$$

But $x = r\sin\theta\cos\phi$ and $r = \sqrt{x^2+y^2+z^2}$. The above becomes

$$\begin{aligned}\frac{dr}{dx} &= \frac{r\sin\theta\cos\phi}{r}\\ &= \sin\theta\cos\phi \end{aligned} \tag{6}$$

And from (4)

$$\begin{aligned}\frac{d}{d\theta}(\cos\theta) &= \frac{d}{dx}\frac{z}{\sqrt{x^2+y^2+z^2}}\\ -\sin\theta d\theta &= -\frac{1}{2}\frac{z(2x)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dx\end{aligned}$$

But $x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$ and $x = r\sin\theta\cos\phi$. The above becomes

$$\begin{aligned}-\sin\theta d\theta &= -\frac{1}{2}\frac{r\cos\theta\left(2r\sin\theta\cos\phi\right)}{\left(r^2\right)^{\frac{3}{2}}}dx\\ &= \frac{-r^2\cos\theta\sin\theta\cos\phi}{r^3}dx\\ &= \frac{-\cos\theta\sin\theta\cos\phi}{r}dx\end{aligned}$$

Hence

$$\begin{aligned}d\theta &= \frac{\cos\theta\cos\phi}{r}dx\\ \frac{d\theta}{dx} &= \frac{1}{r}\cos\theta\cos\phi\end{aligned} \tag{7}$$

And from (5)

$$\begin{aligned}\frac{d}{d\phi}\left(\tan\phi\right) &= \frac{d}{dx}\left(\frac{y}{x}\right)\\ \frac{1}{\cos^2\phi}d\phi &= y\left(\frac{-1}{x^2}\right)dx\end{aligned}$$

But $y = r\sin\theta\sin\phi$ and $x = r\sin\theta\cos\phi$. Therefore

$$\begin{aligned}\frac{1}{\cos^2\phi}d\phi &= \frac{-r\sin\theta\sin\phi}{r^2\sin^2\theta\cos^2\phi}dx\\ \frac{d\phi}{dx} &= \frac{-r\sin\theta\sin\phi\cos^2\phi}{r^2\sin^2\theta\cos^2\phi}\\ &= \frac{-\sin\phi}{r\sin\theta}\end{aligned} \tag{8}$$

The above completes all the terms needed to find $\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{dr}{dx} + \frac{\partial}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial}{\partial \phi}\frac{d\phi}{dx}$. Hence, using (6,7,8) above gives

$$\boxed{\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}} \tag{9}$$

Now the same thing is repeated to find $\frac{\partial}{\partial y}$ in spherical coordinates. From (3)

$$dr = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2+z^2}}dy$$

But $y = r\sin\theta\sin\phi$ and $r = \sqrt{x^2+y^2+z^2}$. The above becomes

$$\begin{aligned}\frac{dr}{dy} &= \frac{r\sin\theta\sin\phi}{r}\\ &= \sin\theta\sin\phi \end{aligned} \tag{10}$$

And from (4)

$$\begin{aligned}\frac{d}{d\theta}\cos\theta &= \frac{d}{dy}\frac{z}{\sqrt{x^2+y^2+z^2}}\\ -\sin\theta d\theta &= -\frac{1}{2}\frac{z(2y)}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}dy\end{aligned}$$

But $x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$ and $y = r\sin\theta\sin\phi$. The above becomes

$$\begin{aligned}-\sin\theta d\theta &= -\frac{1}{2}\frac{r\cos\theta\left(2r\sin\theta\sin\phi\right)}{\left(r^2\right)^{\frac{3}{2}}}dy\\ &= \frac{-r^2\cos\theta\sin\theta\sin\phi}{r^3}dy\\ &= \frac{-\cos\theta\sin\theta\sin\phi}{r}dy\end{aligned}$$

Hence

$$\begin{aligned}d\theta &= \frac{\cos\theta\sin\phi}{r}dy\\ \frac{d\theta}{dy} &= \frac{1}{r}\cos\theta\sin\phi\end{aligned} \tag{11}$$

And from (5)

$$\begin{aligned}\frac{d}{d\phi}\left(\tan\phi\right) &= \frac{d}{dy}\left(\frac{y}{x}\right)\\ \frac{1}{\cos^2\phi}d\phi &= \left(\frac{1}{x}\right)dy\end{aligned}$$

But $x = r\sin\theta\cos\phi$. Therefore

$$\begin{aligned}\frac{1}{\cos^2\phi}d\phi &= \frac{1}{r\sin\theta\cos\phi}dy\\ \frac{d\phi}{dy} &= \frac{\cos^2\phi}{r\sin\theta\cos\phi}\\ &= \frac{1}{r}\frac{\cos\phi}{\sin\theta} \qquad (12)\end{aligned}$$

The above completes all the terms needed to find $\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{dr}{dy} + \frac{\partial}{\partial\theta}\frac{d\theta}{dy} + \frac{\partial}{\partial\phi}\frac{d\phi}{dy}$. Hence, using (10,11,12) above gives

$$\boxed{\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}} \qquad (13)$$

Now the same thing is repeated to find $\frac{\partial}{\partial z}$ in spherical coordinates. From (3)

$$dr = \frac{1}{2}\frac{2z}{\sqrt{x^2+y^2+z^2}}dz$$

But $z = r\cos\theta$ and $r = \sqrt{x^2+y^2+z^2}$. The above becomes

$$\begin{aligned}\frac{dr}{dz} &= \frac{r\cos\theta}{r}\\ &= \cos\theta \qquad (14)\end{aligned}$$

And from (4)

$$\begin{aligned}\frac{d}{d\theta}\cos\theta &= \frac{d}{dz}\frac{z}{\sqrt{x^2+y^2+z^2}}\\ -\sin\theta d\theta &= \left(\frac{1}{\sqrt{x^2+y^2+z^2}} + z\left(-\frac{1}{2}\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}(2z)\right)\right)dz\\ &= \left(\frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)dz\\ &= \left(\frac{\left(x^2+y^2+z^2\right)-z^2}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)dz\end{aligned}$$

But $x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$. The above becomes

$$\begin{aligned}-\sin\theta d\theta &= \left(\frac{r^2 - r^2\cos^2\theta}{r^3}\right)dz\\ &= \frac{1-\cos^2\theta}{r}dz\end{aligned}$$

Hence

$$\frac{d\theta}{dz} = -\frac{1-\cos^2\theta}{r\sin\theta}$$
$$= -\frac{\sin^2\theta}{r\sin\theta}$$
$$= -\frac{1}{r}\sin\theta \tag{15}$$

And from (5)

$$\frac{d}{d\phi}\left(\tan\phi\right) = \frac{d}{dz}\left(\frac{y}{x}\right)$$

Hence, since RHS does not depend on z then

$$\frac{d\phi}{dz} = 0 \tag{16}$$

The above completes all the terms needed to find $\frac{\partial}{\partial z} = \frac{\partial}{\partial r}\frac{dr}{dz} + \frac{\partial}{\partial\theta}\frac{d\theta}{dz} + \frac{\partial}{\partial\phi}\frac{d\phi}{dz}$. Hence, using (14,15,16) above gives

$$\boxed{\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}} \tag{17}$$

The above completes all derivations needed to find L_x, L_y, L_z in Spherical coordinates. Eqs (9,13,17). Here they are in one place.

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{9}$$
$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{13}$$
$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta} \tag{17}$$

Given Eq(1A) found earlier (repeated below)

$$\begin{aligned} L_x &= -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) \\ L_y &= -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \\ L_z &= -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) \end{aligned} \tag{1A}$$

And given (9,13,17), then (1A) becomes

$$L_x = -i\hbar\left(y\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) - z\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)$$

$$L_y = -i\hbar\left(z\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - x\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\right)$$

$$L_z = -i\hbar\left(x\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)$$
$$+ i\hbar\left(y\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)$$

But $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$. The above becomes

$$L_x = -i\hbar\left(r\sin\theta\sin\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) - r\cos\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)$$

$$L_y = -i\hbar\left(r\cos\theta\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right) - r\sin\theta\cos\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\right)$$

$$L_z = -i\hbar\left(r\sin\theta\cos\phi\right)\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$+ i\hbar\left(r\sin\theta\sin\phi\right)\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

Simplifying gives

$$L_x = -i\hbar\left(\left(r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r} - \sin^2\theta\sin\phi\frac{\partial}{\partial\theta}\right) - \left(r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r} + \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} + \cos\theta\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\right)$$

$$L_y = -i\hbar\left(r\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial r} + \cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$
$$+ i\hbar\left(r\sin\theta\cos\phi\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}r\sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)$$

$$L_z = -i\hbar\left(r\sin^2\theta\cos\phi\sin\phi\frac{\partial}{\partial r} + \sin\theta\cos\phi\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \cos^2\phi\frac{\partial}{\partial\phi}\right)$$
$$+ i\hbar\left(r\sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r} + \sin\theta\sin\phi\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \sin^2\phi\frac{\partial}{\partial\phi}\right)$$

Or

$$L_x = -i\hbar\left(r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r} - \sin^2\theta\sin\phi\frac{\partial}{\partial\theta} - r\cos\theta\sin\theta\sin\phi\frac{\partial}{\partial r} - \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$L_y = -i\hbar\left(r\cos\theta\sin\theta\cos\phi\frac{\partial}{\partial r} + \cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} - r\sin\theta\cos\phi\cos\theta\frac{\partial}{\partial r} + \sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)$$

$$L_z = -i\hbar\left(-r\sin^2\theta\cos\phi\sin\phi\frac{\partial}{\partial r} + \cos^2\phi\frac{\partial}{\partial\phi} - r\sin^2\theta\sin\phi\cos\phi\frac{\partial}{\partial r} + \sin^2\phi\frac{\partial}{\partial\phi}\right)$$

Or

$$
\begin{aligned}
L_x &= -i\hbar\left(-\sin^2\theta\sin\phi\frac{\partial}{\partial\theta} - \cos^2\theta\sin\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(\cos^2\theta\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} + \sin^2\theta\cos\phi\frac{\partial}{\partial\theta}\right)\\
L_z &= -i\hbar\left(\sin\theta\cos\phi\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \cos^2\phi\frac{\partial}{\partial\phi} - \sin\theta\sin\phi\cos\theta\cos\phi\frac{\partial}{\partial\theta} + \sin^2\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L_x &= -i\hbar\left(-\left(\sin^2\theta + \cos^2\theta\right)\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(\left(\cos^2\theta + \sin^2\theta\right)\cos\phi\frac{\partial}{\partial\theta} - \cos\theta\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi}\right)\\
L_z &= -i\hbar\left(\cos^2\phi\frac{\partial}{\partial\phi} + \sin^2\phi\frac{\partial}{\partial\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L_x &= -i\hbar\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\\
L_y &= -i\hbar\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)\\
L_z &= -i\hbar\left(\frac{\partial}{\partial\phi}\right)
\end{aligned}
\tag{18}
$$

The above are L_x, L_y, L_z in spherical coordinates. Therefore

$$
\begin{aligned}
\vec{L}\cdot\vec{L} &= L^2\\
&= L_x^2 + L_y^2 + L_z^2
\end{aligned}
$$

But

$$
\begin{aligned}
L_x^2 &= -\hbar^2\left(\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\left(-\sin\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right)\right)\\
&= -\hbar^2\left(\sin^2\phi\frac{\partial^2}{\partial^2\theta} + \sin\phi\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\right) + \frac{\cos\theta}{\sin\theta}\cos\phi\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right) + \frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi\frac{\partial^2}{\partial^2\phi}\right)\\
&= -\hbar^2\left(\sin^2\phi\frac{\partial^2}{\partial^2\theta} - \frac{\sin\phi\cos\phi}{\sin^2\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta}{\sin\theta}\cos\phi\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi\frac{\partial^2}{\partial^2\phi}\right)
\end{aligned}
$$

And

$$
\begin{aligned}
L_y^2 &= -\hbar^2\left(\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)\left(\cos\phi\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right)\right) \\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta} - \cos\phi\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\right) - \frac{\cos\theta}{\sin\theta}\sin\phi\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right) + \frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right) \\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta} - \cos\phi\sin\phi\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi}\right) - \frac{\cos\theta}{\sin\theta}\sin\phi\left(-\sin\phi\frac{\partial}{\partial\theta}\right) + \frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right) \\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta} + \cos\phi\sin\phi\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\phi}\right) + \frac{\cos\theta}{\sin\theta}\sin^2\phi\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right) \\
&= -\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta} + \frac{\cos\phi\sin\phi}{\sec^2\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta}{\sin\theta}\sin^2\phi\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right)
\end{aligned}
$$

And

$$
L_z^2 = -\hbar^2\left(\frac{\partial^2}{\partial^2\phi}\right)
$$

Hence

$$
\begin{aligned}
L^2 = &-\hbar^2\left(\sin^2\phi\frac{\partial^2}{\partial^2\theta} - \frac{\sin\phi\cos\phi}{\sec^2\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta}{\sin\theta}\cos\phi\cos\phi\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi\frac{\partial^2}{\partial^2\phi}\right) \\
&-\hbar^2\left(\cos^2\phi\frac{\partial^2}{\partial^2\theta} + \frac{\cos\phi\sin\phi}{\sec^2\theta}\frac{\partial}{\partial\phi} + \frac{\cos\theta}{\sin\theta}\sin^2\phi\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi\frac{\partial^2}{\partial^2\phi}\right) \\
&-\hbar^2\left(\frac{\partial^2}{\partial^2\phi}\right)
\end{aligned}
$$

Or

$$
\begin{aligned}
L^2 = &-\hbar^2\left(\frac{\partial^2}{\partial^2\theta}\left(\sin^2\phi + \cos^2\phi\right) + \frac{\partial^2}{\partial^2\phi}\left(\frac{\cos^2\theta}{\sin^2\theta}\cos^2\phi + \frac{\cos^2\theta}{\sin^2\theta}\sin^2\phi + 1\right)\right) \\
&-\hbar^2\frac{\partial}{\partial\phi}\left(-\frac{\sin\phi\cos\phi}{\sec^2\theta} + \frac{\cos\phi\sin\phi}{\sec^2\theta}\right) \\
&-\hbar^2\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{\sin\theta}\cos\phi\cos\phi + \frac{\cos\theta}{\sin\theta}\sin^2\phi\right)
\end{aligned}
$$

Which simplifies to

$$\begin{aligned}
L^2 &= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\left(\cos^2\phi + \sin^2\phi\right) + \frac{\partial^2}{\partial^2\phi} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\left(\cos^2\phi + \sin^2\phi\right)\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi} + \frac{\partial^2}{\partial^2\phi} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta} + \left(1 + \frac{\cos^2\theta}{\sin^2\theta}\right)\frac{\partial^2}{\partial^2\phi} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta} + \left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\right)\frac{\partial^2}{\partial^2\phi} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\\
&= -\hbar^2\left(\frac{\partial^2}{\partial^2\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2} &= \frac{1}{\hbar^2 r^2}\left(-\hbar^2\left(\frac{\partial^2}{\partial^2\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right)\\
&= -\frac{1}{r^2}\frac{\partial^2}{\partial^2\theta} - \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi} - \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}
\end{aligned} \tag{20}$$

Therefore

$$\begin{aligned}
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2} &= \frac{1}{r^2}\left(2r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta} + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi} + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\\
&= \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial^2\theta} + \frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}
\end{aligned}$$

But the term in the RHS above is indeed the Laplacian in spherical coordinates. Therefore in spherical coordinates

$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{\vec{L}\cdot\vec{L}}{\hbar^2 r^2}}$$

Which is what we are asked to show.

5 Problem 5

Consider $\psi(x,t)$ for $0 \leq x \leq L$. Given $\psi(0,t) = \psi(L,t) = 0$ and

$$\psi(x,0) = \begin{cases} A\sin\left(\frac{2\pi x}{L}\right) & 0 \leq x \leq \frac{L}{2} \\ 0 & \frac{L}{2} \leq x \leq L \end{cases}$$

Find $\psi(x,t)$ that satisfies the following partial differential equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\frac{\partial^2\psi}{\partial t^2} \tag{1}$$

Where $A, L, \hbar, \mu$ are positive constants.

Solution

Using separation of variables, assuming the solution is

$$\psi(x,t) = X(x)T(t)$$

Where $X(x)$ is function that depends on space only and $T(t)$ is function that depends on t only. Substituting the above into the PDE (1) gives

$$i\hbar XT' = -\frac{\hbar^2}{2\mu}X''T$$

Diving both sides by $XT \neq 0$ gives

$$\begin{aligned} i\hbar\frac{T'}{T} &= -\frac{\hbar^2}{2\mu}\frac{X''}{X} \\ -\frac{2\mu i}{\hbar}\frac{T'}{T} &= \frac{X''}{X} \end{aligned}$$

Since both sides are equal, and left side depends on t only and right side depends on x only, then both must be equal to a constant. Let this constant be $-\lambda$. This gives the following two ODE's to solve

$$-\frac{2\mu i}{\hbar}\frac{T'}{T} = -\lambda \tag{2}$$

$$\frac{X''}{X} = -\lambda \tag{3}$$

Starting with the spatial ODE in order to determine the eigenvalues λ

$$X''(x) + \lambda X(x) = 0 \tag{4}$$

With the boundary conditions transferred from the PDE as

$$X(0) = 0$$
$$X(L) = 0$$

There are three cases to consider. $\lambda < 0, \lambda = 0, \lambda > 0$.

case $\lambda < 0$

Let $\lambda = -\mu^2$ for some real μ. Then the ODE (4) becomes $X''(x) - \mu^2 X(x) = 0$. The roots of the characteristic equation are $\pm\mu$. Hence the solution is

$$\begin{aligned} X(x) &= Ae^{\mu x} + Be^{-\mu x} \\ &= A\cosh(\mu x) + B\sinh(\mu x) \end{aligned}$$

At $x = 0$, the above becomes

$$0 = A$$

Hence the solution now reduces to

$$X(x) = B\sinh(\mu x)$$

At $x = L$, this becomes

$$0 = B\sinh(\mu L)$$

But $\mu L \neq 0$ since $L > 0$ and $\mu \neq 0$. Therefore the only option is that $B = 0$. But this gives trivial solution $X(x) = 0$. Therefore $\lambda < 0$ is not possible.

case $\lambda = 0$

The ODE (4) now becomes

$$X''(x) = 0$$

This has solution $X = Ax + B$. At $x = 0$ this gives $0 = B$. Therefore the solution now reduces to $X(x) = Ax$. At $x = L$ this gives $0 = AL$, which implies $A = 0$. But this gives trivial solution $X(x) = 0$. Therefore $\lambda = 0$ is not possible.

case $\lambda > 0$

In this case, the roots of the characteristic equation of ODE (4) are $\pm i\sqrt{\lambda}$. Hence the solution can written as (by using Euler relation to convert complex exponentials to trigonometric functions) as

$$X(x) = A\cos\left(\sqrt{\lambda}\,x\right) + B\sin\left(\sqrt{\lambda}\,x\right)$$

At $x = 0$ the above gives

$$0 = A$$

Hence the solution now reduces to

$$X(x) = B\sin\left(\sqrt{\lambda}\,x\right)$$

At $x = L$

$$0 = B \sin\left(\sqrt{\lambda}\, L\right)$$

For non-trivial solution this requires that $\sin\left(\sqrt{\lambda}\, L\right) = 0$ or $\sqrt{\lambda}\, L = n\pi$ for $n = 1, 2, \cdots$. Therefore the eigenvalues are

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad n = 1, 2, \cdots$$

This completes the solution to the spatial part. The eigenfunctions are therefore

$$X_n(x) = B_n \sin\left(\frac{n\pi}{L} x\right) \qquad n = 1, 2, \cdots \tag{5}$$

Now the time domain part ODE is solved. This is ODE (2) above. Now that the eigenvalues are known, ODE (2) becomes

$$\begin{aligned} -\frac{2\mu i}{\hbar} \frac{T_n'}{T_n} &= -\lambda_n \\ T_n' &= \frac{T_n \hbar}{2\mu i} \lambda_n \\ T_n' - \frac{\hbar}{2\mu i} \lambda_n T_n &= 0 \end{aligned}$$

This is linear first order ODE. The integrating factor is $I = e^{\frac{-\lambda_n \hbar}{2\mu i} t}$. The above now becomes

$$\frac{d}{dt}\left(T_n e^{\frac{-\lambda_n \hbar}{2\mu i} t}\right) = 0$$

Integrating gives

$$\begin{aligned} T_n e^{\frac{-\lambda_n \hbar}{2\mu i} t} &= C_n \\ T_n(t) &= C_n e^{\frac{\lambda_n \hbar}{2\mu i} t} \\ &= C_n e^{-\frac{i}{2} \frac{\lambda_n \hbar}{\mu} t} \end{aligned}$$

But λ_n are the eigenvalues, given by $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n = 1, 2, \cdots$. Rewriting the above gives

$$T_n(t) = C_n e^{-\frac{i}{2} \frac{\hbar n^2 \pi^2}{\mu L^2} t} \tag{6}$$

But since the solution was assumed to be $\psi(x, t) = X(x) T(t)$, then

$$\psi_n(x, t) = X_n(x) T_n(t)$$

But the general solution is a linear combination of all the solutions $\psi_n(x,t)$. Therefore

$$\begin{aligned}\psi(x,t) &= \sum_{n=1}^{\infty} \psi_n(x,t)\\ &= \sum_{n=1}^{\infty} X_n(x) T_n(t)\end{aligned}$$

And using (5,6) in the above, gives

$$\psi(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) C_n e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}$$

But the two constants $B_n C_n$ can be merged into one, say D_n. Therefore the above becomes

$$\psi(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t} \tag{7}$$

The above is the general solution. What is left is to determine D_n. This is done from initial conditions. At $t = 0$ the above becomes

$$\begin{cases} A\sin\left(\frac{2\pi x}{L}\right) & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases} = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right)$$

The above says that D_n are the Fourier sine series coefficients of the initial conditions. To determine D_n, orthogonality of eigenfunctions $\sin\left(\frac{n\pi}{L}x\right)$ is used.

Multiplying both sides of the above by $\sin\left(\frac{m\pi}{L}x\right)$ and integration both sides from $x = 0$ to $x = L$ gives

$$\int_0^L \begin{cases} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right)dx & 0 \le x \le \frac{L}{2} \\ 0 & \frac{L}{2} \le x \le L \end{cases} = \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi}{L}x\right)dx$$

$$\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right)dx = \sum_{n=1}^{\infty} D_n \int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

Case $m = 1$

The sum above now collapses to one term only when $m = n = 1$, since the sin functions are orthogonal to each others, which gives

$$\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx = D_1\int_0^{L}\sin^2\left(\frac{\pi}{L}x\right)dx$$

$$\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx = D_1\frac{L}{2}$$

$$D_1 = \frac{2}{L}\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx \tag{8}$$

The integral $\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx$, is evaluated using the relation

$$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

the integral becomes

$$\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}x\right)dx = A\int_0^{\frac{L}{2}} \frac{1}{2}\left(\cos\left(\frac{2\pi x}{L} - \frac{\pi}{L}x\right) - \cos\left(\frac{2\pi x}{L} + \frac{\pi}{L}x\right)\right)dx$$
$$= \frac{A}{2}\left(\int_0^{\frac{L}{2}} \cos\left(\frac{2\pi x}{L} - \frac{\pi}{L}x\right)dx - \int_0^{\frac{L}{2}} \cos\left(\frac{2\pi x}{L} + \frac{\pi}{L}x\right)dx\right)$$
$$= \frac{A}{2}\left(\int_0^{\frac{L}{2}} \cos\left(\frac{\pi x}{L}\right)dx - \int_0^{\frac{L}{2}} \cos\left(\frac{3\pi x}{L}\right)dx\right)$$
$$= \frac{A}{2}\left(\left[\frac{\sin\left(\frac{\pi x}{L}\right)}{\frac{\pi}{L}}\right]_0^{\frac{L}{2}} - \left[\frac{\sin\left(\frac{3\pi x}{L}\right)}{\frac{3\pi}{L}}\right]_0^{\frac{L}{2}}\right)$$
$$= \frac{A}{2}\left(\frac{L}{\pi}\left[\sin\left(\frac{\pi x}{L}\right)\right]_0^{\frac{L}{2}} - \frac{L}{3\pi}\left[\sin\left(\frac{3\pi x}{L}\right)\right]_0^{\frac{L}{2}}\right)$$
$$= \frac{A}{2}\left(\frac{L}{\pi}\sin\left(\frac{\pi\frac{L}{2}}{L}\right) - \frac{L}{3\pi}\sin\left(\frac{3\pi\frac{L}{2}}{L}\right)\right)$$
$$= \frac{A}{2}\left(\frac{L}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{L}{3\pi}\sin\left(\frac{3}{2}\pi\right)\right)$$
$$= \frac{A}{2}\left(\frac{L}{\pi} + \frac{L}{3\pi}\right)$$
$$= \frac{L}{\pi}\frac{A}{2}\left(1 + \frac{1}{3}\right)$$
$$= \frac{L}{\pi}\frac{A}{2}\left(\frac{4}{3}\right)$$
$$= \frac{L}{\pi}\frac{2A}{3}$$

Hence Eq. (8) becomes

$$D_1 = \frac{2}{L}\left(\frac{L}{\pi}\frac{2A}{3}\right)$$
$$= \frac{4A}{3\pi}$$

Case $m = 2$

The sum above now collapses to one term only, since the sin functions are orthogonal to each others, so only for $n = 2$ the sum gives a result. Hence

$$\begin{aligned}\int_0^{\frac{L}{2}} A\sin^2\left(\frac{2\pi x}{L}\right)dx &= D_2\int_0^L \sin^2\left(\frac{2\pi}{L}x\right)dx\\ A\frac{L}{4} &= D_2\frac{L}{2}\\ D_2 &= \frac{1}{2}A\end{aligned}$$

<u>case $m \geq 3$</u>

The sum now collapses to case when $m = n$, since the sin functions are orthogonal to each others. Hence

$$\begin{aligned}\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{m\pi}{L}x\right)dx &= D_m\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx\\ &= D_m\frac{L}{2}\end{aligned}$$

Therefore (now calling $m = n$ since a dummy index)

$$D_n = \frac{2}{L}\int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)dx \tag{9}$$

The integral $I = \int_0^{\frac{L}{2}} A\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{n\pi}{L}x\right)dx$, is evaluated using the relation

$$\sin A\sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$$

The integral I becomes, where here $A = \frac{2\pi x}{L}, B = \frac{n\pi}{L}x$

$$
\begin{aligned}
I &= \frac{A}{2}\int_0^{\frac{L}{2}} \cos\left(\frac{2\pi x}{L} - \frac{n\pi x}{L}\right) - \cos\left(\frac{2\pi x}{L} + \frac{n\pi x}{L}\right)dx \\
&= \frac{A}{2}\left(\int_0^{\frac{L}{2}} \cos\left(\frac{(2-n)\pi x}{L}\right)dx - \int_0^{\frac{L}{2}} \cos\left(\frac{(2+n)\pi x}{L}\right)dx\right) \\
&= \frac{A}{2}\left(\left[\frac{\sin\left(\frac{(2-n)\pi x}{L}\right)}{\frac{(2-n)\pi}{L}}\right]_0^{\frac{L}{2}} - \left[\sin\left(\frac{\frac{(2+n)\pi x}{L}}{\frac{(2+n)\pi}{L}}\right)\right]_0^{\frac{L}{2}}\right) \\
&= \frac{A}{2}\left(\frac{L}{(2-n)\pi}\left[\sin\left(\frac{(2-n)\pi x}{L}\right)\right]_0^{\frac{L}{2}} - \frac{L}{(2+n)\pi}\left[\sin\left(\frac{(2+n)\pi x}{L}\right)\right]_0^{\frac{L}{2}}\right) \\
&= \frac{A}{2}\left(\frac{L\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right)}{(2-n)\pi} - \frac{L\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)}{(2+n)\pi}\right) \\
&= \frac{LA}{2\pi}\left(\frac{\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right)}{(2-n)} - \frac{\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)}{(2+n)}\right) \\
&= \frac{LA}{2\pi}\left(\frac{(2+n)\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right) - (2-n)\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)}{(2-n)(2+n)}\right) \\
&= \frac{LA}{2\pi(2-n)(2+n)}\left((2+n)\sin\left(\frac{(2-n)\pi\frac{L}{2}}{L}\right) - (2-n)\sin\left(\frac{(2+n)\pi\frac{L}{2}}{L}\right)\right) \\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left((2+n)\sin\left(\frac{2\pi\frac{L}{2} - n\pi\frac{L}{2}}{L}\right) - (2-n)\sin\left(\frac{2\pi\frac{L}{2} + n\pi\frac{L}{2}}{L}\right)\right)
\end{aligned}
$$

Hence

$$
\begin{aligned}
I &= \frac{LA}{2\pi\left(4-n^2\right)}\left((2+n)\sin\left(\pi-\frac{n}{2}\pi\right)-(2-n)\sin\left(\pi+\frac{n}{2}\pi\right)\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(2\sin\left(\pi-\frac{n}{2}\pi\right)+n\sin\left(\pi-\frac{n}{2}\pi\right)-2\sin\left(\pi+\frac{n}{2}\pi\right)+n\sin\left(\pi+\frac{n}{2}\pi\right)\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(2\left[\sin\left(\pi-\frac{n}{2}\pi\right)-\sin\left(\pi+\frac{n}{2}\pi\right)\right]+n\left[\sin\left(\pi-\frac{n}{2}\pi\right)+\sin\left(\pi+\frac{n}{2}\pi\right)\right]\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(-2\left[\sin\left(\pi+\frac{n}{2}\pi\right)-\sin\left(\pi-\frac{n}{2}\pi\right)\right]+n\left[\sin\left(\pi+\frac{n}{2}\pi\right)+\sin\left(\pi-\frac{n}{2}\pi\right)\right]\right)
\end{aligned}
$$

Using $\sin\left(x+y\right)-\sin\left(x-y\right)=2\cos x\sin y$ and $\sin\left(x+y\right)+\sin\left(x-y\right)=2\sin x\cos y$ on the above gives (where $x=\pi, y=\frac{n}{2}\pi$ in this case)

$$
\begin{aligned}
I &= \frac{LA}{2\pi\left(4-n^2\right)}\left(-2\left[2\cos\pi\sin\frac{n}{2}\pi\right]+n\left[2\sin\pi\cos\frac{n}{2}\pi\right]\right)\\
&= \frac{LA}{2\pi\left(4-n^2\right)}\left(-2\left[-2\sin\frac{n}{2}\pi\right]\right)\\
&= \frac{2LA}{\pi\left(4-n^2\right)}\left(\sin\frac{n}{2}\pi\right)
\end{aligned}
$$

Hence (9) becomes

$$
\begin{aligned}
D_n &= \frac{2}{L}\left(\frac{2LA}{\pi\left(4-n^2\right)}\left(\sin\frac{n}{2}\pi\right)\right)\\
&= \frac{4A}{\pi\left(4-n^2\right)}\sin\frac{n}{2}\pi\\
&= \frac{-4A}{\pi\left(n^2-4\right)}\sin\left(\frac{n}{2}\pi\right)
\end{aligned}
$$

Now all coefficients of the Fourier sine series are found. Therefore the solution (7) becomes

$$
\begin{aligned}
\psi(x,t) &= \psi_1(x,t)+\psi_2(x,t)+\sum_{n=3}^{\infty}D_n\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}\\
&= D_1\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t}+D_2\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t}+\sum_{n=3}^{\infty}\left(\frac{-4A}{\pi\left(n^2-4\right)}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}\\
&= \frac{4A}{3\pi}\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t}+\frac{1}{2}A\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t}+\sum_{n=3}^{\infty}\left(\frac{-4A}{\pi\left(n^2-4\right)}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}
\end{aligned}
$$

Therefore the final solution is

$$\psi(x,t) = \frac{4A}{3\pi}\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t} + \frac{1}{2}A\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t} - \frac{4A}{\pi}\sum_{n=3}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(n^2-4\right)}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}$$

When $n = 4, 6, 8, \cdots$ then $\sin\left(\frac{n\pi}{2}\right) = 0$. Therefore only odd terms survive

$$\psi(x,t) = \frac{4A}{3\pi}\sin\left(\frac{\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar\pi^2}{\mu L^2}t} + \frac{A}{2}\sin\left(\frac{2\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar 4\pi^2}{\mu L^2}t} - \frac{4A}{\pi}\sum_{n=3,5,7,\cdots}^{\infty}\frac{\sin\left(\frac{n\pi}{2}\right)}{\left(n^2-4\right)}\sin\left(\frac{n\pi}{L}x\right)e^{-\frac{i}{2}\frac{\hbar n^2\pi^2}{\mu L^2}t}$$